THE NONEXISTENCE OF LINKED BLOCK DESIGNS WITH
LATIN SQUARE ASSOCIATION SCHEMES

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0. Summary. The listing of partially balanced linked block designs by Roy and
Laha (1957) contains no Latin square designs. The listing of designs with Latin
square association schemes by Clatworthy (1956), which includes those given by
Bose, Clatworthy and Shrikhande (1954) and by Bose and Shimamoto (1952), and
the later listings by Chang and Liu (1964) and by Clatworthy (1967) contain no
linked block designs. The question then arises whether any linked block designs
exist having the Latin square association scheme. In this note a partial answer to
the question is given. It is shown that there do not exist any linked block designs
which are partially balanced with two associate classes and have the $L_i$ association
scheme for $i = 2, 3$ or 4.

1. Introduction. In the $L_2$ association scheme with $v = n^2$ varieties, the varieties
are arranged in a square: two varieties are said to be first associates if they lie in the
same row or in the same column; otherwise they are second associates. In the $L_i$
scheme for $i > 2$, $i - 2$ mutually orthogonal Latin squares are superimposed on the
square array; two varieties are then said to be first associates if they lie in the same
row or column, or if they correspond to the same letter in one of the Latin squares;
otherwise they are second associates. The number of varieties in each associate
class are

$$n_1 = i(n - 1), \quad n_2 = (n - 1)(n - i + 1),$$

and the equation $r(k - 1) = \lambda_1 n_1 + \lambda_2 n_2$ becomes

$$r(k - 1) - i(n - 1)\lambda_1 - (n - 1)(n - i + 1)\lambda_2 = 0. \tag{1}$$

For any design with the $L_i$ scheme, the latent roots, $\theta_i$, of $NN^T$ and their multi-
plicities, $x_i$, were shown by Connor and Clatworthy (1954) to be

$$\theta_0 = rk, \quad x_0 = 1;$$

$$\theta_1 = r + (n - i)\lambda_1 - (n - i + 1)\lambda_2, \quad x_1 = i(n - 1);$$

$$\theta_2 = r - i\lambda_1 + (i - 1)\lambda_2, \quad x_2 = (n - 1)(n - i + 1).$$

Since linked block designs are the duals of balanced incomplete block designs,
there are two possibilities, either $\theta_1 = 0$ and $b = x_2 + 1$, or $\theta_2 = 0$ and $b = x_1 + 1$.
We shall call these Case 1 and Case 2, respectively.

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In Case 1 we solve the equations \( \theta_i = 0 \) and (1) and get

\[
\lambda_1 = \frac{r(k-n)}{n(n-1)}, \quad \lambda_2 = \frac{r(ni-n+(n-i)k)}{n(n-1)(n-i+1)}; \\
r = \frac{bk}{v} = \frac{(n^2-(n-1)i)k}{n^2} = k - \frac{(n-1)ik}{n^2}.
\]

But \( r \) is an integer and \( n^2 \), \( n-1 \) are relatively prime so that \( ik/n^2 \) must be an integer; \( k \) must be of the form \( n^2t/i \) where \( t \) and \( k \) are integers.

In case 2 we solve \( \theta_2 = 0 \) and (1) obtaining

\[
\lambda_1 = \frac{r(n^2-ni+k(i-1))}{in(n-1)}, \quad \lambda_2 = \frac{r(k-n)}{n(n-1)}; \quad r = (i(n-1)+1)k/n^2.
\]

Since the complement of a linked block design is itself a linked block design (their duals being complementary BIB designs), it is enough to consider only designs with \( 2k \leq v \), i.e., \( 2k \leq n^2 \).

For each scheme our technique will be to take the possible values (if any) of \( k \leq v/2 \) which make \( r \) an integer and substitute in (2) to obtain \( \lambda_1 \) or in (3) to obtain \( \lambda_2 \). In either case we shall then show that the fraction obtained for \( \lambda_i \) cannot be reduced to an integer. Hence no designs exist.

2. The \( L_2 \) scheme. Case 1. We have \( k = n^2t/2 \). If \( n \) is odd, \( k \geq n^2 = v \) and there are no designs. Suppose \( n \) is even. Then there is the possibility \( k = n^2/2 \), in which case \( r = (n^2-2n+2)/2 \) and

\[
4\lambda_1 = \frac{(n^2-2n+2)(n^2-2n)}{n(n-1)} = (n-1)(n-2) + \frac{n-2}{n-1}.
\]

But \( n-1, n-2 \) are relatively prime and so this expression for \( 4\lambda_1 \) cannot be an integer unless \( n = 2 \). If \( n = 2 \) we can have a design with \( b = k = 2, v = 4, r = 1, \lambda_1 = 0, \lambda_2 = 1 \), but it is not a linked block design since the two blocks are disjoint.

Case 2. We have \( r = (2n-1)k/n^2 \). But \( 2n-1 \) and \( n^2 \) are relatively prime and so \( k/n^2 \) must be an integer; hence \( k \geq n^2 \), and there are no designs.

3. The \( L_3 \) scheme. Case 1. We must have \( 3k/n^2 \) an integer, and this implies \( k \geq n^2 \) unless \( n \) is divisible by 3. Let \( n = 3s \), where \( s \) is an integer. The only possibility with \( 2k \leq v \) is \( k = 3s^2 \) and \( r = 3s^2-3s+1 \). Then \( \lambda_1 = (3s^2-3s+1)(s-1)/(3s-1) \) and \( 3\lambda_1 = 3s^2-5s+3-2s/(3s-1) \). However, except in the trivial case of \( s = 1 \), which gives three disjoint blocks, \( 0 < 2s/(3s-1) < 1 \), and so \( 3\lambda_1 \) cannot be an integer.

Case 2. We have \( r = (3n-2)k/n^2 \). If \( n \) is odd, \( 3n-2 \) and \( n^2 \) are relatively prime, so that \( k \geq n^2 \), which allows no designs. Suppose that \( n = 2s \) where \( s \) is an integer, and \( s > 1 \); then \( r = (3s-1)k/2s^2 \). If \( s \) is even, \( 3s-1 \) and \( 2s^2 \) are relatively prime, and the only possibility is \( k = 2s^2, r = 3s-1 \), in which case \( \lambda_2 = (3s-1)(s-1)/(2s-1) \), and \( 2\lambda_2 = 3s-2-s/(2s-1) \), which cannot be an integer. If \( s \) is odd, there
is the possibility of $k = s^2$, $2r = 3s - 1$. Then $4\lambda_2 = (3s - 1)(s - 2)/(2s - 1)$ and $8\lambda_2 = 3s - 4 - 3s/(2s - 1)$, which cannot be an integer.

4. The $L_4$ scheme. It can be shown by the same methods that there are no linked block designs with the $L_4$ scheme. The proof is omitted.

REFERENCES


