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NOTES

ON SOME PROBLEMS INVOLVING RANDOM NUMBER OF RANDOM VARIABLES[1]

BY P. TODOROVIC

Colorado State University

1. Introduction. Frequently in various investigations, considerations on stochastic models involve sequences of random number of random variables. For instance, in many applications the number of experiments or observations in some interval of time $(0, t]$ is a chance variable. Of particular interest, for example, is the question of extreme values and sum of these observations in $(0, t]$.

Let $\tau(\nu)$ and ξ_ν denote the time of νth observation and result of the observation, respectively, and suppose that for all $\nu = 1, 2, \cdots, 0 < \tau(\nu) < \tau(\nu+1)$, $\tau(\nu) \to \infty$ if $\nu \to \infty$ and $\xi_\nu > 0$. Assuming that the number of points $\tau(\nu)$ in $(0, t]$ is a chance variable, then $\tau(\nu)$ are random variables (rv's) as well. In addition it is supposed that $\tau(\nu)$ are continuous rv's.

In the following, attention is restricted to the next four functionals:

$$\inf_{\tau(\nu) \leqq t} \xi_\nu, \qquad \sup_{\tau(\nu) \leqq t} \xi_\nu \tag{1.1}$$

$$X(t) = \textstyle\sum_{\tau(\nu) \leqq t} \xi_\nu, \qquad T(x) = \inf\{t; X(t) > x\}. \tag{1.2}$$

An attempt is made to determine a reasonable description of the extremes (1.1). In addition, the mathematical expectations and one-dimensional distribution functions (df's) of the processes (1.2) are determined.

2. Notations and definitions. Consider the probability space $(\Omega, \mathscr{A}, P)$. By definition $\mathscr{A}$ is the smallest σ-field which contains all subsets of Ω of the form $\{\tau(\nu) \leqq t\}$ and $\{X_\nu \leqq x\}$, where $X_\nu = \sum_{k=1}^{\nu} \xi_k$. It is supposed that X_ν are continuous rv's such that with probability one, $X_\nu \to \infty$ if $\nu \to \infty$.

Let $E_\nu^{\,t}$ and $G_\nu^{\,x}$ be defined as follows:

$$E_\nu^{\,t} = \{\tau(\nu) \leqq t < \tau(\nu+1)\}, \qquad G_\nu^{\,x} = \{X_\nu \leqq x < X_{\nu+1}\} \tag{2.1}$$

then for all $i \neq j = 0, 1, \cdots, E_i^{\,t} \cap E_j^{\,t} = \theta$, $G_i^{\,x} \cap G_j^{\,x} = \theta$, $\bigcup_{\nu=0}^{\infty} E_\nu^{\,t} = \bigcup_{\nu=0}^{\infty} G_\nu^{\,x} = \Omega$, where θ denotes the empty set. By virtue of (2.1) it follows that:

$$P(E_\nu^{\,t}) = P\{\tau(\nu) \leqq t\} - P\{\tau(\nu+1) \leqq t\}. \tag{2.2}$$

$$P(G_\nu^{\,x}) = P\{X_\nu \leqq x\} - P\{X_{\nu+1} \leqq x\}. \tag{2.3}$$

Since $\{\tau(\nu)\}$ and $\{X_\nu\}$ are strictly increasing sequences of continuous rv's it follows

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that $P(E_\nu^t) > 0$, $P(G_\nu^x) > 0$, for all $t > 0$, $x > 0$ and $\nu = 0, 1, \cdots$. In addition, these probabilities are continuous functions of t and x, respectively.

Finally, assuming $\tau(0) \equiv 0$ and $X_0 \equiv 0$, from (2.2) and (2.3) it follows that the distribution functions of $\tau(\nu)$ and X_ν are

$$A_\nu(t) = 1 - \sum_{j=0}^{\nu-1} P(E_j^t) \qquad B_\nu(x) = 1 - \sum_{j=0}^{\nu-1} P(G_j^x). \tag{2.4}$$

3. On extreme values of random number of random observations. Consider a phenomenon in an interval of time $(0, t]$ and suppose that the number, as well as the results of observations, are chance variables. What can one say about the extreme observations? The following is the most natural approach to this problem: Suppose that the series of N observations of the phenomenon (for example, N repetition of the same experiment) in $(0, t]$ is available, then one would consider the extremes in $(0, t]$ for each of these observations. In this way, N values for minimal and maximal results are obtained and from these data corresponding frequency distributions may be determined.

In the following, an attempt is made to determine the two distributions theoretically. To this end, consider the sequence:

$$\xi_1, \xi_2, \cdots \qquad (\xi_0 \equiv 0) \tag{3.1}$$

and denote by $\eta(t) = \sup\{\nu; \tau(\nu) \leqq t\}$, i.e., $P\{\eta(t) = \nu\} = P(E_\nu^t)$. Then the following theorem (bearing in mind that on the set E_ν^t variable $\eta(t) = \nu$) gives the distribution functions of $\inf_{\tau(\nu) \leqq t} \xi_\nu$ and $\sup_{\tau(\nu) \leqq t} \xi_\nu$ as the mathematical expectation of the conditional probabilities:

(3.2) (a) $P\{\inf_{\tau(\nu) \leqq t} \xi_\nu \leqq z \mid \eta(t)\}$,

(b) $P\{\sup_{\tau(\nu) \leqq t} \xi_\nu \leqq z \mid \eta(t)\}$,

where $z \geqq 0$.

THEOREM 1. *Let* $\underline{F}(z \mid t)$ *and* $\bar{F}(z \mid t)$ *stand for the mathematical expectation of* (3.2.a) *and* (3.2.b), *respectively, then*

$$\underline{F}(z \mid t) = 1 - \sum_{k=1}^{\infty} P[\textstyle\bigcap_{\nu=1}^{k} \{\xi_\nu > z\} \cap E_k^t] \tag{3.3}$$

$$\bar{F}(z \mid t) = \sum_{k=0}^{\infty} P[\textstyle\bigcap_{\nu=0}^{k} \{\xi_\nu \leqq z\} \cap E_k^t]. \tag{3.4}$$

PROOF.

$$\begin{aligned} EP\{\inf_{\tau(\nu) \leqq t} \xi_\nu \leqq z \mid \eta(t)\} &= 1 - EP\{\inf_{\tau(\nu) \leqq t} \xi_\nu > z \mid \eta(t)\} \\ &= 1 - \sum_{k=1}^{\infty} P[\{\inf_{0 < \nu \leqq k} \xi_\nu > z\} \cap E_k^t] \end{aligned}$$

Similarly,

$$EP\{\sup_{\tau(\nu) \leqq t} \xi_\nu \leqq z \mid \eta(t)\} = \sum_{k=0}^{\infty} P[\{\sup_{0 \leqq \nu \leqq k} \xi_\nu \leqq z\} \cap E_k^t]$$

which proves the theorem.

It is apparent that (3.3) and (3.4) are nondecreasing functions with respect to the variable z, such that $\underline{F}(0 \mid t) = \bar{F}(0 \mid t) = P(E_0{}^t)$, $\underline{F}(\infty \mid t) = \bar{F}(\infty \mid t) = 1$ and so they are required distribution functions.

Let (3.1) be the sequence of independent rv's with the common distribution function $H(z)$ and $\eta(t)$ independent of $\{\xi_v\}$, then (3.3) and (3.4) become

$$\underline{F}(z \mid t) = 1 - \sum_{k=1}^{\infty} [1-H(z)]^k P(E_k{}^t) \tag{3.5}$$

$$\bar{F}(z \mid t) = \sum_{k=0}^{\infty} [H(z)]^k P(E_k{}^t). \tag{3.6}$$

EXAMPLE. Assuming that $\eta(t)$ is Poissonian (3.5) and (3.6) become $\underline{F}(z \mid t) = 1 + e^{-\lambda t} - \exp\{-\lambda t H(z)\}$ and $\bar{F}(z \mid t) = \exp\{-\lambda t(1-H(z))\}$.

4. On the processes $X(t)$ and $T(x)$. Wald [2] proved a notable theorem for the sum $S_n = \sum_{k=1}^{n} \xi_k$ of rv's ξ_k, where the chance variable $n = 1, 2, \cdots,$. Namely, if $E\{n\}$ and $E\{\xi_k\} = a$ exist, and $P\{\xi_v \leqq x \mid n = m\} = P\{\xi_v \leqq x\}$ for all $v > m$, then $E\{S_n\} = aE\{n\}$.

Consider the stochastic process $X(t)$; on the basis of the definition $X(t)$ is the sum of random number of rv's in $(0, t]$, where, distinguished from the Wald case, $\eta(t)$ and $\{\xi_v\}$ are not independent and $\eta(t)$ depends on time.

THEOREM 2. *For all $t \in (0, \infty)$, for which the following series converges,*

$$\sum_{k=1}^{\infty} A_k(t) E\{\xi_k \mid \tau(k) \leqq t\}; \tag{4.1}$$

$E\{X(t)\}$ exists and is equal to (4.1).

PROOF. It is apparent that

$$E\{X(t)\} = \int_\Omega E\{X(t) \mid \eta(t)\}\, dP = \sum_{v=0}^{\infty} \sum_{k=0}^{v} \int_{E_v{}^t} E\{\xi_k \mid \eta(t)\}\, dP$$

or, after simple transformation,

$$\begin{aligned} E\{X(t)\} &= \sum_{k=1}^{\infty} \sum_{v=k}^{\infty} \int_{E_v{}^t} E\{\xi_k \mid \eta(t)\}\, dP \\ &= \sum_{k=1}^{\infty} \int_{\{\tau(k) \leqq t\}} E\{\xi_k \mid \eta(t)\}\, dP \end{aligned}$$

and the theorem is proved.

THEOREM 3. *Let $F_t(x) = P\{X(t) \leqq x\}$, then for every $t \geqq 0$ and $x \geqq 0$*

$$F_t(x) = P(E_0{}^t) + \sum_{v=1}^{\infty} P(E_v{}^t) P\{X_v \leqq x \mid E_v{}^t\}. \tag{4.2}$$

PROOF. It is apparent that

$$\begin{aligned} F_t(x) = EP\{X(t) \leqq x \mid \eta(t)\} &= \sum_{v=0}^{\infty} P\{\textstyle\sum_{k=0}^{v} \xi_k \leqq x \mid E_v{}^t\} P(E_v{}^t) \\ &= P(E_0{}^t) + \sum_{v=1}^{\infty} P(E_v{}^t) P\{X_v \leqq x \mid E_v{}^t\} \end{aligned}$$

which proves the theorem.

REMARK. Theorem 3 may be proven in another way, bearing in mind that $X(t)$ is the sum of the random number of positive rv's in $(0, t]$. Since the sample functions of $X(t)$ are nondecreasing step functions, the following relationship is obvious

$$\{X(t) \leqq x\} \cap E_v{}^t = \{X_v \leqq x\} \cap E_v{}^t.$$

Therefore,

$$P\{X(t) \leqq x\} = \sum_{v=0}^{\infty} P(E_v^t) P\{X_v \leqq x \mid E_v^t\}.$$

THEOREM 4. *For all* $x \in (0, \infty)$, *for which the following series converges,*

$$\sum_{k=0}^{\infty} P(G_k^x) E\{\tau(k+1) \mid G_k^x\}; \tag{4.3}$$

$E\{T(x)\}$ *exists and is equal to* (4.3).

PROOF. Let $\mu(x) = \sup\{v; X_v \leqq x\}$, i.e., $P\{\mu(x) = v\} = P(G_v^x)$. By virtue of the definition $T(x) = \sup_{X_{v-1} \leqq x < X_v} \tau(v)$. Therefore,

$$\begin{aligned} E\{T(x)\} &= \int_\Omega E\{\sup_{X_{v-1} \leqq x < X_v} \tau(v) \mid \mu(x)\}\, dP \\ &= \sum_{k=0}^{\infty} E\{\tau(k+1) \mid G_k^x\} P(G_k^x) \end{aligned}$$

since on the set $G_k^x \sup_{X_{v-1} \leqq x < X_v} \tau(v) = \tau(k+1)$, which proves the theorem.

THEOREM 5. *Let* $Q_x(t) = P\{T(x) \leqq t\}$, *then for every* $t \geqq 0$ *and* $x \geqq 0$

$$Q_x(t) = 1 - F_t(x). \tag{4.4}$$

PROOF.

$$\begin{aligned} P\{T(x) \leqq t\} &= EP\{T(x) \leqq t \mid \mu(x)\} \\ &= \sum_{k=0}^{\infty} \int_{G_k} P\{\sup_{X_{v-1} \leqq x < X_v} \tau(v) \leqq x \mid \mu(x)\}\, dP \\ &= \sum_{k=0}^{\infty} P\{\tau(k+1) \leqq t \mid G_k^x\} P(G_k^x). \end{aligned}$$

Since $\{\tau(k+1) \leqq t\} = \bigcup_{v=k+1}^{\infty} E_v^t$, it follows that

$$\begin{aligned} Q_x(t) &= \sum_{k=0}^{\infty} \sum_{v=k+1}^{\infty} P(G_k^x \cap E_v^t) = 1 - \sum_{v=0}^{\infty} \sum_{k=v}^{\infty} P(E_v^t \cap G_k^x) \\ &= 1 - \sum_{v=0}^{\infty} P[E_v^t \cap \{X_v \leqq x\}] \end{aligned}$$

and the theorem is proved.

NOTE. If in Theorem 2 it is supposed that (3.1) is the sequence of identically distributed rv's independent of $\eta(t)$, then (4.1) becomes

$$E\{X(t)\} = E\{\xi\} \sum_{k=1}^{\infty} A_k(t) = E\{\xi\} E\{\eta(t)\}$$

and this is the Wald case.

In many applications, the events E_v^t and $\{X_v \leqq x\}$ may be supposed to be independent. Under this condition, (4.2) may be written as follows:

$$F_t(x) = P(E_0^t) + \sum_{v=1}^{\infty} \sum_{j=v}^{\infty} P(E_v^t) P(G_j^x).$$

If one assumes that the following conditions are satisfied

$$\sum_{r=2}^{\infty} P(E_r^{t,t+\Delta t}) = o(\Delta t) \qquad \text{if} \quad \Delta t \to 0$$

$$\lim_{\Delta t \to 0} \frac{P(E_1^{t,t+\Delta t} \mid E_v^t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(E_1^{t-\Delta t,t} \mid E_v^{t-\Delta t})}{\Delta t} = \lambda_1(t, v)$$

then

$$\frac{dP(E_v^t)}{dt} = \lambda_1(t, v-1)P(E_{v-1}^t) - \lambda_1(t, v)P(E_v^t) \qquad v = 1, 2, \cdots$$

$$\frac{dP(E_0^t)}{dt} = -\lambda_1(t, 0)P(E_0^t)$$

where $E_r^{t,t+\Delta t} = \{\eta(t+\Delta t) - \eta(t) = r\}$.

By virtue of the following relations

$$P(E_v^{t+\Delta t}) - P(E_v^t) = \sum_{r=1}^{v} P(E_{v-r}^t \cap E_r^{t,t+\Delta t}) - \sum_{r=1}^{\infty} P(E_v^t \cap E_r^{t,t+\Delta t})$$

$$P(E_v^t) - P(E_v^{t-\Delta t}) = \sum_{r=1}^{v} P(E_{v-r}^{t-\Delta t} \cap E_r^{t-\Delta t,t}) - \sum_{r=1}^{\infty} P(E_v^{t-\Delta t} \cap E_r^{t-\Delta t,t})$$

and the conditions the assertion follows. Similarly, if

$$\sum_{r=2}^{\infty} P(G_r^{x,x+\Delta x}) = o(\Delta x) \qquad \text{if} \quad \Delta x \to 0$$

$$\lim_{\Delta x \to 0} \frac{P(G_1^{x,x+\Delta x} \mid G_v^x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{P(G_1^{x-\Delta x,x} \mid G_v^{x-\Delta x})}{\Delta x} = \lambda_2(x, v)$$

then

$$\frac{dP(G_v^x)}{dx} = \lambda_2(x, v-1)P(G_{v-1}^x) - \lambda_2(x, v)P(G_v^x) \qquad v = 1, 2, \cdots$$

$$\frac{dP(G_0^x)}{dx} = -\lambda_2(x, 0)P(G_0^x)$$

where $G_r^{x,x+\Delta x} = \{\mu(x+\Delta x) - \mu(x) = r\}$.

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REFERENCES

[1] GUMBEL, E. J. (1967). *Statistics of Extremes*, 2nd ed. Columbia Univ. Press, New York.
[2] WALD, A. (1944). On cumulative sums of random variables. *Ann. Math. Statist.* **15** 283–296.