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THE DISTRIBUTION OF THE RATIOS OF MEANS TO THE SQUARE ROOT OF THE SUM OF VARIANCES OF A BIVARIATE NORMAL SAMPLE

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1. Introduction. Let (X_i, Y_i) $i = 1, 2, \cdots, m$, be independent observations on a random vector (X, Y) which has a bivariate normal distribution with

$$EX = EY = 0, \qquad EX^2 = EY^2 = \sigma^2, \qquad EXY = \rho\sigma^2.$$

Let

$$\bar{X} = m^{-1}\sum_{i=1}^m X_i, \qquad \bar{Y} = m^{-1}\sum_{i=1}^m Y_i, \qquad s_1{}^2 = m^{-1}\sum_{i=1}^m (X_i - \bar{X})^2,$$

$$s_2{}^2 = m^{-1}\sum_{i=1}^m (Y_i - \bar{Y})^2.$$

Recently Siddiqui [6] has considered the distribution of $(m-1)^{\frac{1}{2}}\bar{X}/s_1$, $(m-1)^{\frac{1}{2}}\bar{Y}/s_2$. For $m > 3$, he obtained asymptotic results. We define $Z = ms_1{}^2 + ms_2{}^2$, $s^2 = (2m-1)^{-1}Z$, so that s^2 is an unbiased estimator of σ^2 based on both X and Y observations. In this note we consider the distribution of (T_1, T_2), where $T_1 = m^{\frac{1}{2}}s^{-1}\bar{X}$, $T_2 = m^{\frac{1}{2}}s^{-1}\bar{Y}$. It is noted that (T_1, T_2) are independent of the scale parameter. We have obtained the probability density function (pdf) of (T_1, T_2) and the distribution function of (T_1, T_2). Also marginal and limiting distributions are discussed.

2. The probability density function of Z**.** The following lemma is used to determine the pdf of Z.

LEMMA. *Let (X_i, Y_i), $i = 1, 2, \cdots, m$; $m > 3$ be the observations from the bivariate normal distribution with the zero means, correlation coefficient ρ and common variance σ^2; then the distribution of Z, defined in Section 1, can be expressed as the distribution function of $[U_1(1+\rho)\sigma^2 + U_2(1-\rho)\sigma^2]$, where U_1, U_2 are independent and identically distributed chi-square random variables with $(m-1)$ degrees of freedom.*

Using Lemma 2 of [1] it can be easily shown that

$$M_Z(t) = E\,e^{tZ} = [1-2t(1+\rho)\sigma^2]^{-\frac{1}{2}(m-1)}[1-2t(1-\rho)\sigma^2]^{-\frac{1}{2}(m-1)} \tag{1}$$

which is the same as moment generating function of $[U_1(1+\rho)\sigma^2 + U_2(1-\rho)\sigma^2]$. The distribution of Z does not depend on the means of (X_i, Y_i).

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The pdf of Z can be derived by using this lemma. Let $Z = Y_1 + Y_2$, with $Y_1 = U_1(1+\rho)\sigma^2$, $Y_2 = U_2(1-\rho)\sigma^2$. By the independence of Y_1, Y_2 the joint pdf of Y_1, Y_2 can be written as

$$f(y_1, y_2) = \frac{\exp[-y_1/(2(1+\rho)\sigma^2) - y_2/(2(1-\rho)\sigma^2)]}{\sigma^{2(m-1)}\{\Gamma((m-1)/2)\}^2\{4(1-\rho^2)\}^{\frac{1}{2}(m-1)}}(y_1 y_2)^{\frac{1}{2}(m-3)} \tag{2}$$

$$0 < y_1 < \infty, \quad 0 < y_2 < \infty.$$

Making the transformation $z = y_1 + y_2$, $y_2 = y_2$, where $0 < y_i < z < \infty$, $i = 1, 2$ and $0 < z < \infty$, the joint pdf of Z and Y_2 is obtained. When integrated over y_2 with m even and odd separately the pdf of Z follows. First, let m be odd, then $\frac{1}{2}(m-3)$ is an integer, say k

$$f(z) = \frac{\exp[-z/(2(1+\rho)\sigma^2)]}{\sigma^{4(k+1)}\{\Gamma(k+1)\}^2\{4(1-\rho^2)\}^{k+1}}\int_0^z y_2{}^k(z-y_2)^k \exp\left[\frac{-\rho y_2}{(1-\rho^2)\sigma^2}\right]dy_2.$$

Expanding the binomial $(z-y_2)^k$ and letting $v = \rho y_2/((1-\rho^2)\sigma^2)$ and integrating with respect to v, the pdf of Z is

$$\begin{aligned} f(z) = K_1 \exp\{-z/(2(1+\rho)\sigma^2)\} &\textstyle\sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i}(z)^i\sigma^{-(2i+2)}\Gamma(2k-i+1) \\ &\cdot[(1-\rho^2)/\rho]^{2k-i+1}[1-\textstyle\sum_{j=0}^{2k-i} j!^{-1}[\rho z/((1-\rho^2)\sigma^2)]^j \\ &\cdot \exp[-\rho z/((1-\rho^2)\sigma^2)]] \qquad 0 < z < \infty, \end{aligned} \tag{3}$$

where $K_1 = [(\Gamma(k+1))^2(4(1-\rho^2))^{k+1}]^{-1}$.

For the purpose of evaluating $f(z)$ when m is even, Z could be considered as a linear combination of chi-square variates each of which has one degree of freedom, in symbols,

$$Z = \sigma^2(1+\rho)(X_1{}^2 + X_2{}^2 + \cdots + X_{m-1}^2) + \sigma^2(1-\rho)(X_m{}^2 + X_{m+1}^2 + \cdots + X_{2(m-1)}^2),$$

where X_i, $i = 1, 2, \cdots, 2(m-1)$, are independent observations from a normal distribution with mean zero and variance σ^2. Let $Z' = Z/((1+\rho)\sigma^2)$, then,

$$Z' = X_1{}^2 + X_2{}^2 + \cdots + X_{(m-1)}^2 + (1-\rho)/(1+\rho)(X_m{}^2 + X_{(m+1)}^2 + \cdots + X_{2(m-1)}^2).$$

If $f(z')$ denotes the density of Z' and $f_k(t)$ denotes the density function of chi-square variate with k degrees of freedom, then by [4] and [5], the density function of Z' can be written as a linear combination of chi-square densities. From the density of Z', the pdf of Z follows in the form

$$f(z) = \sum_{i=0}^{\infty} q_i f_{(2m+2i-2)}\left(\frac{z}{(1+\rho)\sigma^2}\right)\frac{1}{(1+\rho)\sigma^2} \qquad 0 < z < \infty; \tag{4}$$

where the q_i's are constants depending on m and ρ. Equation (4) can be reduced to

$$f(z) = \sum_{i=0}^{\infty} \frac{q_i}{2^{m+i-1}\Gamma(m+i-1)}((1+\rho)\sigma^2)^{-(m+i-1)}z^{m+i-2}\exp\left[\frac{-z}{2(1+\rho)\sigma^2}\right] \tag{5}$$

$$0 < z < \infty,$$

where

$$q_i = \binom{-\frac{1}{2}(m-1)}{i}\left(\frac{2\rho}{1+\rho}\right)^i\left(\frac{1+\rho}{1-\rho}\right)^{\frac{1}{2}(m-1)}.$$

3. The joint probability density function of (T_1, T_2)**.** The joint pdf of $\bar{X}$, $\bar{Y}$ is

$$f(\bar{x}, \bar{y}) = \frac{m}{2\pi(1-\rho^2)^{\frac{1}{2}}\sigma^2}\exp\left[-\frac{m}{2\sigma^2}\left(\frac{\bar{x}^2-2\rho\bar{x}\bar{y}+\bar{y}^2}{1-\rho^2}\right)\right] \quad -\infty < \bar{x}, \bar{y} < \infty. \tag{6}$$

Noting the independence of $(\bar{X}, \bar{Y})$ and Z, and making the transformation

$$T_1 = \frac{m^{\frac{1}{2}}\bar{X}}{\left[\dfrac{ms_1^2+ms_2^2}{2(m-1)}\right]^{\frac{1}{2}}}, \qquad T_2 = \frac{m^{\frac{1}{2}}\bar{Y}}{\left[\dfrac{ms_1^2+ms_2^2}{2(m-1)}\right]^{\frac{1}{2}}} \tag{7}$$

and integrating out z from the joint pdf of t_1, t_2, z, the joint pdf of t_1, t_2 for m odd and equal to $2k+3$ can be put in the form

$$\begin{aligned} f(t_1, t_2) = \frac{K_1}{2\pi(1-\rho^2)^{\frac{1}{2}}4(k+1)}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left[\frac{1-\rho^2}{\rho}\right]^{2k-i+1}\Gamma(2k-i+1) \\ \cdot\left\{\Gamma(i+2)\left[\frac{1}{2(1+\rho)}+\frac{t_1^2-2\rho t_1 t_2+t_2^2}{8(k+1)(1-\rho^2)}\right]^{-(i+2)}\right. \\ \left. -\sum_{j=0}^{2k-i}\left(\frac{\rho}{1-\rho^2}\right)^j\frac{\Gamma(i+j+2)}{j!}\left[\frac{1}{2(1-\rho)}+\frac{t_1^2-2\rho t_1 t_2+t_2^2}{8(k+1)(1-\rho^2)}\right]^{-(i+j+2)}\right\} \\ -\infty < t_i < \infty, \quad i = 1, 2. \end{aligned} \tag{8}$$

By the same procedure used to get equation (8), $f(t_1, t_2)$ can be derived from (5) and (6) for m even in the form,

$$\begin{aligned} f(t_1, t_2) = \sum_{i=0}^{\infty}\frac{(m+i-1)(1-\rho^2)^{-\frac{1}{2}}q_i}{\pi 2^{m+i+1}(m-1)(1+\rho)^{m+i-1}}\left[\frac{1}{2(1+\rho)}+\frac{t_1^2-2\rho t_1 t_2+t_2^2}{4(m-1)(1-\rho^2)}\right]^{-(m+i)} \\ -\infty < t_i < \infty, \quad i = 1, 2. \end{aligned} \tag{9}$$

4. The distribution function of T_1, T_2**.** The distribution function of T_1, T_2 can be obtained by integrating the density function with respect to t_1, t_2 over any region of the type, $h_1 < t_1 < h_2$, $h_1' < t_2 < h_2'$, finite or infinite. The integral of the distribution function is denoted by

$$P_m(h, h', \rho) = \int_{-\infty}^{h'}\int_{-\infty}^{h} f(t_1, t_2)\,dt_1\,dt_2. \tag{10}$$

For m odd, substituting (8) into (10) $P_m(h, h', \rho)$ can be written as

$$P_m(h, h', \rho) = \frac{K_1}{4(k+1)}\left\{\sum_{i=0}^{k} \Gamma(i+2)c_i\left(\frac{1-\rho^2}{\rho}\right)^{2k-i+1}[2(1+\rho)]^{i+2}\right.$$

$$\cdot P_i(n_1, \rho, h, h') - \sum_{i=0}^{k} c_i \sum_{j=0}^{2k-i}\frac{\Gamma(i-j+2)}{j!}\left(\frac{1-\rho^2}{\rho}\right)^{2k-i-j+1} \tag{11}$$

$$\left.\cdot [2(1-\rho)]^{i+j+2}P_{i+j}(n_2, \rho, h, h')\right\},$$

where $c_i = \binom{k}{i}(-1)^{k-i}\Gamma(2k-i+1)$,

$$P_i(n_\alpha, \rho, h, h') = \frac{1}{2(1-\rho^2)^{\frac{1}{2}}}\int_{-\infty}^{h'}\int_{-\infty}^{h}\left[1+\frac{t_1{}^2-2\rho t_1 t_2+t_2{}^2}{n_\alpha(1-\rho^2)}\right]^{-(i+2)} dt_1\, dt_2, \qquad \alpha = 1, 2,$$

$n_1 = 4(k+1)/(1+\rho)$, $n_2 = 4(k+1)/(1-\rho)$. In order to evaluate $P_i(n_\alpha, \rho, h, h')$ we use the transformation

$$r\cos\theta = (t_1-\rho t_2)/(1-\rho^2)^{\frac{1}{2}}, \qquad r\sin\theta = t_2, \tag{12}$$

then

$$\begin{aligned} P_i(n_\alpha, \rho, h, h') = {}& \tfrac{1}{8}n_\alpha(i+1)^{-1}(1+\operatorname{sgn} h)(1+\operatorname{sgn} h') \\ & -\textstyle\int_{c(h,h',\rho)}^{\pi}\int_{h'\csc\theta}^{\infty}\phi(r)\,dr\,d\theta - \int_{c(h',h,\rho)}^{\pi}\int_{h\csc\theta}^{\infty}\phi(r)\,dr\,d\theta \end{aligned} \tag{13}$$

where $\phi(r) = (2\pi)^{-1}r(1+r^2/n_\alpha)^{-(i+2)}$,

$$c(h, h', \rho) = \arctan[h(1-\rho^2)^{\frac{1}{2}}/(h-\rho h')], \qquad \operatorname{sgn} h = +1 \text{ if } h \geqq 0, \quad = -1 \text{ if } h < 0.$$

We assume that for any real A and B ($B \neq 0$) $0 \leqq \arctan(A/B) \leqq 2\pi$ and the angle is to be interpreted as lying in the interval $(0, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \pi)$, $(\pi, \frac{3}{2}\pi)$ or $(\frac{3}{2}\pi, 2\pi)$ according as the signs of A and B are $(+, +)$, $(+, -)$, $(-, -)$, or $(-, +)$. Integrating with respect to r, (13) reduces to

$$\begin{aligned} P_i(n_\alpha, \rho, h, h') = {}& \tfrac{1}{2}n_\alpha(i+1)^{-1}[\tfrac{1}{4}(1+\operatorname{sgn} h)(1+\operatorname{sgn} h') - Q_{i+1}(n_\alpha, \rho, h, h') \\ & - Q_{i+1}(n_\alpha, \rho, h', h)], \end{aligned} \tag{14}$$

where $Q_{i+1}(n_\alpha, \rho, h, h') = (2\pi)^{-1}\int_{c(h,h',\rho)}^{\pi}(1+h'^2 n_\alpha^{-1}\csc^2\theta)^{-(i+1)}\,d\theta$. Using the recursion formula for $Q_{i+1}(n_\alpha, \rho, h, h')$ given in Dunnette and Sobel [3],

$$\begin{aligned} P_i(n_\alpha, \rho, h, h') = {}& \tfrac{1}{2}n_\alpha(i+1)^{-1}[(2\pi)^{-1}\arctan((1-\rho^2)^{\frac{1}{2}}/-\rho) \\ & + \tfrac{1}{4}h'(n_\alpha\pi)^{-\frac{1}{2}}\textstyle\sum_{s=1}^{i+1}\Gamma(s-\frac{1}{2})(\Gamma(s))^{-1}(1+h'^2/n_\alpha)^{-(s-\frac{1}{2})} \\ & \cdot\{1+\operatorname{sgn}(h-\rho h')I_{x(n_\alpha,h,h')}(\tfrac{1}{2}, s-\tfrac{1}{2})\} + \tfrac{1}{4}h(n_\alpha\pi)^{-\frac{1}{2}} \\ & \cdot\textstyle\sum_{s=1}^{i+1}\Gamma(s-\frac{1}{2})(\Gamma(s))^{-1}(1+h^2/n_\alpha)^{-(s-\frac{1}{2})} \\ & \cdot\{1+\operatorname{sgn}(h'-\rho h)I_{x(n_\alpha,h',h)}(\tfrac{1}{2}, s-\tfrac{1}{2})\}], \end{aligned} \tag{15}$$

where $x(n_\alpha, h, h') = (h-\rho h')^2/[(h-\rho h')^2+(1-\rho^2)(n_\alpha+h'^2)]$, and

$$I_{x(n_\alpha,h,h')}(\tfrac{1}{2}, s-\tfrac{1}{2}) = \Gamma(s)[\Gamma(\tfrac{1}{2})\Gamma(s-\tfrac{1}{2})]^{-1}\int_0^{x(n_\alpha,h,h')} y^{-\frac{1}{2}}(1-y)^{s-\frac{3}{2}}\,dy.$$

In the case of m even, the distribution can be put in the form

$$P_m(h, h', \rho) = \textstyle\sum_{i=0}^{\infty}(m-1)^{-1}(m+i-1)(1+\rho)q_i P_{m+i-2}(n_3, \rho, h, h'), \tag{16}$$

where $n_3 = 2(m-1)/(1+\rho)$.

5. Marginal density function of T_1. The marginal density of T_1 is evaluated by integrating over T_2 in (8) or (9) according as m is even or odd. For m odd and equal to $2k+3$,

$$\begin{aligned} f(t_1) = \frac{K_1}{2(2k+2)^{\frac{1}{2}}\pi^{\frac{1}{2}}}\Bigg\{ & \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\frac{1-\rho^2}{\rho}\right)^{2k-i+1}\cdot\Gamma(2k-i+1)\Gamma(i+\tfrac{3}{2}) \\ & \cdot\left(\frac{1}{2(1+\rho)}+\frac{t_1^2}{8(k+1)}\right)^{-(i+\frac{3}{2})} - \sum_{i=0}^{k}\sum_{j=0}^{2k-i}\binom{k}{i}(-1)^{k-i}(j!)^{-1} \\ & \cdot\left(\frac{1-\rho^2}{\rho}\right)^{2k-i-j+1}\Gamma(2k-i+1)\Gamma(i+j+\tfrac{3}{2}) \\ & \cdot\left(\frac{1}{2(1-\rho)}+\frac{t_1^2}{8(k+1)}\right)^{-(i+j+\frac{3}{2})}\Bigg\} \qquad -\infty < t_1 < \infty. \end{aligned} \tag{17}$$

For m even,

$$f(t_1) = \frac{1}{\pi^{\frac{1}{2}}}\sum_{i=0}^{\infty}\frac{(m+i-1)\Gamma(m+i-\frac{1}{2})q_i}{2^{m+i}(m-1)^{\frac{1}{2}}(1+\rho)^{m+i-1}\Gamma(m+i)}\left(\frac{1}{2(1+\rho)}+\frac{t_1^2}{4(m-1)}\right)^{-(m+i-\frac{1}{2})} \qquad -\infty < t_1 < \infty. \tag{18}$$

6. The limiting distribution of (T_1, T_2). If t is replaced by $t/\{(2m-2)\sigma^2\}$ in expression (1), upon taking the limit, it follows that

$$\lim_{m\to\infty} M_{\{Z/(2m-2)\sigma^2\}}(t) = \lim_{m\to\infty}\left[1-\frac{(1+\rho)\sigma^2}{(m-1)\sigma^2}t\right]^{-\frac{1}{2}(m-1)} \cdot\left[1-\frac{(1-\rho)\sigma^2}{(m-1)\sigma^2}t\right]^{-\frac{1}{2}(m-1)} = e^t.$$

Hence $Z/\{(2m-2)\sigma^2\}$ converges stochastically to one. Also, the sequence $(m^{\frac{1}{2}}\bar{X}, m^{\frac{1}{2}}\bar{Y})$ converges in distribution to the central normal distribution with correlation ρ and common variance σ^2. By Cramér [2] page 254, it follows that the limiting distribution of T_1, T_2 is central normal distribution with correlation ρ, and common variance σ^2.

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REFERENCES

[1] Baldessari, B. (1967). The distribution of a quadratic form of normal random variables. *Ann. Math. Statist.* **38** 1700–1704.

[2] Cramér, H. (1958). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[3] Dunnett, C. M. and Sobel, J. (1954). A bivariate generalization of student's t-distribution with tables for certain special cases. *Biometrika* **41** 153–168.

[4] Press, S. J. (1969). On serial correlation. *Ann. Math. Statist.* **40** 188–196.

[5] Robbins, H. and Pitman, E. J. G. (1949). Application of the method of mixtures to quadratic forms in normal variates. *Ann. Math. Statist.* **20** 552–560.

[6] Siddiqui, M. M. (1967). A bivariate t-distribution. *Ann. Math. Statist.* **38** 162–166.