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STRUCTURAL ANALYSIS FOR THE FIRST ORDER AUTOREGRESSIVE STOCHASTIC MODELS

BY M. SAFIUL HAQ

University of Western Ontario

1. Summary and introduction. The response variables of a first order autoregressive stochastic process with autocorrelation ρ can be constructed as a location-scale transform of a set of error variables whose distribution depends on the autocorrelation parameter, and the model can be treated as a composite response model with an error quantity ρ (Fraser 1968, page 192). For known value of ρ the model is a conditional structural model (Fraser 1968, page 188). Inference concerning ρ is naturally based on the marginal likelihood function of ρ obtained from the marginal probability distribution of the orbit of the response. The results are specialized to cover the normal error distribution.

For the same model with error variables having a periodic structure a suitable transformation reduces the error variable into uncorrelated variables and the resultant transformed model can be treated as a location-scale structural model. The orbit of the transformed response depends on ρ and so inference concerning ρ can be made from the marginal likelihood of ρ obtained from the marginal probability distribution of the orbit. It has been found that without any approximation being used the marginal likelihood function of ρ thus obtained depends on the first order circular serial correlation coefficient in general, and for normal distribution in particular.

For both the cases the general approximate distribution of the serial correlation coefficient has been derived by likelihood modulation. Using Anderson's (1942) result the general exact distribution of the circular serial correlation coefficient has also been obtained.

2. First order autoregressive model. Consider the set of responses

$$
\begin{aligned}
x_1 &= \mu + \sigma e_1 \\
x_2 &= \mu + \sigma e_2 \\
&\vdots \\
x_n &= \mu + \sigma e_n,
\end{aligned} \tag{2.1}
$$

where the responses are assumed to have been obtained by a location-scale transformation of the error variables e_α's, with $E(e_\alpha) = 0$, $(\alpha = 1, 2, \cdots, n)$, and covariance matrix of e_α as

$$
\operatorname{cov}(e_\alpha, e_\beta) = \Sigma = \begin{bmatrix} 1 & \rho & \cdots & \rho^{n-1} \\ \rho & 1 & \cdots & \rho^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{bmatrix}; \tag{2.2}
$$

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so that

$$\Sigma^{-1} = (1-\rho^2)^{-1}\begin{bmatrix} 1 & -\rho & 0 & 0 & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & & \cdots & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and the probability of $(e_1, \cdots, e_n)$ is given by

$$f(e_1, \cdots, e_n : \rho)\, d\, e_1 \cdots d\, e_n. \tag{2.3}$$

The random variables e_α constitute a first order autoregressive stochastic process with autocorrelation parameter ρ. The location-scale transformation $[\mu, \sigma]$ is an element of the group of transformation G: $\{g = [a, c]/-\infty < a < \infty, c > 0\}$. The transformations g put the observed response into its orbit $G\mathbf{x} = \{a\mathbf{1} + c\mathbf{x}\}$. The model thus described is a composite response model with the additional quantity ρ for the error variable. For known value of ρ the model is a conditional structural model. Thus following Fraser (1968, Chapter 1) the conditional probability element for $\bar{e}$ and $s(e)$ on the orbit is obtained as (for known ρ)

$$\psi_\rho(a) f(\bar{e}\mathbf{1} + s(e)\mathbf{a} : \rho) s^{n-2}(e)\, d\,\bar{e}\, ds(e); \tag{2.4}$$

where

$$\bar{e} = \textstyle\sum_{\alpha=1}^n e_\alpha/n, \qquad s^2(e) = \sum_{\alpha=1}^n (e_\alpha - \bar{e})^2,$$

$$\mathbf{a} = (a_1, \cdots, a_n),$$

$$a_j = (e_j - \bar{e})/s(e)$$

$$= (x_j - \bar{x})/s(x), \qquad (j = 1, 2, \cdots, n)$$

and $\psi_\rho^{-1}(\mathbf{a}) = \int_G f(\bar{e}\mathbf{1} + s\mathbf{a} : \rho) s^{n-2}\, d\bar{\mathbf{e}}\, ds$;
and the structural distribution of μ and σ is obtained as

$$\psi_\rho(\mathbf{a})\sigma^{-(n+1)} f(\sigma^{-1}(\mathbf{x} - \mu\mathbf{1}) : \rho) s^{n-1}(x)\, d\mu\, d\sigma. \tag{2.5}$$

For known value of ρ the expressions (2.4) and (2.5) are the basis of inference about μ and σ. These expressions have been obtained from (2.3) and properties of invariant differentials; and no integration with respect to the orbital variables $a_1, \cdots, a_n$ is needed. Thus, following Fraser (1968, Chapter 4, Section 3, and problem 11) the marginal probability element for the orbital variables is obtained by dividing (2.3) by (2.4) adjusted by the factor $n^{\frac{1}{2}}$ necessary to measure the Euclidean volume on the orbit in R^n, as

$$n^{\frac{1}{2}}\psi_\rho^{-1}(\mathbf{a}) s^{-(n-2)}(e) \textstyle\prod_{i=1}^n d\, e_i/(d(n^{\frac{1}{2}}\bar{e})\, ds(e)), \tag{2.6}$$

which when expressed with respect to volume at $\mathbf{x}$ is

$$n^{\frac{1}{2}}\psi_\rho^{-1}(\mathbf{a}(\mathbf{x})) s^{-(n-2)}(x) \textstyle\prod_{i=1}^n dx_i/(d(n^{\frac{1}{2}}\bar{x})\, ds(x)) = n^{\frac{1}{2}}\psi_\rho^{-1}(\mathbf{a}(\mathbf{x})) s^{-(n-2)}(x)\, dv. \tag{2.7}$$

The marginal likelihood of ρ is then

(2.8) $$L(\rho \mid \mathbf{x}) = R^{+}(\mathbf{x})\psi_{\rho}^{-1}(\mathbf{a}(\mathbf{x}))$$

where $R^{+}(\mathbf{x})$ is the map that carries a point $\mathbf{x}$ into the single entity, the set $R^{+} = (0, \infty)$. The marginal likelihood function thus obtained provides the basis for inference about the parameter ρ.

3. The normal error. The assumption of normality is quite prominent in the analysis of the responses from an autoregressive stochastic model. So in this section the composite response model with autoregressive error variable is discussed with the error variable having normal distribution with mean zero and the covariance matrix Σ of Section 2:

(3.1) $$\begin{aligned} f(e_1, \cdots e_n : \rho)\, d\mathbf{e} &= (2\pi)^{-n/2}\,|\Sigma|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\mathbf{e}'\Sigma^{-1}\mathbf{e}\}\, d\mathbf{e} \\ &= (2\pi)^{-n/2}(1-\rho^2)^{-\frac{1}{2}(n-1)} \exp\{-\tfrac{1}{2}\mathbf{e}'\Sigma^{-1}\mathbf{e}\}\, d\mathbf{e}. \end{aligned}$$

The conditional probability element of $\bar{e}$ and $s(e)$ on the orbit is obtained as

(3.2) $$\begin{aligned} &\psi_{\rho}(\mathbf{a})(2\pi)^{-n/2}(1-\rho^2)^{-\frac{1}{2}(n-1)} s^{n-2}(e) \\ &\qquad \cdot \exp\{-\tfrac{1}{2}(\bar{e}\mathbf{1} + s(e)\mathbf{a})'\Sigma^{-1}(\bar{e}\mathbf{1} + s(e)\mathbf{a})\}\, d\bar{e}\, ds(e). \end{aligned}$$

The quadratic expression in the exponent of (3.2) can be simplified:

$$\begin{aligned} &(\bar{e}\mathbf{1} + s(e)\mathbf{a})'\Sigma^{-1}(\bar{e}\mathbf{1} + s(e)\mathbf{a}) \\ &= \bar{e}\mathbf{1}'\Sigma^{-1}\bar{e}\mathbf{1} + 2\bar{e}\mathbf{1}'\Sigma^{-1}s(e)\mathbf{a} + s^2(e)\mathbf{a}'\Sigma^{-1}\mathbf{a} \\ &= (1+\rho)^{-1}N(\bar{e}^2 + 2\bar{e}\rho s(e)P) + (1-\rho^2)^{-1}s^2(e)Q \\ &= (1+\rho)^{-1}N(\bar{e} + \rho s(e)P)^2 + (1+\rho)^{-1}s^2(e)[(1-\rho)^{-1}Q - NP^2\rho^2], \end{aligned}$$

where

$$\begin{aligned} N &= n - n\rho + 2\rho \\ P &= (a_1 + a_n)/N \\ Q &= 1 + \rho^2 - \rho^2(a_1^2 + a_n^2) - 2\rho r \\ r &= \textstyle\sum_{\alpha=1}^{n-1}(e_\alpha - \bar{e})(e_{\alpha+1} - \bar{e})/s^2(e) \\ &= \textstyle\sum_{\alpha=1}^{n-1}(x_\alpha - \bar{x})(x_{\alpha+1} - \bar{x})/s^2(x). \end{aligned} \tag{3.3}$$

Using these results the probability element for $\bar{e}$ and $s(e)$ can be written as

(3.4) $$\begin{aligned} &\psi_{\rho}(\mathbf{a})(2\pi)^{-n/2}(1-\rho^2)^{-\frac{1}{2}(n-1)} s^{n-2}(e) \\ &\qquad \cdot \exp\left\{-\frac{N}{2(1+\rho)}(\bar{e} + \rho s(e)P)^2 - \frac{s^2(e)}{2(1+\rho)}[Q/(1-\rho) - NP^2\rho^2]\right\}. \end{aligned}$$

Integrating over $\bar{e}$ and $s(e)$ the normalizing constant $\psi_{\rho}(\mathbf{a})$ can be obtained as

(3.5) $$\psi_{\rho}(\mathbf{a}) = N^{\frac{1}{2}}A_{n-1}(1+\rho)^{-\frac{1}{2}}[Q - (1-\rho)\rho^2P^2N]^{\frac{1}{2}(n-1)}$$

where $A_{n-1} = 2\pi^{\frac{1}{2}(n-1)}/\Gamma(\frac{1}{2}(n-1))$ is area of the unit sphere in R^{n-1}. The normalizing constant depends on **a** through r, and a_1 and a_n only. For known value of ρ the structural density for μ and σ is obtained as

$$\begin{aligned}(3.6)\quad &\psi_\rho(\mathbf{a})(2\pi)^{-n/2}(1-\rho^2)^{-(n-1)/2}\sigma^{-(n+1)}s^{n-1}(x)\exp\{-N(2\sigma^2(1+\rho))^{-1}\\ &\qquad\cdot[\bar{x}-\mu+\rho s(x)P]^2-s^2(x)(2\sigma^2(1+\rho))^{-1}[Q(1-\rho)^{-1}-NP^2\rho^2\end{aligned}$$

and the marginal density of σ is obtained as

$$\begin{aligned}(3.7)\quad &\psi_\rho(\mathbf{a})(2\pi(1-\rho^2))^{-\frac{1}{2}(n-1)}(1+\rho)^{\frac{1}{2}}N^{-\frac{1}{2}}\sigma^{-n}s^{n-1}(x)\\ &\qquad\cdot\exp\{-s^2(x)(2\sigma^2(1+\rho))^{-1}[(1-\rho)^{-1}Q-NP^2\rho^2]\}\end{aligned}$$

and the marginal density of μ is obtained as

$$\begin{aligned}(3.8)\quad &\left[\frac{N(1-\rho)}{\pi}\right]^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n-1))}[s^2(x)\{Q-(1-\rho)\rho^2NP^2\}]^{-\frac{1}{2}}\\ &\qquad\cdot[1+s^{-2}(x)N(1-\rho)(\bar{x}-\mu+\rho s(x)P)^2(Q-(1-\rho)\rho^2NP^2)^{-1}]^{-n/2}.\end{aligned}$$

Thus it is seen that for known value of ρ the marginal distribution of σ is rescaled inverted chi, and the marginal distribution of μ is rescaled t, the rescaling factor depending on the autocorrelation parameter ρ.

3.1. *Inference about* ρ. The marginal probability element for the orbital variable $a_1, \cdots, a_n$ at the observed response is

$$\begin{aligned}(3.9)\quad &\psi_\rho^{-1}(\mathbf{a})n^{\frac{1}{2}}s^{-(n-2)}(x)\,dv\\ &\qquad= N^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}(A_{n-1})^{-1}n^{\frac{1}{2}}[Q-(1-\rho)\rho^2NP^2]^{-\frac{1}{2}(n-1)}s^{-(n-2)}(x)\,dv.\end{aligned}$$

The marginal likelihood of ρ is

$$(3.10)\qquad L^{**}(\rho:\mathbf{a}) = R^{+}(\mathbf{a})N^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}[Q-(1-\rho)\rho^2NP^2]^{-\frac{1}{2}(n-1)};$$

which when expressed as a ratio relative to $\rho = 0$ reduces to

$$(3.11)\qquad L^{*}(\rho:\mathbf{a}) = N^{-\frac{1}{2}}[n(1+\rho)]^{\frac{1}{2}}\cdot[Q-(1-\rho)\rho^2NP^2]^{-\frac{1}{2}(n-1)}.$$

The marginal likelihood function $L^*(\rho\,|\,\mathbf{a})$ derived from the marginal probability element for the orbital statistic **a** depends on the orbit through a_1, a_n and r. For moderately large n, a_1 and a_n may be considered small and the factor $N^{-\frac{1}{2}}[n(1+\rho)]^{\frac{1}{2}}$ approaches the quantity $(1-\rho)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}$. Hence the marginal likelihood function can be written as approximately

$$(3.12)\qquad L(\rho:r) = [(1-\rho)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}]\cdot(1+\rho^2-2\rho r)^{-\frac{1}{2}(n-1)}.$$

If $g(r:0)$ is the density function for the distribution of r when $\rho = 0$, the general distribution of r can be written as

$$(3.13)\qquad g(r:\rho)\,dr = [(1-\rho)^{-\frac{1}{2}}(1+\rho)^{\frac{1}{2}}]\cdot(1+\rho^2-2\rho r)^{-\frac{1}{2}(n-1)}g(r:0)\,dr.$$

The quantity r may be called a first order serial correlation. First order serial correlation coefficient is usually defined as (Kendall 1966, page 361)

(3.14) $$\sum_{\alpha=1}^{n-1}(x_\alpha-\bar{x}_1)(x_{\alpha+1}-\bar{x}_2)/\{\sum_{\alpha=1}^{n-1}(x_\alpha-\bar{x}_1)^2\sum_{\alpha=2}^{n}(x_\alpha-\bar{x}_2)^2\}^{\frac{1}{2}}$$

where $\bar{x}_1 = \sum_{\alpha=1}^{n-1} x_\alpha/(n-1)$, $\bar{x}_2 = \sum_{\alpha=2}^{n} x_\alpha/(n-1)$.

But in practice a more convenient estimate is used:

(3.15) $$r_1 = \sum_{\alpha=1}^{n}(x_\alpha-\bar{x})(x_{\alpha+1}-\bar{x})/\sum_{\alpha=1}^{n}(x_\alpha-\bar{x})^2$$

with $x_{n+1} = x_1$.

r_1 approximates (3.14) and is known as circular serial correlation coefficient. Anderson (1942) has derived the exact distribution of r_1 when $\rho = 0$. Moran (1948) used a statistic somewhat similar to r:

(3.16) $$r_2 = \frac{n}{n-1}\sum_{\alpha=1}^{n-1}(x_\alpha-\bar{x})(x_{\alpha+1}-\bar{x})\bigg/\sum_{\alpha=1}^{n}(x_\alpha-\bar{x})^2.$$

r_2 differs from r only by a multiplicative constant factor $n/(n-1)$. The exact distribution of r_1 derived by Anderson can be utilized to find the distribution of r_2 when $n = 2m$ (Kendall & Stuart 1966, page 440). Several people have worked on the approximate distribution of serial correlation, mostly in the circular form. Following Moran (1948) the distribution of r when $\rho = 0$ is obtained approximately as

(3.17) $$g(r:0)\,dr = \frac{\Gamma(\frac{1}{2}(n+2))}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}(n+1))}(1-r^2)^{\frac{1}{2}(n-1)}\,dr, \qquad -1 < r < 1;$$

and $t = (n+1)^{\frac{1}{2}}r/(1-r^2)^{\frac{1}{2}}$ has a t distribution with $n+1$ degrees of freedom.

The approximate general distribution of the correlation coefficient is then

(3.18) $$\frac{\Gamma(\frac{1}{2}(n+2))}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}(n+1))}(1+\rho)^{\frac{1}{2}}/(1-\rho)^{\frac{1}{2}}\cdot\frac{(1-r^2)^{\frac{1}{2}(n-1)}}{(1+\rho^2-2\rho r)^{\frac{1}{2}(n-1)}}\,dr, \qquad -1 < r < 1.$$

The method followed here is exactly the same as Fraser (1968, Chapter 4, Section 3) where he has derived the distribution of correlation coefficient for samples from bivariate normal distribution.

4. The transformed model. Let $u_\alpha(\alpha = \cdots -2, -1, 0, 1, 2, \cdots)$ be independently and identically distributed random variables with $E(u_\alpha) = 0$, $\mathrm{Var}(u_\alpha) = \sigma_u^2$. Let $e_\alpha = \rho e_{\alpha-1} + u_\alpha$ (where $|\rho| < 1$), then $E(e_\alpha) = 0$; $\mathrm{Var}(e_\alpha) = \sigma_u^2(1-\rho^2)^{-1} = \sigma^2$ (say) and $\mathrm{Cov}(e_\alpha, e_\beta) = \rho^{|\alpha-\beta|}\sigma^2$. The random variables e_α's constitute a first order autoregressive stochastic process. Thus a sample from a first-order stochastic process can be transformed into uncorrected random variables by a linear transformation: $u_\alpha = e_\alpha - \rho e_{\alpha-1}$. For a sample of size n the assumption of periodic structure is necessary so that the $(n+1)$th response may be identified with the first response and the transformation becomes one to one. Data from time series is often analyzed under such assumptions. Assuming that the composite response model described in Section 2 has such a periodic structure the model can be transformed as follows:

$$R\mathbf{x} = R\mu\mathbf{1} + \sigma R\mathbf{e} \quad \text{or} \quad \mathbf{x}_\rho = \mu_\rho \mathbf{1} + \sigma\mathbf{u}, \tag{4.1}$$

where
$$R = \begin{bmatrix} -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \cdots & 0 \\ 0 & 0 & -\rho & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 1 & 0 & 0 & \cdots & -\rho \end{bmatrix}$$

is a $n \times n$ transformation matrix. The transformation maps a point $\mathbf{x}$ in R^n to $\mathbf{x}_\rho$ in R^n. The transformed responses may be considered to have been obtained from the transformed error variables u_α's ($\alpha = 1, 2, \cdots, n$) by the transformation $[\mu_\rho, \sigma]$ in G. u_α's are uncorrelated with $E(u_\alpha) = 0$, and variance $(u_\alpha) = \sigma^2(1-\rho^2)$. Further assume that u_α's have a probability distribution

$$\prod_{\alpha=1}^n g(u_\alpha : \rho)\, du_\alpha; \tag{4.2}$$

then for a known value of ρ the transformed model is a structural model. The elements of G generate an orbit for the transformed responses through $\mathbf{x}_\rho$. So following Fraser (1968) as in Section 2, the conditional probability element of $\bar{u}$ and $s(u)$ on the orbit is obtained as

$$\phi_\rho(D_\rho(u)) \prod_{\alpha=1}^n g(\bar{u} + s(u)\, d_{\rho\alpha} : \rho) s^{n-2}(u)\, d\bar{u}\, ds(u); \tag{4.3}$$

where $\phi_\rho(D_\rho(u))$ is the normalizing constant and

$$\begin{aligned} D_\rho(u) &= (d_{\rho 1}, \cdots, d_{\rho n})' \\ &= [(u_1 - \bar{u})/s(u), \cdots, (u_n - \bar{u})/s(u)]' \\ &= [(x_{\rho 1} - \bar{x}_\rho)/s(x_\rho), \cdots, (x_{\rho n} - \bar{x}_\rho)/s(x_\rho)]' \\ &= D_\rho(\mathbf{x}_\rho); \end{aligned} \tag{4.4}$$

and the structural distribution of μ_ρ and σ from (4.3) is obtained as

$$\phi_\rho(D_\rho(u)) \prod_{\alpha=1}^n g(\sigma^{-1}(\bar{x}_\rho - \mu_\rho + s(x_\rho)\, d_{\rho\alpha}) : \rho) s^{n-1}(x_\rho) \sigma^{-(n+1)}\, d\mu_\rho\, d\sigma. \tag{4.5}$$

For known value of ρ the expression (4.5) is the basis of inference about μ and σ.

4.1. *Inference about* ρ. The conditional probability element of $\bar{u}$ and $s(u)$ on the orbit is given by the expression (4.3), which when adjusted for the Euclidean volume on the orbit in R^n reduces to

$$\phi_\rho(D_\rho) n^{-\frac{1}{2}} \prod_{\alpha=1}^n g(\bar{u} + s(u)\, d_{\rho\alpha} : \rho) s^{n-2}(u)\, d(n^{\frac{1}{2}}\bar{u})\, ds(u); \tag{4.6}$$

and as in Section 2 the marginal probability element for the orbital variables is obtained by dividing the joint probability element (4.2) by the conditional probability element (4.6) and is obtained as

$$n^{\frac{1}{2}} \phi_\rho^{-1}(D_\rho(u)) s^{-(n-2)}(u)\, d\mathbf{u} / d(n^{\frac{1}{2}}\bar{u})\, ds(u). \tag{4.7}$$

The marginal probability element of the orbit at $\mathbf{x}_\rho$, rather than at $\mathbf{u}$, is

$$n^{\frac{1}{2}}\phi_\rho^{-1}(D_\rho(x_\rho))s^{-(n-2)}(x_\rho)\,d\mathbf{x}_\rho/(d(n^{\frac{1}{2}}\bar{x}_\rho)\,ds(x_\rho)). \tag{4.8}$$

Under the transformation described at (4.1) the Jacobian of the transformation from $\mathbf{x}_\rho$ to $\mathbf{x}$ is $(1-\rho^n)$. The same transformation changes the differential $dn^{\frac{1}{2}}\bar{x}_\rho$, $ds(x_\rho)$ along the orbit to $n^{\frac{1}{2}}(1-\rho)\,d\bar{x}$, $(1+\rho^2-2\rho r_1)^{\frac{1}{2}}\,ds(x)$, where r_1 is the circular serial correlation coefficient defined at (3.15). The Jacobian derivation uses the relations:

$$\bar{x}_\rho = (1-\rho)\bar{x}$$
$$s^2(x_\rho) = s^2(x)(1+\rho^2-2\rho r_1).$$

Also under the inverse transformation the reference point $D_\rho(x)$ is mapped into the point

$$\begin{aligned} D(x) &= R^{-1}D_\rho(x) \\ &= R^{-1}\left[\frac{x_2-\bar{x}-\rho(x_1-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}, \cdots, \frac{x_1-\bar{x}-\rho(x_n-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}\right]' \\ &= (1+\rho^2-2\rho r_1)^{-\frac{1}{2}}[(x_1-\bar{x})/s(x), \cdots, (x_n-\bar{x})/s(x)]'. \end{aligned}$$

The inverse image of the reference point indexes the inverse image of the orbit and depends on the autocorrelation parameter. It is interesting to note that the factor involving ρ in $D(x)$ comes as a multiple and thus the inverse image of the orbit does not change direction with ρ. So in terms of the observed response the probability element cross-sectional to the inverse image of the orbit is obtained as

$$\begin{aligned} &n^{\frac{1}{2}}\phi_\rho^{-1}\left\{\frac{x_2-\bar{x}-\rho(x_1-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}, \cdots, \frac{x_1-\bar{x}-\rho(x_n-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}\right\} \\ &\quad\cdot(1+\rho^2-2\rho r_1)^{-\frac{1}{2}(n-1)}s^{-\frac{1}{2}(n-2)}(x)(1-\rho^n)(1-\rho)^{-1}\,d\mathbf{x}/(dn^{\frac{1}{2}}\bar{x}\,ds(x)). \end{aligned} \tag{4.9}$$

Thus the marginal likelihood of ρ is obtained as

$$\begin{aligned} L(\rho\,|\,x) &= R^+(\mathbf{x})n^{\frac{1}{2}}\phi_\rho^{-1}\left\{\frac{x_2-\bar{x}-\rho(x_1-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}, \cdots, \frac{x_1-\bar{x}-\rho(x_n-\bar{x})}{s(x)(1+\rho^2-2\rho r_1)^{\frac{1}{2}}}\right\} \\ &\quad\cdot(1-\rho^n)(1-\rho)^{-1}(1+\rho^2-2\rho r_1)^{-\frac{1}{2}(n-1)}. \end{aligned} \tag{4.10}$$

It is observed that in general the marginal likelihood function of ρ depends on r_1, the first order circular serial correlation coefficient, and does not require any approximation.

5. Transformed model: Normal error. Now consider that the error variables have the probability element

$$(2\pi(1-\rho^2))^{-\frac{1}{2}n}\exp\left\{-\frac{1}{2(1-\rho^2)}\sum_{\alpha=1}^{n}u_\alpha^2\right\}\prod_{\alpha=1}^{n}du_\alpha. \tag{5.1}$$

The conditional probability element of $\bar{u}$ and $s(u)$ is obtained as

$$(5.2)\qquad \phi_\rho(D_\rho)(2\pi(1-\rho^2)^{-\frac{1}{2}n}\exp\left\{-\frac{1}{2(1-\rho^2)}(n\bar{u}^2+s^2(u))\right\}s^{n-2}(u)\,d\bar{u}\,ds(u).$$

$\bar{u}$ and $s(u)$ are independent. The normalizing constant is independent of $\bar{u}$ and $s(u)$ and is obtained as

$$(5.3)\qquad \phi_\rho(D_\rho) = A_{n-1}n^{\frac{1}{2}}.$$

Hence the marginal likelihood of ρ as obtained from (4.10) is

$$(5.4)\qquad L^*(\rho\,|\,\mathbf{x}) = R^+(\mathbf{x})(1-\rho^n)(1-\rho)^{-1}(1+\rho^2-2\rho r_1)^{-\frac{1}{2}(n-1)};$$

which when expressed as a ratio relative to $\rho = 0$ reduces to

$$(5.5)\qquad L(\rho\,|\,\mathbf{x}) = (1-\rho^n)(1-\rho)^{-1}(1+\rho^2-2\rho r_1)^{-\frac{1}{2}(n-1)}.$$

The exact distribution of r_1 when $\rho = 0$ has been obtained by Anderson (1942) which is difficult to use in practice. However we quote the general form of the frequency function here as stated in [4] (page 440):

$$(5.6)\qquad f(r_1) = \tfrac{1}{2}(n-3)\textstyle\sum_{i=1}^{m}(\lambda_i-r_1)^{\frac{1}{2}(n-5)}/\prod_{i\neq j=1}^{\frac{1}{2}(n-1)}(\lambda_i-\lambda_j), \qquad \lambda_{m+1}\leqq r_1\leqq\lambda_m, \quad n\text{ odd};$$

$$= \tfrac{1}{2}(n-3)\textstyle\prod_{i=1}^{m}(\lambda_i-r_1)^{\frac{1}{2}(n-5)}/[\prod_{i\neq j=1}^{\frac{1}{2}(n-2)}(\lambda_i-\lambda_j)(1+\lambda_i)^{\frac{1}{2}}] \qquad \lambda_{m+1}\leqq r_1\leqq\lambda_m, \quad n\text{ even}.$$

Where λ's are the roots of the equation

$$\begin{vmatrix} -\left(\lambda+\frac{1}{n}\right) & \frac{1}{2}\left(1-\frac{2}{n}\right) & -\frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{2}\left(1-\frac{2}{n}\right) \\ \frac{1}{2}\left(1-\frac{2}{n}\right) & -\left(\lambda+\frac{1}{n}\right) & \frac{1}{2}\left(1-\frac{2}{n}\right) & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & \frac{1}{2}\left(1-\frac{2}{n}\right) & -\left(\lambda+\frac{1}{n}\right) & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{2}\left(1-\frac{2}{n}\right) & -\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{2}\left(1-\frac{2}{n}\right) & -\left(\lambda+\frac{1}{n}\right) \end{vmatrix} = 0.$$

This frequency function when modulated by the likelihood function (5.5) gives the exact distribution of r_1:

$$g(r_1\,|\,\rho) = (1-\rho^n)(1-\rho)^{-1}(1+\rho^2-2\rho r_1)^{-\frac{1}{2}(n-1)}f(r_1).$$

Rubin (1945) derived an approximate distribution of r_1 when $\rho = 0$ as

$$(5.7)\qquad \frac{\Gamma(\frac{1}{2}(n+2))}{\Gamma(\frac{1}{2}(n+1))\pi^{\frac{1}{2}}}(1-r_1{}^2)^{\frac{1}{2}(n-1)}\,dr_1; \qquad -1<r_1<1;$$

which is actually the distribution of ordinary correlation coefficient based on $(n+3)$ observations. The approximation is good for $n \geqq 10$. This when modulated by the likelihood function (5.5) gives the approximate general distribution of the circular serial correlation coefficient.

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