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A LIMIT THEOREM FOR CONDITIONED RECURRENT RANDOM WALK ATTRACTED TO A STABLE LAW[1]

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1. Introduction. Consider an ensemble of independent particles whose motion describes a random walk on Z^d, the d-dimensional lattice of integers. If **A** is an arbitrary subset of Z^d and the random walk is assumed recurrent (consequently $d \leqq 2$), then as time passes it becomes increasingly unlikely that any given particle has avoided **A**. Suppose, however, that at each stage attention is restricted to only those particles whose past history is such that **A** has been avoided. Then it is of interest to investigate the possible distortive effects of this conditioning on the asymptotic behavior of the particle motion. Suppose **A** is finite and $\tilde{g}_{\mathbf{A}}(0) \neq 0$ (the function $\tilde{g}_{\mathbf{A}}(x)$ of potential-theoretic interest is defined below and the connection between this condition and the motion of the random walk established) and suppose that the underlying distribution F governing the particle transitions is attracted to a stable law $G_\alpha(1 \leqq \alpha \leqq 2$ is the index of the stable law). The principal result of the paper (Theorem 2.1) states that the conditional distribution of the particles whose past motion has avoided the set **A** is also attracted to a limit distribution H_α. Except for the case $d = 1$ with G_α a Cauchy distribution and the case $d = 2$ with G_α a normal distribution, the distributions G_α and H_α are in general different. For $d = 1$ and $\alpha = 2$, under certain further restrictions on **A**, G_α turns out to be a two-sided Rayleigh distribution. It is the case, however, that the same constants normalizing the particle position may be used in the statement of the attraction of the conditioned motion to H_α as in the statement of the attraction of the unconditioned motion to the stable law G_α. In preparation we first review some basic definitions and record some preliminary facts about recurrent lattice random walk.

We let $p: Z^d \times Z^d \to [0, 1]$ be the transition function of the random walk. Thus,

(i) $$p(x_1, x_2) = p(0, x_2 - x_1) \quad \text{for} \quad x_1, x_2 \in Z^d$$

(ii) $$\textstyle\sum_{x \in Z^d} p(0, x) = 1,$$

and we inductively define

$$p_n(x_1, x_2) = \textstyle\sum_{y \in Z^d} p_{n-1}(x_1, y) p(y, x_2) \quad \text{for} \quad n = 2, 3 \cdots.$$

An underlying probability space (Ω, P, B) is assumed to have been constructed, on which a sequence of independent random variables X_i, $i = 1, 2, \cdots$ (the increments of the random walk) are defined, such that $P[X_i = x] = p(0, x)$, $i = 1, 2, \cdots$.

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We denote by F the common distribution of the increments. The starting point x_0 is fixed but arbitrary, and we use the notation P^x for the underlying probability measure to indicate that $x_0 = x$. Similarly we let $E^x f = \int_\Omega f dP^x$ for any $f: \Omega \to R$ (the reals) for which the integral on the right is defined. When $x = 0$ we simply write $P[A]$.

We will call $S_n = x_0 + \sum_{i=1}^n X_n$ the nth partial sum of the random walk. Unless explicitly stated otherwise the following assumptions will be made. All random walks are (i) aperiodic, (ii) recurrent.

Aperiodicity is defined in [8], but for our purposes the following characterization is more convenient. If $\phi(t)$ with $t = (t_1, \cdots t_d)$ is the characteristic function of X_1, then the associated random walk is *aperiodic* iff $\phi(t) = 1$ implies each coordinate of t is an integral multiple of 2π.

A random walk is said to be *recurrent* if for each x, $\sum_{n=1}^\infty p_n(0,x) = \sum_{n=1}^\infty P[S_n = x] = \infty$, and *transient* if the alternative is true. Again as a matter of convenience in application we require the following equivalent statement (see [8] once again). A random walk is recurrent iff

$$\text{(I)} \qquad \int_{C^d} \operatorname{Re} \frac{dt}{1-\phi(t)} = \infty, \qquad \text{where } C^d = \{t : |t_i| \leqq \pi,\ i = 1, 2, \cdots d\}.$$

We now make a list of useful definitions and related notation. Let $\mathbf{A}$ be an arbitrary subset of Z^d, then

$$\begin{aligned} (1.1) \qquad T_{\mathbf{A}} &= \min[n > 0 : S_n \in \mathbf{A}] \qquad \text{if such an } n \text{ exists,} \\ &= \infty, \qquad \text{otherwise;} \end{aligned}$$

and $r_n(x, \mathbf{A}) = P^x[T_{\mathbf{A}} > n]$, for $x \in Z^d$, $n \geqq 0$, ($T = T_{\{0\}}$ and $r_n = P[T > n]$). We remark that by recurrence $T_{\mathbf{A}}$ is finite with probability one and $\lim_{n\to\infty} r_n(x, \mathbf{A}) = 0$.

$$\begin{aligned} & Q_{\mathbf{A}}^{(0)}(x,y) = \delta(x,y) \text{(the Kronecker } \delta) \\ (1.2) \qquad & Q_{\mathbf{A}}^{(n)}(x,y) = P^x[S_n = y; T_{\mathbf{A}} \geqq n] \\ & Q^{(n)}(x,y) = Q_{\{0\}}^{(n)}(x,y); \qquad f_n = Q^{(n)}(0,0). \end{aligned}$$

(1.3) $\tilde{g}_{\mathbf{A}}(x,y) = \sum_{n=0}^\infty Q_{\mathbf{A}}^{(n)}(x,y) =$ the expected number of visits to y starting at x up to and including the first visit to $\mathbf{A}$.

$$\begin{aligned} (1.4) \qquad \tilde{\Pi}_{\mathbf{A}}(x,y) &= P^x[S_{T_{\mathbf{A}}} = y] \quad \text{for } \ y \in \mathbf{A} \\ &= 0 \qquad\qquad\quad\ \ \text{for } \ y \in \mathbf{A}. \end{aligned}$$

It is shown in [8] by applying the integral criterion (I) that for $d \geqq 3$ all random walks are transient, and so from now on $d = 1$ or 2. We now record some important results about recurrent random walks for later use.

Let $|y|$ denote the ordinary d-dimensional Euclidean distance from y to the origin. If a random walk is recurrent and $d = 1$ with $\sigma^2 = \sum_{-\infty}^\infty x^2 p(0,x) = \infty$ or $d = 2$, then for $\mathbf{A}$ finite

$$(1.5) \qquad \tilde{g}_{\mathbf{A}}(x) = \lim_{|y|\to\infty} \tilde{g}_{\mathbf{A}}(x,y) \quad \text{exists;}$$

while for $d = 1$ with $\sigma^2 < \infty$

$$\tilde{g}_A(x) = \tfrac{1}{2}\lim_{y\to+\infty}[\tilde{g}_A(x,y)+\tilde{g}_A(x,-y)] \quad \text{exists.} \tag{1.6}$$

For the proof see [8]. In addition it is shown in [4] that for $x\in \mathbf{A}$, $\tilde{g}_A(x,y) = \tilde{\Pi}_{-A}(-y,-x)$. Thus for $d = 1$ with $\sigma^2 = \infty$ or $d = 2$ and $x \in \mathbf{A}$

$$\tilde{g}_A(x) = \lim_{|y|\to\infty}\tilde{\Pi}_{-A}(-y,-x); \tag{1.7}$$

while for $d = 1$ with $\sigma^2 < \infty$

$$\tilde{g}_A(x) = \tfrac{1}{2}\lim_{y\to+\infty}[\tilde{\Pi}_{-A}(-y,-x)+\tilde{\Pi}_{-A}(y,-x)]. \tag{1.8}$$

Our primary aim in this paper then is to investigate the probability measures $P^y[\,\cdot \mid T_A > n]$ and the behavior as $n \to \infty$ of the associated sample paths when A is a finite set under the condition that $\tilde{g}_A(x) > 0$ for appropriate $x \in Z^d$. By checking the definitions one sees that this is in the nature of a condition that the random walk starting at x can *escape* to infinity along a path which avoids the set $\mathbf{A}$. If we let I_A denote the smallest d-dimensional interval containing the finite set $\mathbf{A}$, $E_A = Z^d - I_A$ and for $d = 1$

$$x_+(\mathbf{A}) = \max\{x\in\mathbf{A}\}, \qquad x_-(\mathbf{A}) = \min\{x\in\mathbf{A}\},$$
$$\mathbf{A}_+ = \{x > x_+\}, \qquad \mathbf{A}_- = \{x < x\},$$

the precise statement is the following

PROPOSITION. *Let* $\mathbf{A}$ *be a finite subset of* Z^d. *If* $d = 2$ *or* $d = 1$ *with* $\sigma^2 < \infty$, *then for* $x \in Z^d$

$$\tilde{g}_A(x) > 0 \quad \text{iff} \quad P^x[T_{E_A} < T_A] > 0; \tag{1.9}$$

if $d = 1$ *with* $\sigma^2 = \infty$, *then*

$$\tilde{g}_A(x) > 0 \quad \text{iff} \quad \textit{both} \quad P^x[T_{A_+} < T_A] > 0 \quad \textit{and} \quad P^x[T_{A_-} < T_A] > 0. \tag{1.10}$$

PROOF. We first show the sufficiency of the conditions in (1.9) and (1.10). Inasmuch as for $x \notin \mathbf{A}$

$$P^x[T_{E_{A\cup\{x\}}} < T_{A\cup\{x\}}] > 0 \quad \text{iff} \quad P^x[T_{E_A} < T_A] > 0$$

with a similar result for $\mathbf{A}_\pm$ and $\tilde{g}_{A\cup\{x\}}(x,y) \leqq \tilde{g}_A(x,y)$, there is no loss of generality in assuming $x \in \mathbf{A}$.

The main tool in the proof is the following adaptation of a result in [9]. If $\mathbf{A}_n$ is a collection of finite subsets of Z^d increasing to Z^d, then for each fixed y and $\varepsilon > 0$ there is an $n(y,\varepsilon)$ such that $n \geqq n(y,\varepsilon)$ and $y \in \mathbf{A}_n$ imply $\lim_{|x|\to\infty}\tilde{\Pi}_{A_n}(x,y) < \varepsilon$. We apply this result to the reversed random walk, i.e. the random walk with transition function p_* such that $p_*(0,x) = p(x,0)$. Let $\mathbf{A}_1 = I_A$, but otherwise the $\mathbf{A}_n$ may be an arbitrary sequence of finite sets increasing to Z^d. Then

$$\lim_{|y|\to\infty} P_*^y[T_{A_n - I_A} < T_{I_A}] > 0 \tag{1.11}$$

for n sufficiently large. Now if $x_1, x_2 \in \mathbf{A}_+$ or $x_1, x_2 \in \mathbf{A}_-$ in the case $d = 1$, or $x_1, x_2 \in E_{\mathbf{A}}$ in the case $d = 2$, we have

$$P_*^{x_1}[T_{\{x_2\}} < T_{I_{\mathbf{A}}}] > 0. \tag{1.12}$$

For $d = 1$ this is a consequence of the fact that by recurrence there is some path from x_1 to x_2, and thus, by choosing an appropriate permutation of the transitions comprising this path there must in fact be such a path which does not enter $\mathbf{A}$. When $x_1, x_2 \in \mathbf{A}_+$, for example, this may be achieved by taking a permutation which puts all of the transitions to the right (positive steps) before all of those to the left (negative steps). The argument for $d = 2$ follows similar lines.

By hypothesis, for $d = 1$ with $\sigma^2 < \infty$ and for $d = 2$ there exists an $x_2 \in E_{\mathbf{A}}$ such that

$$P_*^{x_2}[S_{T_{\mathbf{A}}} = x] > 0; \tag{1.13}$$

while for $d = 1$ with $\sigma^2 = \infty$ there exist $x_2 \in \mathbf{A}_+$ and $x_2' \in \mathbf{A}_-$ such that

$$P_*^{x_2}[S_{T_{\mathbf{A}}} = x] > 0 \quad \text{and} \quad P_*^{x_2'}[S_{T_{\mathbf{A}}} = x] > 0. \tag{1.14}$$

Combining (1.11) through (1.14) we obtain for $d = 1$ with $\sigma^2 = \infty$ and $d = 2$

$$\lim_{|y| \to \infty} P_*^y[S_{T_{\mathbf{A}}} = x] > 0. \tag{1.15}$$

For $d = 1$ with $\sigma^2 < \infty$ we require the additional fact that

$$\lim_{y \to -\infty} \tilde{\Pi}_{Z_+}(y, x) = P[\xi \geqq x]/E[\xi], \tag{1.16}$$

where Z_+ is the positive integers and ξ is the positive ladder random variable. (See [8] for the definition and a proof of (1.16).) A similar result holds for $y \to +\infty$ for Z_-, the negative integers, and the negative ladder random variable. Combining this result with (1.12) and (1.13) we obtain for $d = 1$, $\sigma^2 < \infty$

$$\tfrac{1}{2} \lim_{y \to \infty} (P_*^y[S_{T_{\mathbf{A}}} = x] + P_*^{-y}[S_{T_{\mathbf{A}}} = x]) > 0. \tag{1.17}$$

It is shown in [4] that $P_*^y[S_{T_{\mathbf{A}}} = x] = P^{-y}[S_{T_{-\mathbf{A}}} = -x]$. In view of (1.4) and (1.8) therefore, (1.15) and (1.17) imply $\tilde{g}_{\mathbf{A}}(x) > 0$.

The necessity of the conditions in (1.9) and (1.10) is a straightforward consequence of the definitions of the quantities involved. This completes the proof of the proposition.

We will confine our attention until Section 4 to the case $d = 1$ in which the distribution F of the increments of the random walk belongs to the domain of attraction of a stable law of index α $(0 < \alpha \leqq 2)$, i.e., there exist constants $B_n > 0$ and A_n and a distribution G_α on R^d such that

$$\lim_{n \to \infty} P[S_n/B_n - A_n \leqq x] = G_\alpha(x). \tag{1.18}$$

Here again we require some preliminary discussion. It was shown by P. Lévy and Y. Khintchine (a discussion appears in [2]) that if G_α has characteristic function ϕ_α, then the following representation theorem holds:

$$\ln \phi_\alpha(t) = i\gamma t - c\,|t|^\alpha(1 + i\beta\,|t|^{-1} t\omega(t, \alpha)), \tag{1.19}$$

where α, β, γ and c are constants such that $0 < \alpha \leqq 2$, $-1 \leqq \beta \leqq 1$ *and* $c \geqq 0$ *with*

$$\omega(t, \alpha) = \tan \tfrac{1}{2}\pi\alpha \qquad \alpha \neq 1$$
$$= 2\pi^{-1} \ln |t| \qquad \alpha = 1.$$

It can be shown by applying the criterion (I) to the Lévy-Khintchine representation that the assumption of recurrence for a stable law implies $1 \leqq \alpha \leqq 2$.

The following criterion (again discussed in [2]) is very useful for determining when a distribution F is attracted to a stable law.

THEOREM 1.1. (*Gnedenko and Doeblin*) *In order that the distribution F belong to the domain of attraction of a stable law of index α, $0 < \alpha \leqq 2$, it is necessary and sufficient that for* $0 < \alpha < 2$

(i) $\quad F(-x)/[1-F(x)] \to c_1/c_2 \quad$ *as* $\quad x \to \infty, \quad$ *with* $\quad c_1, c_2 \geqq 0,$

(ii) $\quad x^\alpha[1-F(x)+F(-x)] = L(x) \quad$ *with* $\quad L(x)$ *slowly varying,*

while for $\alpha = 2$

$$x^2 \textstyle\int_{|y| \geqq x} dF(y) / \int_{|y| \leqq x} y^2 \, dF(y) \to 0 \quad as \quad x \to \infty.$$

Several comments concerning the theorem are necessary.

(A) A nonnegative function $L(x)$ on $(0, \infty)$ is said to be slowly varying if for every fixed $y > 0$ $\lim_{x\to\infty}[L(yx)/L(x)] = 1$. Karamata showed in his classic paper [3] that L is slowly varying if and only if it can be represented in the form

$$L(x) = c(x) \exp\left[\textstyle\int_1^x t^{-1}\zeta(t)\, dt\right] \tag{1.20}$$

with $\lim_{x\to\infty} c(x) = c$, $0 < c < \infty$ and $\lim_{x\to\infty}\zeta(x) = 0$. It is easily shown from this characterization that given any $\delta > 0$, there is an x_0 such that $x \geqq x_0$ implies $L(x) < x^\delta$. We will make use of this fact later on.

(B) Under conditions (i) and (ii) $F(x)$ in fact belongs to the domain of attraction of the stable law of index α with $\beta = (c_1 - c_2)/(c_1 + c_2)$ for $\alpha \neq 1$ and $\beta = (c_2 - c_1)/(c_1 + c_2)$ for $\alpha = 1$. The constants c and γ of the limit stable law are not uniquely determined. Once it is known that there is some choice of constants A_n and B_n for which (1.18) holds, then it is not difficult to see that for any $a > 0$ and b there is a choice of the A_n and B_n for which (1.18) holds with $G_\alpha(x)$ replaced by $G_\alpha(ax+b)$. We can however make the following statement. Let $\chi(x) = 1 - F(x) + F(-x)$ and define the inverse function $\chi^{-1}(x) = \inf[y : \chi(y) \leqq x]$. Then it is embodied in the Kolmogorov-Gnedenko proof of Theorem (1.1) that for $0 < \alpha < 2$ the choice

$$B_n = \chi^{-1}(n^{-1}(c_1 + c_2)) \tag{1.21}$$

is possible, in which case for $\alpha = 1$, $c = \frac{1}{2}[(c_1 + c_2)\pi]$; while for $0 < \alpha \leqq 2$, $\alpha \neq 1$, $c = (c_1 + c_2)\Gamma(1-\alpha)\cos\frac{1}{2}\pi\alpha$.

Our analysis will require more detailed information about the sequences A_n and

B_n. We have already indicated a choice for the B_n in the case $0 < \alpha < 2$. For $\alpha = 2$ we have from [2] (Section 26 Theorem 4 and its proof)

$$B_n^2 = n[\int_{|x|<C_n} x^2\, dF(x) - (\int_{|x|<C_n} x\, dF(x))^2], \tag{1.22}$$

with C_n a sequence of constants the only properties of which concern us are: (i) $C_n \to \infty$ as $n \to \infty$ and (ii) $C_n = o(B_n)$ as $n \to \infty$. For $0 < \alpha < 2$ it follows from Lemma 2-A of [11] that under the hypotheses of Theorem 1.1

$$\lim_{x\to\infty} [\chi^{-1}(x)/\chi^{-1}(xy)] = y^{1/\alpha} \quad \text{for fixed} \quad y > 0.$$

Hence $B_n = n^{1/\alpha}\hat{L}(n)$ where $\hat{L}(x)$ is slowly varying.

The treatment of the case $\alpha = 2$ requires a different approach. The argument we give now holds in fact for any law attracted to a symmetric stable law when the A_n may be taken to be zero (which we will see below to be the case for $1 < \alpha \leqq 2$ if $\int_{-\infty}^{\infty} x dF(x) = 0$ and for $0 < \alpha < 1$ in general).

Let $F^{(n)} = F*F*\cdots*F$ (n times) be the nth fold convolution of F and let G_α be the limit stable law. Since all stable laws have continuous densities (Section 36 of [2])

$$F^{(n)}(B_n x) \to G_\alpha(x) \quad \text{uniformly in } x \text{ as} \quad n \to \infty.$$

Let $k_0 \in N$ be fixed and let $X_1, X_2 \cdots$ be independent, identically distributed with distribution F. Then as $n \to \infty$

$$P\left[\frac{X_1+X_2+\cdots+X_n}{B_n}+\cdots+\frac{X_{(k_0-1)n+1}+\cdots+X_{k_0 n}}{B_n} \leqq \frac{B_{k_0 n}}{B_n}x\right] \to G_\alpha(x)$$

on the one hand, and on the other differs for large n from $G_\alpha^{(k_0)}(xB_{k_0 n}/B_n)$ by very little (inasmuch as for probability distributions $F_{n,1}$ and $F_{n,2}$, $F_{n,1} \to F_1$ and $F_{n,2} \to F_2$ imply $F_{n,1}*F_{n,2} \to F_1*F_2$ (see, e.g., Lemma 1 of Section 17 in [6])). But by the stability and symmetry of G_α, this latter expression is the same as $G_\alpha(xk_0^{-1/\alpha}B_{k_0 n}/B_n)$, and therefore $B_{k_0 n}/B_n \to k_0^{1/\alpha}$ as $n \to \infty$. Thus B_n has the form $B_n = n^{1/\alpha}\hat{L}(n)$ with $\hat{L}(n)$ slowly varying for $\alpha = 2$ as well. We note that if we extend the definition of B_n to R_+ by setting $B(x) = B_{[x]}$, then $B(x)$ remains of the form $x^{1/\alpha}\hat{L}(x)$ with $\hat{L}(x)$ slowly varying.

For the sequence A_n we use another basic result of Kolmogorov and Gnedenko, Theorem 4 of Section 25 in [2], stating that the most general choice of A_n is

$$A_n = n\int_{|x|<t} x\, dF(B_n x) - \gamma_n(t), \tag{1.23}$$

where $t > 0$ is fixed and $\gamma_n(t)$ is any convergent sequence. From this characterization it is shown in the Appendix of [1] that

$$\begin{aligned} A_n &= nB_n^{-1}\int_{-\infty}^{\infty} x\, dF(x), && 1 < \alpha < 2, \\ &= 0, && \alpha < 1, \\ &= n\,\mathrm{Im}[1 - \phi(B_n^{-1})], && \alpha = 1 \end{aligned} \tag{1.24}$$

are possible choices. First we remark that if F lies in the domain of attraction of a stable law of index α, then it is known (Theorem (3) of Section 35 in [2]) that it possesses all δth order moments for $0 \leqq \delta < \alpha$. In particular then, for $1 < \alpha \leqq 2$ the mean μ exists and our assumption of recurrence implies $\mu = 0$. Consequently, we may take $A_n \equiv 0$. However, when $\alpha = 1$ a first moment may or may not exist and it is necessary to introduce the notion of *normal* attraction.

DEFINITION. A distribution F is said to belong to the domain of *normal* attraction of a stable law G_α of index α, $0 < \alpha \leqq 2$ if it belongs to its domain of (general) attraction with $B_n = n^{1/\alpha}$ a possible choice for the norming constants.

An equivalent way to formulate the criterion for normal attraction is to require for $\alpha < 2$ that the slowly varying function in the relation $x^\alpha \chi(x) = L(x)$ have a non-zero finite limit as $x \to \infty$, and to require a finite variance for $\alpha = 2$ (see Theorems 4 and 5 of Section 35 in [2]).

We now state the following result (proved in [1]).

THEOREM 1.2. *Let F belong to the domain of normal attraction of a stable law of index* 1. *Then*

(i) $$\lim_{t\to\infty} \operatorname{Im}\left[t^{-1}(1-\phi(t))\right] = \mu \quad \textit{if and only if}$$

(ii) $$\lim_{x\to\infty} \int_{-x}^{x} \zeta\, dF(\zeta) = \mu.$$

In particular, it follows from (1.24) that the centering constants A_n may be taken to be zero if and only if (ii) holds for μ finite.

Finally we record the following result. The proof is essentially a generalization of the argument relating the asymptotic tail behavior of the distribution of a symmetric stable law to the behavior of its characteristic function near the origin. A detailed proof appears in [1].

THEOREM 1.3. *Let F be the probability distribution of a random variable X. Assume F satisfies the hypotheses of Theorem* 1.1. *Then in the notation of that theorem*:

(i) *For* $1 < \alpha < 2$ *and* $E[X] = 0$

$$\lim_{t\to 0\pm} \frac{1-\phi(t)}{\chi(1/|t|)} = \frac{c}{c_1+c_2}(1 \pm i\beta \tan \tfrac{1}{2}\pi\alpha).$$

(ii) *For* $0 < \alpha < 1$

$$\lim_{t\to 0\pm} \frac{1-\phi(t)}{\chi(1/|t|)} = \frac{c}{c_1+c_2}(1 \pm i\beta \tan \tfrac{1}{2}\pi\alpha).$$

(iii) *For* $\alpha = 2$ *and* $E[X] = 0$

$$\lim_{t\to 0} \frac{1-\phi(t)}{t^2 \int_{-1/|t|}^{1/|t|} x^2\, dF(x)} = \frac{1}{2}.$$

(iv) *For* $\alpha = 1$.

$$\lim_{t\to 0} \operatorname{Re}\left[\frac{1-\phi(t)}{\chi(1/|t|)}\right] = \tfrac{1}{2}\pi;$$

and if for $\alpha = 1$ the additional hypothesis is made that F is normally *attracted to its limit stable law, then*

$$\lim_{t\to 0\pm} \operatorname{Im}\left[\frac{1-\phi(t)}{\ln|t|\,\chi(1/|t|)}\right] = \pm\beta.$$

2. Basic limit theorem. The discussion in this section will be limited to the special class of one-dimensional recurrent random walks whose increments have their common distribution belonging to the domain of attraction of a stable law of index α, $1 \leqq \alpha \leqq 2$.

THEOREM 2.1. *Let $S_n = x_0 + \sum_{k=1}^{n} X_k$ be the* nth *partial sum of an integer-valued, recurrent, aperiodic walk. If F is the distribution of X_1, we assume*

(i) *F belongs to the domain of attraction of a stable law of index α, $1 < \alpha \leqq 2$. Thus there is a sequence $B_n > 0$ such that*

$$\lim_{n\to\infty} P[S_n/B_n \leqq x] = G_\alpha(x) \qquad x \in R,$$

where G_α is a probability distribution with a characteristic function ϕ_α of the form

$$\ln \phi_\alpha(t) = -c\,|t|^\alpha(1 + it\,|t|^{-1}\beta \tan \tfrac{1}{2}\pi\alpha) \equiv -b\,|t|^\alpha$$

with $c > 0$ and $-1 \leqq \beta \leqq 1$.

Alternatively, we assume

(ii) *F belongs to the domain of normal attraction of a stable law of index $\alpha = 1$ and, in addition,* $\lim_{x\to\infty} \int_{-x}^{x} \zeta\, dF(\zeta) = \mu < \infty$.

Thus there is a sequence $B_n > 0$ such that $\lim_{n\to\infty} P[S_n/B_n \leqq x] = G_1(x)$, $x \in R$, *where G_1 has characteristic function ϕ_1 such that* $\ln \phi_1(t) = -c|t|$ with $c > 0$.

Then in both cases (i) *and* (ii) *if $T = T_{\{0\}}$, we have that*

$$\lim_{n\to\infty} P[S_n/B_n \leqq x \mid T > n] = H_\alpha(x),$$

where H_α is a probability distribution with a bounded continuous density h_α with characteristic function Ψ_α given by

$$\Psi_\alpha(t) = 1 - b\,|t|^\alpha \int_0^1 x^{(1/\alpha)-1} \phi_\alpha[t(1-x)^{1/\alpha}]\, dx. \tag{2.2}$$

The two special cases $\alpha = 2$ (normal) and $\alpha = 1$ (Cauchy) permit an explicit evaluation of the limit law:

$$h_2(x) = \tfrac{1}{2}\sigma^{-2}x^2 \exp\left[-\tfrac{1}{2}\sigma^{-2}x^2\right] \quad \text{(a two-sided Rayleigh density)},$$

where $\sigma^2 = 2c$ is the variance of the limit normal law with characteristic function ϕ_2:

$$h_1(x) = \frac{1}{\pi}\frac{c}{c^2 + x^2}.$$

Before proceeding to the proof we make several remarks and prove a lemma.

First, in the statement of the theorem we have made use of our earlier remark that for $\alpha > 1$ the sequence A_n of centering constants may be taken to be identically zero. For $\alpha = 1$ we note that by Theorem 1.2 Condition 2.1 guarantees that $\lim_{t\to 0} \operatorname{Im}[t^{-1}(1-\phi(t))] = \mu$. It then follows from Condition (I) for recurrence and statement (iv) of Theorem 1.3 that the underlying random walk must be recurrent and that $\beta = 0$ in the limit stable law (and therefore $\gamma = -\mu$). Consequently, $\lim_{n\to\infty} P[S_n/B_n - \mu \leqq x] = G_1(x+\mu)$. Therefore taking G_1 as the limit law we may assume $A_n = 0$ for $\alpha = 1$ as well.

The second fact we need is that without any loss of generality we may assume B_n to be of the form $B_n = n^{1/\alpha}\tilde{L}(n)$, $0 < \alpha \leqq 2$, where $\tilde{L}$ is a slowly varying function. In the particular case $\alpha = 1$, we further require $\lim_{n\to\infty} \tilde{L}(n)$ is finite and non-zero (i.e., the attraction must be normal). Finally, we remark that in the Cauchy case the theorem states the interesting result that conditioning on the event $[T > n]$ plays no role in the limit.

We now prove the following lemma.

LEMMA 2.1. *Under the hypotheses of Theorem* 2.1 *for* $1 < \alpha \leqq 2$

$$\lim_{n\to\infty} \frac{nr_n}{B_n} = \frac{\sin \pi/\alpha}{\pi g_\alpha(0)}, \tag{2.3}$$

where g_α is the density of G_α; for $\alpha = 1$ $\lim_{n\to\infty} r_n \ln n = \frac{1}{2}\pi^2 l$ where $l = \lim_{n\to\infty} \tilde{L}(n)$ is finite and non-zero.

PROOF. Let $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and $U(x) = \sum_{n=0}^{\infty} u_n x^n$. Then $R(x) = [(1-x)U(x)]^{-1}$. By the local limit theorem for lattice variables attracted to a stable law[2] (recalling our assumption of aperiodicity) we have that

$$\lim_{n\to\infty} B_n p_n(0,0) = \lim_{n\to\infty} B_n u_n = g_\alpha(0).$$

By a standard Abelian theorem it easily follows that

$$U(x) \sim \frac{g_\alpha(0)\Gamma(1-1/\alpha)(1-x)^{(1/\alpha)-1}}{\tilde{L}(1/1-x)} \qquad \text{as} \quad x \to 1-.$$

Hence

$$R(x) \sim \frac{\tilde{L}(1/1-x)}{g_\alpha(0)\Gamma(1-1/\alpha)}(1-x)^{-(1/\alpha)} \qquad \text{as} \quad x \to 1-. \tag{2.4}$$

Finally, since r_n is a nonincreasing sequence, we may apply Karamata's Tauberian theorem for $\alpha \neq 1$ to (2.4) to obtain

$$\begin{aligned} r_n &\sim \frac{\tilde{L}(n)}{n^{1-1/\alpha}} \frac{1}{g_\alpha(0)\Gamma(1-1/\alpha)\Gamma(1/\alpha)} \\ &= \frac{\tilde{L}(n)}{n^{1-1/\alpha}} \frac{\sin \pi/\alpha}{\pi g_\alpha(0)} \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Since $B_n = n^{1/\alpha}\tilde{L}(n)$ we have demonstrated (2.3) for $1 < \alpha \leqq 2$.

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[2] As stated in Section 49 and Section 50 of [2] the theorem is valid except when $\alpha = 2$, $\sigma^2 = \infty$. However, the latter case is proved by C. Stone in [10].

The case $\alpha = 1$ is contained in the proof of Theorem 18 in [5]. This completes the proof.

PROOF OF THEOREM 2.1. For $x \neq 0$, a simple decomposition of the event $[T \leqq n]$ gives

(2.5) $$P[S_n/B_n \in dx \mid T > n] = r_n^{-1} P[S_n/B_n \in dx] - r_n^{-1} \sum_{k=1}^{n-1} f_k P[S_{n-k}/B_n \in dx].$$

After a summation by parts this expression reduces to

(2.6) $$\sum_{k=0}^{n-1} r_n^{-1} r_k (P[S_{n-k}/B_n \in dx] - P[S_{n-k-1}/B_n \in dx]).$$

If we let $\Psi_{\alpha,n}$ be the characteristic function of this probability measure, we obtain

$$\Psi_{\alpha,n}(t) = \sum_{k=0}^{n-1} r_n^{-1} r_k [\phi^{n-k}(t/B_n) - \phi^{n-k-1}(t/B_n)]$$
$$- \sum_{k=0}^{n-1} r_n^{-1} r_k (P[S_{n-k} = 0] - P[S_{n-k-1} = 0]);$$

and since $\sum_{k=0}^{n} r_k u_{n-k} = 1$,

(2.7) $$\Psi_{\alpha,n}(t) = 1 - \sum_{k=0}^{n-1} r_n^{-1} r_k [1 - \phi(t/B_n)] \phi^{n-k-1}(t/B_n).$$

Now we recall that $B_n = B(n) = n^{1/\alpha} \tilde{L}(n)$ with $\tilde{L}$ a slowly varying function. Therefore

$$\frac{B(n-(k+1))}{B(n)} = \frac{B(n(1-(k+1)/n))}{B(n)} \sim \left(1 - \frac{k+1}{n}\right)^{1/\alpha} \text{for large } n \text{ and } \quad 0 \leqq k \leqq n(1-\varepsilon).$$

By our hypotheses $\lim_{n\to\infty} \phi^n(t/B_n) = \phi_\alpha(t)$, with the convergence uniform for t on finite intervals. Thus

(2.8) $$\lim_{n\to\infty} (\phi^{n-(k+1)}(t/B_n) - \phi_\alpha[t(1-n^{-1}(k+1))^{1/\alpha}]) = 0,$$

with the convergence uniform in $0 \leqq k \leqq (1-\varepsilon)n$.

It is an easy consequence of Theorem 1.3 that

(2.9) $$\lim_{n\to\infty} n[1-\phi(t/B_n)] = c(1 + i \operatorname{sgn}(t) \beta (\tan \tfrac{1}{2}\pi\alpha) |t|^\alpha) \quad \text{for} \quad 1 < \alpha \leqq 2;$$

and $\lim_{n\to\infty} n[1-\phi(t/B_n)] = c|t|$ for $\alpha = 1$.

By Lemma (2.1) we have that uniformly for $\varepsilon n \leqq k \leqq n$, $r_k/r_n \sim (n/k)^{1-(1/\alpha)}$ as $n \to \infty$ for $1 < \alpha \leqq 2$, and $r_k/r_n \to 1$ as $n \to \infty$ for $\alpha = 1$.

Therefore, we see that $1 - \sum_{k=\varepsilon n}^{(1-\varepsilon)n} r_n^{-1} r_k [1-\phi(t/B_n)] \phi^{n-(k+1)}(t/B_n)$ is approximated by (their difference tends to zero as $n \to \infty$)

$$1 - \sum_{k=\varepsilon n}^{(1-\varepsilon)n} (n/k)^{1-(1/\alpha)} b\, |t|^\alpha n^{-1} \phi_\alpha[t(1-n^{-1}(k+1))^{1/\alpha}],$$

where $b = c(1 + i \operatorname{sgn}(t) \beta \tan \frac{1}{2}\pi\alpha)$ for $1 < \alpha \leqq 2$ and $b = c$ for $\alpha = 1$. This last expression in turn is an approximating sum to the Riemann integral

$$1 - b\, |t|^\alpha \int_\varepsilon^{1-\varepsilon} x^{(1/\alpha)-1} \phi_\alpha[t(1-x)^{1/\alpha}]\, dx \quad \text{for} \quad 1 \leqq \alpha \leqq 2.$$

Now $\int_0^1 x^{(1/\alpha)-1} \phi_\alpha[t(1-x)^{1/\alpha}]\, dx$ is an absolutely convergent integral. Hence, in order to show that for every $t \in R$

(2.10) $$\lim_{n\to\infty} \Psi_{\alpha,n}(t) = 1 - b\, |t|^\alpha \int_0^1 x^{(1/\alpha)-1} \phi_\alpha[t(1-x)^{1/\alpha}]\, dx,$$

it is enough to show

$$\text{(2.11)} \qquad \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\Big|\textstyle\sum_{k=0}^{\varepsilon n} + \sum_{k=(1-\varepsilon)n}^{n} r_n^{-1} r_k[1-\phi(t/B_n)]\phi^{n-k-1}(t/B_n)\Big| = 0.$$

We estimate the expression in (2.11) using Theorem 1.3 and Lemma 2.1: Fsrst

$$\text{(2.12)} \qquad 1-\phi(t/B_n) \leqq K_1\, n^{-1}|t|^{\alpha}$$

for some constant K_1. For $1 < \alpha \leqq 2$, $\lim_{n\to\infty}[r_n n^{1-(1/\alpha)}/\tilde{L}(n)]$ is finite and non-zero.
For $\alpha = 1$, $\lim_{n\to\infty} r_n \ln n$ is finite and non-zero.
Thus we obtain

$$\text{(2.13)} \qquad \sum_{k=0}^{M} \frac{r_k}{r_n}\frac{|t|^{\alpha}}{n} \leqq |t|^{\alpha} \sum_{k=0}^{M} \frac{1}{n^{1/\alpha}\tilde{L}(n)} = \frac{|t|^{\alpha}(M+1)}{n^{1/\alpha}\tilde{L}(n)} \to 0 \quad \text{as} \quad n\to\infty$$

for each fixed positive integer M. Also

$$\text{(2.14)} \qquad |t|^{\alpha} \sum_{k=M}^{\varepsilon n} \frac{r_k}{r_n}\frac{1}{n} = |t|^{\alpha} \sum_{k=M}^{\varepsilon n} \left(\frac{n}{k}\right)^{1-(1/\alpha)} \frac{1}{n}\frac{\tilde{L}(k)}{\tilde{L}(n)}.$$

The expression on the right in (2.14) is asymptotically of the same order as

$$\frac{1}{n^{1/\alpha}\tilde{L}(n)} \int_M^{\varepsilon n} \frac{\tilde{L}(x)}{x^{1-(1/\alpha)}}\,dx.$$

We now appeal to an argument given in the Appendix of [1] which shows that we make take $\tilde{L}$ to be differentiable. We may therefore apply L'Hospital's rule in estimating this ratio. It follows that

$$\text{(2.15)} \qquad \lim_{M\to\infty}\limsup_{n\to\infty} \frac{1}{n^{1/\alpha}\tilde{L}(n)} \int_M^{\varepsilon n} \frac{\tilde{L}(x)}{x^{1-(1/\alpha)}}\,dx = \varepsilon\alpha.$$

Finally,

$$\text{(2.16)} \qquad \sum_{k=(1-\varepsilon)n}^{n} \frac{r_k}{r_n}\frac{1}{n} \leqq \varepsilon K_2,$$

where

$$K_2 = \sup_n \sup_{(1-\varepsilon)n \leqq k \leqq n} \frac{r_k}{r_n} \leqq \frac{1}{(1-\varepsilon)^{1-(1/\alpha)}} \sup_n \sup_{(1-\varepsilon)n\leqq k\leqq n} \frac{\tilde{L}(k)}{\tilde{L}(n)} < \infty,$$

since $\tilde{L}$varies slowly. Combining the estimates (2.12)–(2.16), we get (2.11). Thus for $1 < \alpha \leqq 2$ we have proved $\lim_{n\to\infty} \Psi_{\alpha,n}(t) = \Psi_{\alpha}(t)$.

That Ψ_{α} is in fact the characteristic function of a probability distribution is a result of $|1-\Psi_{\alpha}(t)| \leqq |b||t|^{\alpha}\int_0^1 x^{(1/\alpha)-1}\,dx$, which implies Ψ_{α} is continuous at the origin. Actually a stronger result holds. By an easy dominated convergence argument

$$\text{(2.17)} \qquad \lim_{t\to 0}\frac{1-\Psi_{\alpha}(t)}{|t|^{\alpha}} = b\int_0^1 x^{(1/\alpha)-1}\,dx.$$

The estimates for $\alpha = 1$ are very much the same and we omit the details.

The only two cases in which we attempt an explicit evaluation of H_α are for $\alpha = 1, 2$. A routine computation shows $\Psi_1(t) = \exp(-ct)$. The case $\alpha = 2$ can be handled as follows. The characteristic function $\overline{\Psi}_2$ corresponding to the density h_2 with $h_2(x) = \frac{1}{2}\sigma^{-2}|x|\exp(-\frac{1}{2}\sigma^{-2}x^2)$ is $\overline{\Psi}_2(t) = 1 - t\int_0^\infty \exp[-\frac{1}{2}\sigma^{-2}x^2]\sin xt\,dx$.

Now

$$\Psi_2(t) = 1 - \tfrac{1}{2}t^2\sigma^2 \textstyle\int_0^1 x^{-\frac{1}{2}}\exp[-\frac{1}{2}t^2\sigma^2(1-x)]\,dx.$$

To see that $\overline{\Psi}_2 = \Psi_2$ we define

$$\varphi_2(t) = t^{-1}[1-\Psi_2(t)] \quad \text{and} \quad \bar{\varphi}_2(t) = t^{-1}[1-\overline{\Psi}_2(t)].$$

It is then a simple matter to check that

(i) $$\bar{\varphi}_2(0) = \varphi_2(0) = 0$$

(ii) $$d\bar{\varphi}_2/dt = [1-t\bar{\varphi}_2]\sigma^2; \quad d\varphi_2/dt = [1-t\varphi_2]\sigma^2.$$

To complete the proof of the theorem it remains only to demonstrate that for $1 < \alpha < 2$, Ψ_α corresponds to a probability law with a bounded continuous density. For this we will show

(2.18) $$\textstyle\int_{-\infty}^{\infty} |\Psi_\alpha(t)|\,dt < \infty.$$

Now

$$\begin{aligned}
|\Psi_\alpha(t)| &= \left|1 - b|t|^\alpha \textstyle\int_0^1 x^{(1/\alpha)-1}\exp[-b|t|^\alpha(1-x)]\,dx\right| \\
&\leqq \left|b|t|^\alpha \textstyle\int_0^{1-\delta} x^{(1/\alpha)-1}\exp[-b|t|^\alpha(1-x)]\,dx\right| \\
&\qquad + \left|1 - b|t|^\alpha \textstyle\int_{1-\delta}^1 x^{(1/\alpha)-1}\exp[-b|t|^\alpha(1-x)]\,dx\right| \\
&\leqq |bt^\alpha| \textstyle\int_0^{1-\delta} x^{(1/\alpha)-1}\exp[-c|t|^\alpha(1-x)]\,dx \\
&\qquad + \left|1 - b|t|^\alpha \textstyle\int_{1-\delta}^1 \exp[-b|t|^\alpha(1-x)]\,dx\right| \\
&\qquad + |bt|^\alpha \left|\textstyle\int_{1-\delta}^1 (x^{(1/\alpha)-1}-1)\exp[-b|t|^\alpha(1-x)]\,dx\right|.
\end{aligned}$$

The first two of these terms are dominated by $A|bt^\alpha|\exp[-c|t|^\alpha\delta]$ and $\exp[-c|t|^\alpha\delta]$ respectively, for some constant A. Both of these functions are integrable. The third term is dominated by

$$|bt^\alpha| \textstyle\int_{1-\delta}^1 (x^{(1/\alpha)-1}-1)\exp[-c|t|^\alpha(1-x)]\,dx.$$

However,

$$\begin{aligned}
&\textstyle\int_0^\infty t^\alpha \int_{1-\delta}^1 (x^{(1/\alpha)-1}-1)\exp[-ct^\alpha(1-x)]\,dx\,dt \\
&\qquad = \int_{1-\delta}^1 \frac{x^{(1/\alpha)-1}-1}{(1-x)^{1+(1/\alpha)}}\,dx \int_0^\infty t^\alpha \exp[-ct^\alpha]\,dt \\
&\qquad \leqq K\int_{1-\delta}^1 \frac{dx}{(1-x)^{1/\alpha}} < \infty, \quad \text{for some constant } K.
\end{aligned}$$

This completes the proof of Theorem 2.1.

3. Generalization to finite sets. In this section we prove an analogue of Theorem 2.1 when the event $[T > n]$ is replaced by $[T_{\mathbf{A}} > n]$, where $\mathbf{A} = \{x_1, x_2, \cdots x_M\}$ is a finite set of integers with $\tilde{g}_{\mathbf{A}}(0) \neq 0$. A rather interesting phenomenon occurs in that the limit distribution $H_{\alpha,\mathbf{A}}$ depends on $\mathbf{A}$ only when F has a finite variance. The full statement of the theorem is as follows.

THEOREM 3.1. *Under the same hypotheses as Theorem* 2.1, *if* $\mathbf{A} = \{x_1, x_2, \cdots, x_M\}$ *is a finite set of integers such that* $\tilde{g}_{\mathbf{A}}(0) \neq 0$, *then*

(3.1) $$\lim_{n\to\infty} P[S_n/B_n \leqq x \mid T_{\mathbf{A}} > n] = H_{\alpha,\mathbf{A}}(x) \quad \textit{for every} \quad x \in R.$$

$H_{\alpha,\mathbf{A}}$ is a probability distribution with characteristic function $\Psi_{\alpha,\mathbf{A}}$ with the following properties:

(i) If $1 \leqq \alpha < 2$ or $\alpha = 2$ and F has infinite variance then $\Psi_{\alpha,\mathbf{A}} = \Psi_\alpha$, i.e. the limit distribution is independent of the set $\mathbf{A}$.

(ii) If $\alpha = 2$ and the variance of F is finite

$$\Psi_{2,\mathbf{A}}(t) = \Psi_2(t) - i(\tfrac{1}{2}\pi)^{\frac{1}{2}} E[S_{T_{\mathbf{A}}}][\tilde{g}_{\mathbf{A}}(0)]^{-1}\sigma^{-1} t \exp(-\tfrac{1}{2}t^2\sigma^2)$$

and

$$h_{2,\mathbf{A}}(x) = \tfrac{1}{2}\sigma^{-2}\exp(\tfrac{1}{2}\sigma^{-2}x^2)[|x| - (\sigma^2\tilde{g}_{\mathbf{A}}(0))^{-1} x E[S_{T_{\mathbf{A}}}]].$$

PROOF.

$$\begin{aligned}
&P[S_n/B_n \in dx \mid T_{\mathbf{A}} > n] \\
&\quad = \{P[T_{\mathbf{A}} > n]\}^{-1} \\
&\qquad \cdot (P[S_n/B_n \in dx] - \textstyle\sum_{k=1}^{n}\sum_{i=1}^{M} P[T_{\mathbf{A}} = k; S_{T_{\mathbf{A}}} = x_i] P^{x_i}[S_{n-k}/B_n \in dx]) \\
&\quad = \{P[T_{\mathbf{A}} > n]\}^{-1}(P[S_n/B_n \in dx] - \textstyle\sum_{k=1}^{n} P[T_{\mathbf{A}} = k] P[S_{n-k}/B_n \in dx]) \\
&\qquad + \{P[T_{\mathbf{A}} > n]\}^{-1} \\
&\qquad \cdot \{\textstyle\sum_{k=1}^{n}\sum_{i=1}^{M} P[T_{\mathbf{A}} = k; S_{T_{\mathbf{A}}} = x_i](P[S_{n-k}/B_n \in dx] - P^{x_i}[S_{n-k}/B_n \in dx])\} \\
&\quad = \mu_n^{(1)}(dx) + \mu_n^{(2)}(dx).
\end{aligned}$$

A summation by parts gives

$$\mu_n^{(1)}(dx) = P[S_0 \in dx] - \textstyle\sum_{k=0}^{n-1}\{P[T_{\mathbf{A}} > k]/P[T_{\mathbf{A}} > n]\}$$
$$(P[S_{n-(k+1)}/B_n \in dx] - P[S_{n-k}/B_n \in dx]).$$

Consequently, if we let $\Psi_n^{(1)} = \Psi_{\alpha,\mathbf{A},n}^{(1)}$ and $\Psi_n^{(2)} = \Psi_{\alpha,\mathbf{A},n}^{(2)}$ be the contributions to the characteristic function of the probability measure $P[S_n/B_n \in dx \mid T_{\mathbf{A}} > n]$ corresponding to $\mu_n^{(1)}$ and $\mu_n^{(2)}$ respectively, then

$$\Psi_n^{(1)}(t) = 1 - \textstyle\sum_{k=0}^{n-1}\{P[T_{\mathbf{A}} > k]/P[T_{\mathbf{A}} > n]\}[1 - \phi(t/B_n)]\phi^{n-(k+1)}(t/B_n).$$

By Theorem 4a in [4], when $\tilde{g}_{\mathbf{A}}(0) \neq 0$, $\{P[T_{\mathbf{A}} > k]/P[T_{\mathbf{A}} > n]\}$ has the same asymptotic behavior for large k and n as r_k/r_n; hence

(3.2) $$\lim_{n\to\infty} \Psi_n^{(1)}(t) = \lim_{n\to\infty} \Psi_{\alpha,n}(t) = \Psi_\alpha(t),$$

by Theorem 1.1.

Also we may rewrite $\mu_n^{(2)}(dx)$ as

$$\{P[T_A > n]\}^{-1} \sum_{k=1}^{n} \sum_{i=1}^{M} P[T_A = k; S_{T_A} = x_i] \cdot (P[S_{n-k}/B_n \in dx] - P[(S_{n-k}+x_i)/B_n \in dx]).$$

Thus we conclude that

$$\Psi_n^{(2)}(t) = \{P[T_A > n]\}^{-1} \sum_{k=1}^{n} \sum_{i=1}^{M} P[T_A = k; S_{T_A} = x_i] \cdot [\phi^{n-k}(t/B_n) - \phi^{n-k}(t/B_n) \exp(itx_i/B_n)].$$

Once again performing a summation by parts we find that

$$\begin{aligned}\Psi_n^{(2)}(t) = & \sum_{i=1}^{M} \sum_{k=1}^{n-1} \{P[T_A > k; S_{T_A} = x_i]/P[T_A > n]\} \phi^{n-(k+1)}(t/B_n) \\ & \cdot [1-\phi(t/B_n)][1-\exp(itx_i/B_n)] \\ & + \sum_{i=1}^{M} \{P[T_A > n]\}^{-1} P[S_{T_A} = x_i][1-\exp(itx_i/B_n)]\phi^{n-1}(t/B_n) \\ & - \sum_{i=1}^{M} \{P[T_A > n; S_{T_A} = x_i]/P[T_A > n]\}[1-\exp(itx_i/B_n)].\end{aligned} \tag{3.3}$$

Since $B_n \to \infty$ as $n \to \infty$, the third term on the right in (3.3), which is bounded by $\sum_{i=1}^{M} |1-\exp(itx_i/B_n)|$, tends to zero as $n \to \infty$. The first term is dominated in absolute value by

$$\left|\sum_{i=1}^{M} [1-\exp(itx_i/B_n)] \sum_{k=1}^{n-1} \{P[T_A > k]/P[T_A > n]\} [1 - \phi(t/B_n)]\phi^{n-(k+1)}(t/B_n)\right|.$$

Since the expression in the inner summation was already shown to have a finite limit as $n \to \infty$, the first term on the right in (3.3) tends to zero as well.

To estimate the second term we consider two separate cases:

Case (i). $1 \leqq \alpha < 2$.

First we have

$$|1-\exp(itx_i/B_n)| = O(tx_i/B_n) \quad \text{as} \quad n \to \infty.$$

Also, for $1 < \alpha < 2$,

$$P[T_A > n] \sim \tilde{g}_A(0) r_n \sim \tilde{g}_A(0)\tilde{L}(n)/n^{1-(1/\alpha)} \quad \text{as} \quad n \to \infty;$$

while for $\alpha = 1$,

$$P[T_A > n] \sim \tfrac{1}{2}\pi^2 l \tilde{g}_A(0)/\ln n \quad \text{as} \quad n \to \infty.$$

Consequently,

$$|1-\exp(itx_i/B_n)|/P[T_A > n] \leqq Kn^{1-(2/\alpha)}/[\tilde{L}(n)]^2 \to 0 \tag{3.4}$$

for some constant K when $1 < \alpha < 2$;

$$|1-\exp(itx_i/B_n)|/P[T_A > n] \leqq Kn^{-1} \ln n \to 0 \tag{3.5}$$

when $\alpha = 1$.

For $\alpha = 2$, when the variance of F is infinite by our choice of the sequence B_n (see (1.22)) $B_n^2 = n\int_{-C_n}^{C_n} x^2 \, dF(x)$, and since $C_n \to \infty$ we obtain $n/B_n^2 \to 0$. Thus once again we have that $\lim_{n\to\infty} |1-\exp(itx_i/B_n)|/P[T_A > n] = 0$.

Combining this with (3.4) and (3.5) it follows that the second term in (3.3) tends to zero.

Case (ii). $\alpha = 2$ and F has finite variance.
Then we have

$$\lim_{n\to\infty} \phi^{n-1}(t/B_n) = \exp[-\tfrac{1}{2}t^2\sigma^2], \qquad 1-\exp(itx_i/B_n) \sim -itx_i/B_n = -n^{-\frac{1}{2}}itx_i,$$

and

$$\{P[T_A > n]\}^{-1} \sim [\tilde{g}_A(0)r_n]^{-1} \sim [\tilde{g}_A(0)\sigma]^{-1}(\tfrac{1}{2}\pi)^{\frac{1}{2}}n^{\frac{1}{2}}.$$

Combining these facts with (3.2) gives $\lim_{n\to\infty}[\Psi^{(1)}_{2,A,n}(t)+\Psi^{(2)}_{2,A,n}(t)] = \Psi_{2,A}(t)$. This completes the proof of Theorem 3.1.

4. The 2-dimensional finite variance case. To this point we have restricted our attention to random walks with state space Z. We now consider the 2-dimensional case.

Lévy's formula [7] for the characteristic function of a d-dimensional stable law G_α, $0 < \alpha \leqq 2$ is

$$\int e^{it\cdot x}\, dG_\alpha(x) = \exp\{iA\cdot t - \lambda|t|^\alpha[c_1(t)+ic_2(t)]\}, \tag{4.1}$$

where A is a constant vector, λ a positive constant and

$$\begin{aligned} c_1(t) &= \int |t|^{-1}|\theta\cdot t|^\alpha\, dH(\theta), \\ c_2(t) &= -\int \operatorname{sgn}(\theta\cdot t)\tan(\tfrac{1}{2}\pi\alpha)\,|t^{-1}\theta\cdot t|^\alpha\, dH(\theta) \qquad &&\text{for } \alpha \neq 1, \\ &= 2\pi^{-1}\int(|t|^{-1}\theta\cdot t)\ln(\theta\cdot t)\, dH(\theta) &&\text{for } \alpha = 1, \end{aligned}$$

with H a probability measure on the unit sphere in R^d.

For a *genuinely d-dimensional* (as defined in [8]) recurrent random walk with increments $X_k = (Y_k^{(1)}, \cdots Y_k^{(d)})$ having their common distribution in the domain of attraction of a stable law, it can be shown by applying the criterion in (I) of the Introduction that either $d = 1$ or $d = 2$, $\alpha = 2$. The case $d = 1$ has already been considered so there remains only the case where the distribution F of X_1 belongs to the domain of attraction of a bivariate normal distribution. We require in addition that $E|X_1|^2 < \infty$. By (7.7) of [8], if ϕ is the characteristic function of F, we then have

$$\lim_{t\to 0}[Q(t)]^{-1}[1-\phi(t)] = \tfrac{1}{2}, \tag{4.2}$$

where $Q(t) = Q(t_1, t_2) = E[(X\cdot t)^2]$. We define $\sigma_1^{\,2} = \operatorname{Var} Y_1^{(1)}$, $\sigma_2^{\,2} = \operatorname{Var} Y_1^{(2)}$ and $\rho = [\sigma_1\sigma_2]^{-1}E[\,Y_1^{(1)}\,Y_1^{(2)}]$ and state the following analogue of Lemma 2.1.

LEMMA 4.1. *For an aperiodic, recurrent, genuinely 2-dimensional random walk with* $E|X_1|^2 < \infty$ we have

$$\lim_{n\to\infty} r_n \ln n = 2\pi\sigma_1\,\sigma_2(1-\rho^2)^{\frac{1}{2}}. \tag{4.3}$$

PROOF. As stated in the proof of Lemma 1.1 we have

$$U(x) = (2\pi)^{-2}\int_{C_2}([1-x\phi(t)]^{-1}\,dt.$$

By aperiodicity we have for any fixed $\delta > 0$

$$\lim_{x\to 1-}\frac{U(x)}{\ln(1/1-x)}$$

$$= \lim_{x\to 1-}\frac{1}{(2\pi)^2\ln(1/1-x)}\int_{C^2}\frac{dt}{\frac{1}{2}Q(t)+(1-x)}$$

$$= \lim_{x\to 1-}\frac{1}{(2\pi)^2\ln(1/1-x)}\int_0^{2\pi}\int_0^{\delta(1-x)^{-1/2}}\frac{r}{(\frac{1}{2}r^2)Q(\cos\theta,\sin\theta)+1}\,dr\,d\theta$$

(4.4)

$$= \lim_{x\to 1-}\frac{1}{(2\pi)^2\ln(1/1-x)}\int_0^{2\pi}\frac{\ln([Q(\cos\theta,\sin\theta)/2(1-x)]\,\delta^2+1)}{Q(\cos\theta,\sin\theta)}\,d\theta$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi}\frac{d\theta}{Q(\cos\theta,\sin\theta)} = \frac{1}{2\pi\sigma_1\,\sigma_2(1-\rho^2)^{\frac{1}{2}}}\,.$$

Since $U(x) = (1-x)R(x)$, applying Karamata's Tauberian theorem we obtain (4.3).

THEOREM 4.1. *Under the hypotheses of Lemma* 4.1 *if* $\mathbf{A}$ *is a finite set in* Z^2 *such that* $\tilde{g}_{\mathbf{A}}(0) > 0$, *then*

$$\lim_{n\to\infty} P[S_n/n^{\frac{1}{2}} \leqq x \mid T_{\mathbf{A}} > n] = G(x), \tag{4.5}$$

where G *is the bivariate normal distribution with parameters* σ_1, σ_2 *and* ρ.[3]

Thus we find that conditioning on the event $[T_{\mathbf{A}} > n]$ plays no role in the limit (c.f. (ii) of Theorem 2.1).

PROOF. Proceeding as in the proof of Theorem 2.1 we let ψ_n be the characteristic function of the probability measure P_n, where $P_n(dx) = P[S_n/n^{\frac{1}{2}} \in dx \mid T > n]$. Then by (4.2), the Central Limit Theorem, (2.6), and Lemma 4.1 we have

$$\begin{aligned}\psi(t) &= \lim_{n\to\infty}\psi_n(t)\\ &= \lim_{n\to\infty}\{1-\textstyle\sum_{k=0}^{n-1} r_n{}^{-1} r_k[1-\phi(n^{-\frac{1}{2}}t)]\phi^{n-(k+1)}(n^{-\frac{1}{2}}t)]\\ &= 1-\tfrac{1}{2}Q(t)\textstyle\int_0^1\exp(-\tfrac{1}{2}Q(t)(1-x))\,dx\\ &= \exp(-\tfrac{1}{2}Q(t)),\end{aligned}$$

and (4.5) follows immediately for $\mathbf{A} = \{0\}$. The generalization to finite $\mathbf{A}$ with $\tilde{g}_{\mathbf{A}}(0) \neq 0$ proceeds as in the proof of Theorem 3.1.

[3] We say $(x_1, y_1) \leqq (x_2, y_2)$ if $x_1 \leqq x_2$ and $y_1 \leqq y_2$.

5. Counterexample. We saw in the proposition in the introduction that the condition $\tilde{g}_A(x) > 0$ is equivalent to the requirement that the random walk is able to *escape to infinity* along some path which starts at x and avoids the set **A**. In the case $d = 1, \sigma^2 < \infty$ this was to be interpreted to mean that the random walk could reach both arbitrarily large positive and arbitrarily large negative values without entering **A**. It is clear that Theorem 1.1 would fail to hold if an **A** were so chosen that the condition $T_A > n$ on S_n would force the partial sums of the random walk to be bounded both above and below. However, it is not a priori clear what the appropriate restatement of the theorem should be if the condition $T_A > n$ confines the random walk to a half line. We now give an example showing that Theorem 1.1 as stated may indeed fail in this situation. However, for this particular example an analogous result will hold.

Let F be a distribution function corresponding to a left-continuous (i.e., $p(0,x) = 0$ for $x \leqq -2$), recurrent random walk in the domain of normal attraction of a stable law of index α with $1 < \alpha < 2$. In particular, then

$$x^\alpha[1-F(x)] \to c \quad \text{as} \quad x \to \infty. \tag{5.1}$$

(For simplicity we take $c = 1$.) Take the set **A** to be the point $\{-1\}$. Then by Theorem 1.3

$$\lim_{t\to 0\pm} (1-\phi(t))/|t|^\alpha = \Gamma(1-\alpha)\cos(\tfrac{1}{2}\pi\alpha)(h \pm i\tan(\tfrac{1}{2}\pi\alpha)), \tag{5.2}$$

Defining $R_1(x) = \sum_{n=0}^\infty r_n(1,\{0\})x^n$ and $U_1(x) = \sum_{n=0}^\infty p_n(1,0)x^n$, it is easily seen that

$$R_1(x) = [(1-x)U(x)]^{-1}[U(x) - U_1(x)].$$

Now the potential kernel $a(x)$ defined by $a(x) = \sum_{n=0}^\infty [p_n(0,0) - p_n(x,0)]$ has the Fourier analytic representation $a(x) = (2\pi)^{-1}\int_{-\pi}^{\pi}[1-x\phi(t)]^{-1}(1-e^{it})\,dt$.

Consequently,

$$\lim_{x\to 1}(U(x) - U_1(x)) = a(1) = (2\pi)^{-1}\int_{-\pi}^{\pi}[1-\phi(t)]^{-1}(1-e^{it})\,dt.$$

However, for left continuous random walk with $\sigma^2 = \infty$ it is known that $a(x)$ vanishes on the right half line $x > 0$ (see Section 30 of [8]).

Therefore we may write

$$U(x) - U_1(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-x)\phi(t)(1-e^{it})}{(1-\phi(t))(1-x\phi(t))}\,dt.$$

It follows that for $\delta > 0$ as $x \to 1$

$$\frac{U(x)-U_1(x)}{1-x} \sim \frac{1}{2\pi}\int_{\pi \geqq |t| \geqq \delta}\frac{\phi(t)(1-e^{it})}{(1-\phi(t))^2}\,dt + \frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\phi(t)(1-e^{it})}{(1-\phi(t))(1-x\phi(t))}\,dt.$$

Now it is a straightforward argument using (5.2) to show that for arbitrary $\varepsilon > 0$ and fixed δ small enough,

$$(1-\varepsilon)K \leqq \frac{(1-x)^{2-(2/\alpha)}}{2\pi}\int_{-\delta}^{\delta}\frac{\phi(t)(1-e^{it})}{(1-\phi(t))(1-x\phi(t))}\,dt \leqq (1+\varepsilon)K$$

for x sufficiently close to 1, where

$$K = \frac{1}{\pi[\Gamma(1-\alpha)]^{2/\alpha}} \int_0^\infty \frac{du}{u^{\alpha-1}u^{\alpha+1}} = \frac{csc(2\pi/\alpha)}{\alpha[\Gamma(1-\alpha)]^{2/\alpha}}.$$

Consequently, as $x \to 1$

$$R_1(x) \sim -\frac{csc(2\pi/\alpha)(1-x)^{(1/\alpha)-1}}{\alpha g_\alpha(0)\Gamma(1-1/\alpha)[\Gamma(1-\alpha)]^{2/\alpha}},$$

and by Karamata's Theorem

$$r_n(0,\{-1\}) \sim \frac{n^{-1/\alpha}}{2\pi\alpha g_\alpha(0)\cos\pi/\alpha[\Gamma(1-\alpha)]^{2/\alpha}} \quad \text{as} \quad n \to \infty.$$

One now observes that in contrast to the situation in Lemma 2.1, where r_n tended to zero asymptotically like $n^{(1/\alpha)-1}$, we now find that $r_n(0,\{-1\})$ tends to zero like $n^{-(1/\alpha)}$.[4] It is clear from its proof that Theorem 3.1 may now be restated for our example with

$$\Psi_\alpha(t) = 1 - b|t|^\alpha \textstyle\int_0^1 x^{-(1/\alpha)}\phi_\alpha[t(1-x)^{1/\alpha}]\,dx.$$

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[4] We remark that $1-1/\alpha = 1/\alpha$ for $\alpha = 2$, as one might suspect, since for any left continuous, recurrent random walk with $\sigma^2 < \infty$, it is necessarily true that $g_A(x) > 0$ for $A = \{-1\}$.