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FLUCTUATIONS WHEN $E(|X_1|) = \infty$[1]

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1. Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables. Set $S_n = \sum_{k=1}^n X_k$ and $S_0 \equiv 0$. We define the two random variables, $N(\omega)$ and $M(\omega)$, by $N(\omega) = \sum_{n=0}^{\infty} I_{\{\omega: S_n(\omega) \geqq 0\}}$ and $M(\omega) = \sup_{n \geqq 0}\{S_n(\omega)\}$.

It is the purpose of this paper to study the problem of the finiteness of N, M, $E(N)$, and $E(M)$ in the case when $E(|X_1|) = \infty$. We obtain results which parallel known results for the case when $E(X_1)$ is finite and negative. In [7] it is shown that if $O > E(X_1) > -\infty$ and if k is a positive integer that $E(M^k) < \infty$ if and only if $E((X_1^+)^{k+1}) < \infty$. This difference of unity between the order of these moments appears as the difference between α and $\alpha+1$ in our results. If $0 > E(X_1) > -\infty$ then it can be deduced from [3] and [6] and a truncation argument that $E((X_1^+)^2) < \infty$ if and only if $E(N) = \sum_{n=0}^{\infty} P(S_n \geqq 0) < \infty$. The fact that $E(N) < \infty$ implies $E((X_1^+)^2) < \infty$ when $E(X_1)$ is finite and negative, can also be obtained from Theorem 7 of [8] by setting $l = 2$. This ratio of 1 to 2 in the order of these moments appears as the ratio of α to 2α in our results and suggests conjectures concerning the existence of higher moments of N and X_1^+. In all that follows, $F(x) = P(X_1 \leqq x)$. All slowly varying functions, L, are assumed to have been defined so that $L(x) = L(-x)$.

PROPOSITION 1. *Assume there exists* $x_0 < 0$, *a constant* α *satisfying* $0 < \alpha < 1$, *and a function L slowly varying at* ∞, *such that for all* $x \leqq x_0$, $L(x)/|x|^\alpha$ *is monotone and* $F(x) \geqq L(x)/|x|^\alpha$. *Then*:

(i) $E((X_1^+)^\alpha / L(X_1^+)) < \infty$ *implies both N and M finite* a.s.;
(ii) If $\alpha \neq \frac{1}{2}$, $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$ *implies* $E(N) < \infty$;
(iii) $E((X_1^+)^{1+\alpha}/L(X_1^+)) < \infty$ *implies* $E(M) < \infty$.

PROPOSITION 2. *Assume there exists* $x_0 < 0$, *a constant* α *satisfying* $0 < \alpha < 1$, *and a function L slowly varying at* ∞, *such that for all* $x \leqq x_0$, $L(x)/|x|^\alpha$ *is monotone and* $F(x) \leqq L(x)/|x|^\alpha$. *Then*:

(i) *Either N or M finite* a.s. (*hence both*) *implies* $E((X_1^+)^\alpha/L(X_1^+)) < \infty$;
(ii) $E(N) < \infty$ *implies* $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$;
(iii) $E(M) < \infty$ *implies* $E((X_1^+)^{\alpha+1}/L(X_1^+)) < \infty$.

REMARK 1. If F and G are probability distribution functions with $F \leqq G$ for all x then by induction $F^{n*} \leqq G^{n*}$ for all x and all n. $L(x)/|x|^\alpha$ which is assumed monotone for $x \leqq x_0 < 0$ can be pieced together with F to form a probability

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distribution function G, satisfying $G \leqq F$ in Proposition 1 and $G \geqq F$ in Proposition 2. Forming the key series

$$\sum_{k=1}^{\infty} k^{-1} P(S_k > 0), \qquad \sum_{k=1}^{\infty} P(S_k > 0), \qquad \text{and}$$
$$\sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} x\, dP(S_k \leqq x) = \sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} P(S_k > x)\, dx$$

first with F and then with G shows that the corresponding series satisfy an inequality opposite to that satisfied by F and G. Hence to prove the previous propositions it is enough to prove the following:

THEOREM. *If for $x < 0$, $F(x) = L(x)/|x|^{\alpha}$ where $0 < \alpha < 1$ and L varies slowly at ∞, then*:

(i) *Either N or M finite* a.s. (*hence both*) *if and only if* $E((X_1^+)^{\alpha}/L(X_1^+)) < \infty$;
(ii) $E(N) < \infty$ *if and only if* $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$ *provided that* $\alpha \neq 1/2$.
(iii) $E(M) < \infty$ *if and only if* $E(X_1^+)^{\alpha+1}/L(X_1^+)) < \infty$.

Since L is positive, (a part of the definition of slow variation) and since $L(x)/|x|^{\alpha}$ is monotone $\int_{\{X_1^+ \leqq b\}} (X_1^+)^{\gamma}/L(X_1^+)\, dP$ is finite for each real b and $\gamma = \alpha$, 2α, and $\alpha + 1$. The conditions in (i), (ii), and (iii) involving X_1^+ and $L(X_1^+)$ are, therefore, only conditions on the behavior of $1 - F$ at $+\infty$. On the other hand, our basic assumption in the theorem is a restriction on the behavior of F only at $-\infty$.

The proof of this theorem, which is found in Section 3, contains the proof of (ii) of Proposition 2 for the case $\alpha = \frac{1}{2}$. The fact that this $\alpha = \frac{1}{2}$ singularity is not removable is made clear in Example 4.1 of Section 4. For this example $F(x) = |x|^{-\frac{1}{2}}$ for $x < -4$, and $E((X_1^+)^{2/2}) < \infty$, but $E(N) = \infty$. Example 4.2 shows it is possible to have both $\lim_{n\to\infty} P(S_n \geqq 0) = 0$ and $P(\limsup_{n\to\infty} S_n = +\infty) = 1$. In fact it shows that for arbitrarily large Γ there exists an example depending on Γ for which both $\lim_{n\to\infty} P(S_n \geqq -n^{\Gamma}) = 0$ and $P(\limsup_{n\to\infty} S_n = +\infty) = 1$.

We mention briefly some application. First, let $H(x) = \sum_{n=0}^{\infty} P[S_n \leqq x)$. In the case when $E(|X_1|) = \infty$, our results applied to the reversed random walk give a solution to the problem, when is the renewal function, $H(x)$, finite for all x, or equivalently, for the random walk, $\{S_n, n \geqq 0\}$, when is the expected time spent in the half-line, $(-\infty, x]$, finite for all x. Second, in the case $F(x) = L(x)/|x|^{\alpha}$ for $x < 0$, $0 < \alpha < 1$, and L slowly varying at ∞, (i) of the theorem gives a necessary and sufficient condition for $P\{\limsup_{n\to\infty} S_n = -\infty\} = 1$. In [2] a condition on $E(\exp\{itX_1\})$ is shown to be necessary and sufficient for this type of strong law. However, interpreting the result of [2] in terms of a condition on F appears difficult. Our result would seem to lead to the conjecture; $P(\limsup_{n\to\infty} S_n = -\infty) = 1$ if and only if $E(1/F(-X_1^+)) < \infty$. Third, if $\tau(\omega)$ is defined by $\tau(\omega) = \min\{n: S_n(\omega) = M(\omega)\}$ then $E(N) = E(\tau)$. To see this, apply the monotone convergence theorem to the conclusion of Problem 7 of Chapter 4 [9]. For queuing theory quantities related to M the reader should look at Problem 11 of Chapter 4 [9].

We make several references to [4] and to [9], even though in most cases, these books are not the primary sources for the results referred to. However, these books do contain a complete bibliography. Actually [9] only discusses the lattice case, but as is pointed out in Problem 13 of Chapter 4 [9], the results we need here are true in the non-lattice case as well. In proving the implications of parts (ii) and (iii) of Proposition 2 we have made use of techniques first developed in [1] and [3].

2. Preliminary notation and lemmas. In the case when $E(X_1{}^+) = \infty$ we define characteristic functions $\varphi^+(t)$ and $\varphi^-(t)$ by

$$\varphi^+(t) = \int_{0^+}^{\infty} e^{itx}\, dF(x) + F(0), \qquad (2.1)$$

$$\varphi^-(t) = \int_{-\infty}^{0^+} e^{itx}\, dF(x) + 1 - F(0).$$

If $E(X_1{}^+) < \infty$ then we define $\varphi^+(t)$ in a way which gives a 0 mean for the probability distribution associated with φ^+. Let $a = \inf\{y : \int_{y^+}^{\infty} x\, dF(x) \geqq 0\}$. $a > -\infty$ because $E|X_1{}^-| = \infty$. If $F(a) > F(a^-)$, choose c, $F(a) - F(a^-) \geqq c \geqq 0$, such that $\int_{a^+}^{\infty} x\, dF(x) + ca = 0$. Then define φ^+ and φ^- by

$$\varphi^+(t) = \int_{a^+}^{\infty} e^{itx}\, dF(x) + C\, e^{ita} + F(a) - c, \qquad (2.2)$$

$$\varphi^-(t) = \int_{-\infty}^{a^-} e^{itx}\, dF(x) + (F(a) - F(a^-) - c)\, e^{ita} + 1 - F(a) + c$$

throughout the rest of this paper, whenever $E(X_1{}^+) < \infty$ is part of what is being assumed, the definition in (2.2) is the one which is to be used. Otherwise (2.1) is to be used. In either case, $\varphi^+(t) + \varphi^-(t) = E(e^{itX_1}) + 1 = \varphi(t) + 1$. F^+ and F^- will always be defined by $\varphi^+(t) = \int e^{itx} dF^+(x)$ and $\varphi^-(t) = \int e^{itx} dF^-(x)$. The two cases are treated separately, because in what follows it will be necessary when $E(X_1{}^+) < \infty$ to write $1 - \varphi^+(t)$ as $\int (1 + itx - e^{itx})\, dF^+(x)$ in order that certain standard estimates can be made. The reader should note that when $E(X_1{}^+) < \infty$, $P(X_1{}^+ \leqq x)$ is not the same as $F^+(x)$. Next define $\tilde{\varphi}(t)$ by

$$\tilde{\varphi}(t) = \varphi^+(-t) + \varphi^-(t) - 1. \qquad (2.3)$$

$\tilde{\varphi}(t)$ is the characteristic function of a probability distribution, $\tilde{F}(x)$, with the property that there exists a finite positive constant, r, for which $\tilde{F}(r) = 1$. Consequently we have:

LEMMA 1. *If $E(|X_1{}^-|) = \infty$, if $\tilde{F}$ is defined by $\int e^{itx} d\tilde{F}(x) = \tilde{\varphi}(t)$ as in* (2.3), *and if $\tilde{N}$ and $\tilde{M}$ are defined for $\tilde{F}$ as in Section* 1 *then $\tilde{N}$ and $\tilde{M}$ are finite* a.s. *and $E(\tilde{N})$ and $E(\tilde{M})$ are both finite.*

PROOF. If $P(\tilde{X}_1 \leqq x) = \tilde{F}(x)$ then it is possible to truncate $\tilde{X}_1$ from below in such a way that the truncated random variable, $\tilde{X}_1{}^t$, has a negative mean. For the truncated process we have $E(\tilde{N}^t) < \infty$ and $E(\tilde{M}^t) < \infty$ and $\tilde{N}^t \geqq \tilde{N}$ a.s. and $\tilde{M}^t \geqq \tilde{M}$ a.s., and this proves the lemma.

REMARK 2. Given any probability distribution, F, there exist probability distributions, $D(x)$ and $U(x)$, with the following properties:

(i) $D(x) \leqq F(x) \leqq U(x)$ for all x;
(ii) $D(x) = F(x) = U(x)$ for all x such that $|x|$ is sufficiently large;
(iii) $D(x)$ and $U(x)$ each have an absolutely continuous component.

It is clear from looking at the graph of F that such D and U exist. This fact when combined with the contents of Remark 1 makes it possible to assume from this point on that F has an absolutely continuous component. Therefore, $\varphi(t)$ and $\tilde{\varphi}(t)$ will always be bounded away from 1 on all sets where $|t|$ is bounded away from 0.

LEMMA 2. *Let $F^+(x)$ be such that $\varphi^+(t) = \int e^{itx}\,dF^+(x)$. Assume that*

$$\int (|x|^\gamma / L(x))\,dF^+(x) < \infty$$

where L is slowly varying and positive, where $L(x)/x^\gamma$ is nonincreasing on $(0, +\infty)$, and where $0 < \gamma < 1$ or $1 < \gamma < 2$. Then $|1-\varphi^+(t)|/[L(1/t)|t|^{\gamma+1}]$ is integrable on every bounded interval, $[-\delta, \delta]$.

PROOF. It is enough to prove the lemma for $\delta = 1$. For $0 < \gamma < 1$ we have

$$\int_0^1 \int_{a^-}^\infty \left[\frac{|1-e^{itx}|\,dF^+(x)}{L(1/t)\,|t|^{\gamma+1}}\right] dt \leqq \int_0^1 \int_{a^-}^1 \left[\frac{|x|\,dF^+(x)}{L(1/t)\,|t|^\gamma}\right] dt$$
$$+ \int_1^\infty \int_0^{1/x} \left[\frac{(dt)x\,dF^+(x)}{L(1/t)t^\gamma}\right] + \int_1^\infty \int_{1/x}^1 \left[\frac{2dt\,dF^+(x)}{L(1/t)t^{\gamma+1}}\right].$$

The first integral on the right in this inequality is finite because $\gamma < 1$ and L is slowly varying. In the second and third integrals let $v = 1/t$. $x\int_x^\infty (v^{\gamma-2}\,dv/L(v))$ and $\int_1^x (v^{\gamma-1}\,dv/L(v))$ are both asymptotic to a constant times $x^\gamma/L(x)$ by Theorem 9.1 of Chapter 8 [4]. This completes the proof for $\gamma < 1$. If $1 < \gamma < 2$, write $1-\varphi^+(t) = \int (1+itx-e^{itx})\,dF^+(x)$. Break up the integral as before, bounding $|1+itx-e^{itx}|$ by a constant times t^2x^2 over the region $0 \leqq t \leqq 1/x$ and by $2tx$ over the region $1/x \leqq t < \infty$.

The next lemma we require appears as 3.23 in [5].

Garsia–Lamperti Lemma. Let $G(x)$ be a probability distribution with $G(0) = 1$ and $G(x) = |x|^{-\alpha}L(x)$ where L is slowly varying and $0 < \alpha < 1$. Then

$$\lim_{t\downarrow 0}\left[\frac{1-\int e^{itx}\,dG(x)}{L(1/t)\,|t|^\alpha}\right] = (i)\int_{-\infty}^0 e^{iy}|y|^{-\alpha}\,dy \quad \textit{and} \quad \lim_{t\uparrow 0}\left[\frac{1-\int e^{itx}\,dG(x)}{L(1/t)\,|t|^\alpha}\right] = \bar{I}$$

where $\bar{I}$ is the conjugate of $(i)\int_{-\infty}^0 e^{iy}|y|^{-\alpha}\,dF(x)$.

LEMMA 3. *Let $F(x) = L(x)/|x|^\alpha$ for $x < 0$ as defined in the statement of the theorem and let $F^+(x)$ be defined by $\varphi^+(t) = \int e^{itx}\,dF^+(x)$. If $\int(|x|^\alpha/L(x))\,dF^+(x) < \infty$ then $\lim_{t\to 0}[1-\varphi^+(t)]/[|t|^\alpha L(1/t)] = 0$.*

PROOF. We consider only $\int_0^\infty (1-e^{itx})\,dF^+(x)$. If $a < 0$ the integral over $[a, 0]$ is cared for by $|1-e^{itx}| \leqq |tx|$. If $M(\alpha) = \int_0^\infty (|x|^\alpha/L(x))\,dF^+(x)$ then for every $y > 0$ and $t \neq 0$

$$M(\alpha) \geqq \int_{y/|t|}^\infty (|x|^\alpha/L(x))\,dF^+(x) \tag{2.4}$$
$$\geqq [(y/|t|)^\alpha/L(y/|t|)][1-F^+(y/|t|)].$$

Combining (2.4) with an integration by parts and the fact that L is slowly varying gives for fixed $\varepsilon > 0$

$$\limsup_{t\to 0}\left|\int_0^{\varepsilon/|t|}\frac{(1-e^{itx})\,dF^+(x)}{|t|^\alpha L(1/t)}\right| \leqq \limsup_{t\to 0}\frac{|1-e^{i\varepsilon t/|t|}|\,L(\varepsilon/t)M(\alpha)}{\varepsilon^\alpha L(1/t)}$$

$$(2.5)\qquad +\limsup_{t\to 0}|t|\int_0^{\varepsilon/|t|}\frac{(1-F^+(x))\,dx}{|t|^\alpha L(1/t)}$$

$$\leqq \varepsilon^{1-\alpha}M(\alpha)+\limsup_{t\to 0}\int_0^{\varepsilon}\frac{M(\alpha)L(y/t)\,dy}{y^\alpha L(1/t)}$$

$$= \varepsilon^{1-\alpha}M(\alpha)(1+1/(1-\alpha)).$$

The justification for taking the limit under the integral can be found in the proof of 3.23, [5]. Choose and fix $\varepsilon > 0$ so that the extreme right side of (2.5) is small. For arbitrarily small $\delta > 0$ we have for all sufficiently small $|t|$, $\delta \geqq \int_{\varepsilon/|t|}^\infty (x^\alpha/L(x))\,dF^+(x)$ and hence

$$\limsup_{t\to 0}\left|\int_{\varepsilon/|t|}^\infty\frac{(1-e^{itx})\,dF^+(x)}{|t|^\alpha L(1/t)}\right| \leqq \limsup_{t\to 0}[\delta L(\varepsilon/t)/(\varepsilon^\alpha L(1/t))] = \delta\varepsilon^{-\alpha}$$

This completes the proof of the lemma.

In the proof of the theorem, approximations to $\int_0^b dP(S_n \leqq x)$ and $\int_0^b x\,dP(S_n \leqq x)$ will be required. They will take the form $\int g_b(x)\,dP(S_n \leqq x)$ and $\int m_b(x)\,dP(S_n \leqq x)$ where g_b and m_b are defined in the following way, $g_b = 1$ on $[0, b]$ and $g_b \equiv 0$ on $(-\infty, -1]\cup[b+1, +\infty)$. g_b goes monotonically to 0 on $[-1, 0]$ and $[b, b+1]$ in such a way that both g_b' and g_b'' are continuous for all x. $m_b(x)$ is most easily defined in terms of $m_b'(x)$.

$$\begin{aligned} m_b'(x) &= 1 && \text{for}\quad 1 \leqq x \leqq b\\ &= -(3b/8)(x-b)^2+1 && \text{for}\quad b \leqq x \leqq b+1\\ &= (3b/8)(x-b-2)^2+1-3b/4 && \text{for}\quad b+1 \leqq x \leqq b+3\\ &= -(3b/8)(x-b-4)^2+1 && \text{for}\quad b+3 \leqq x \leqq b+4\\ &= 0 \quad \text{for}\quad x \leqq \tfrac{1}{2} \quad \text{and} \quad x \geqq b+5. \end{aligned}$$

m_b' goes continuously from 0 to 1 (respectively 1 to 0) on $[\frac{1}{2}, 1]$ and $[b+4, b+5]$ in such a way that m_b'' is continuous for all x and in such a way that $\int_{\frac{1}{2}}^1 m_b'(x)\,dx = 1$, $\int_{b+4}^{b+y} m_b'(x)\,dx \geqq -4$ for all $4 \leqq y \leqq 5$, and $\int_{b+4}^{b+5} m_b'(x)\,dx = -4$. Let $m_b(x) = \int_{-\infty}^x m_b'(y)\,dy$. Then $\int_0^b dP(S_n \leqq x) \leqq \int g_b(x)\,dP(S_n \leqq x)$ and $\int_1^b x\,dP(S_n \leqq x) \leqq \int m_b(x)\,dP(S_n \leqq x)$. The m_b approximation does not necessarily overestimate if we include the integral from 0 to 1. However, in all that follows $\sum_{n=1}^\infty P(S_n \in [0, 1]) < \infty$ so that it does not matter.

Next let $\gamma_b(t) = \int e^{itx}g_b(x)\,dx$, $\mu_b(t) = \int e^{itx}m_b(x)\,dx$, and $\bar{\mu}_b(t) = \int [e^{itx}m_b(x)/(ix)]\,dx$. Integrating by parts once gives:

There exists $\bar{C}$ independent of b such that for all t,

$$(2.6)\qquad |\gamma_b(-t)| \leqq \bar{C}|t|^{-1} \quad \text{and} \quad |\bar{\mu}_b(-t)| \leqq \bar{C}|t|^{-1}.$$

Two integrations by parts gives:

There exists $\underline{C}$ independent of b such that for all t,

(2.7) $|\gamma_b(-t)| \leqq \underline{C}|t|^{-2}$, $\quad |\bar{\mu}_b(-t)| \leqq \underline{C}|t|^{-2}$, and $|\mu_b(-t)| \leqq \underline{C}b|t|^{-2}$.

If $\{V_k\}$ is a sequence of independent identically distributed random variables with $\int e^{itx}\,dP(V_k \leqq x) = \varphi^-(t)$ and if $\{B(n)\}$ is chosen so that $\lim_{n\to\infty}(nF(-B(n))) = 1$, then $(B(n))^{-1}\sum_{k=1}^n V_k$ converges in distribution to a non-degenerate stable law supported on $(-\infty, 0]$ and with index α. (See Section 8, Chapter 9 of [4]). From the Karamata characterization of slowly varying functions, L, (See the corollary in Section 9, Chapter 8, [4]), we have $L(x) = A(x)\exp\{\int_0^x(\varepsilon(y)/y)\,dy\}$ where $\varepsilon(x)\to 0$ and $A(x)\to C$, $0 < C < \infty$, as $x \to +\infty$. Let $W(x) = C|x|^{-\alpha}\exp\{\int_0^{|x|}(\varepsilon(y)/y)\,dy\}$; then for all sufficiently large $|x|$, $W(x)$ is strictly monotone. In the following pages $B(n)$ will always be chosen by the formula $B(n) = W^{-1}(1/n)$ where the positive branch of W^{-1} is to be used.

LEMMA 4. *If X is a random variable and A any positive constant then*:

(2.8) $\sum_{n=1}^\infty nP(X \geqq AB(n)) < \infty$ *implies* $E((X^+)^{2\alpha}/L^2(X^+)) < \infty$;

(2.9) $\sum_{n=1}^\infty B(n)P(X \geqq AB(n)) < \infty$ *implies* $E((X^+)^{1+\alpha}/L(X^+)) < \infty$.

PROOF. For all sufficiently large n, $P(X \geqq AB(n)) = P(W(X^+/A) \leqq 1/n) = P(1/W(X^+/A) \geqq n)$. Therefore

$$\sum_{n=1}^\infty nP(X \geqq AB(n)) < \infty \quad \text{implies} \quad E((1/W(X^+/A))^2) < \infty$$

which proves (2.8). From Lemma 2-A, [11], $B(x) = W^{-1}(1/x) = x^{1/\alpha}J(x)$ where J is slowly varying at $+\infty$. From Theorem 9.1, Chapter 8, [4], $f(x) = \int_1^x y^{1/\alpha}J(y)\,dy$ is asymptotic to $\alpha(1+\alpha)^{-1}x^{1+1/\alpha}J(x)$. Hence $\sum_{n=1}^\infty B(n)P(X \geqq AB(n)) < \infty$ implies $\int_1^\infty x^{1/\alpha}J(x)P(1/W(X^+/A) \geqq x)\,dx < \infty$ which implies $\int_1^\infty x^{1+1/\alpha}J(x)\,dP(1/W(X^+/A) \leqq x) < \infty$ which finally implies $E(W^{-1}(W(X^+/A))/W(X^+/A)) < \infty$. This proves (2.9).

3. Proof of the theorem. From P19.2, [9], (i) is equivalent to the statement that $\sum_{k=1}^\infty k^{-1}P(S_k > 0) < \infty$ if and only if $E((X_1{}^+)^\alpha/L(X_1{}^+)) < \infty$. We first assume that the moment is finite. The probability distribution associated with $\tilde{\varphi}$ has support $(-\infty, 0]$ and hence we clearly have

$$\begin{aligned}\lim_{b\to+\infty}\sum_{k=1}^\infty k^{-1}\int g_b(x)\,dP(\tilde{S}_k \leqq x) &= \lim_{b\to+\infty}\lim_{r\uparrow 1}\sum_{k=1}^\infty(2\pi)^{-1}\int \gamma_b(-t)r^k\tilde{\varphi}^k(t)k^{-1}\,dt\\ &= \lim_{b\to+\infty}\lim_{r\uparrow 1}(2\pi)^{-1}\int[-\gamma_b(-t)\ln(1-r\tilde{\varphi}(t))]\,dt\\ &= \lim_{b\to+\infty}(2\pi)^{-1}\int[-\gamma_b(-t)\ln(1-\tilde{\varphi}(t))]\,dt < \infty.\end{aligned}$$

In the above expression, and in the remainder of this paper, to justify taking the limit on r under the integral, write

$$|\ln(1-r\varphi(t))| = |1-r\varphi(t)|\,|\ln(1-r\varphi(t))|\,|1-r\varphi(t)|^{-1}.$$

$y \ln y$ is bounded near 0;

$$|1-r\varphi(t)|^{-1} \leqq 1/\mathscr{R}(1-\varphi^-(t)); \qquad \mathscr{R}(1-\varphi^-(t)) \sim \mathscr{R}(I)L(1/t)|t|^\alpha$$

by the Garsia–Lamperti lemma; $|\gamma_b(-t)| \leqq C|t|^{-2}$ by (2.7); $\alpha < 1$ and L is slowly varying. Hence the dominated convergence theorem is applicable. Therefore, to show that the series converges, we need only show that

$$\lim_{b\to+\infty}|(2\pi)^{-1}\textstyle\int\gamma_b(-t)[\ln(1-\tilde{\varphi}(t))-\ln(1-\varphi(t))]\,dt| < \infty.$$

From Lemma 3, the identity $1-\varphi(t) = 1-\varphi^+(t)+1-\varphi^-(t)$, and the Garsia–Lamperti lemma we have that there exist positive constants, C and h, such that

$$\left|\int_{-h}^{h}\gamma_b(-t)\ln\left(\frac{1-\tilde{\varphi}(t)}{1-\varphi(t)}\right)dt\right| \leqq \left|\int_{-h}^{h}\gamma_b(-t)\ln\left(1+\frac{2\mathscr{I}(\varphi^+(t))}{1-\varphi(t)}\right)dt\right|$$
$$\leqq c\textstyle\int_{-h}^{h}|\gamma_b(-t)||\mathscr{I}(\varphi^+(t))||1-\varphi(t)|^{-1}\,dt.$$

The integrability of $|\mathscr{I}(\varphi^+(t))|/[L(1/t)|t|^{1+\alpha}]$ on $[-h, h]$ which is guaranteed by Lemma 2 together with the estimate $|\gamma_b(t)| \leqq \bar{C}|t|^{-1}$ of (2.6) then gives

$$\limsup_{n\to+\infty}\left|\int_{-h}^{h}\gamma_b(-t)\ln\left(\frac{1-\tilde{\varphi}(t)}{1-\varphi(t)}\right)dt\right| < \infty.$$

The estimate $|\gamma_b(-t)| \leqq \underline{C}|t|^{-2}$ of (2.7) takes care of the integration on $\{t: |t| \geqq h\}$ and completes the proof of the fact that the existence of the moment implies the convergence of the series.

Next assume $\sum_{k=1}^{\infty} k^{-1}P(S_k > 0) < \infty$. Then

$$\lim_{b\to\infty}(2\pi)^{-1}\textstyle\int[-\gamma_b(-t)\ln(1-\varphi(t))]\,dt < \infty.$$

We can write

$$\ln(1-\varphi^+(t)\varphi^-(t))-\ln(1-\varphi(t)) = \ln(1-(1-\varphi^+(t))(1-\varphi^-(t))/(1-\varphi(t)))$$

so that if it can be shown that

$$\limsup_{b\to\infty}\left|\int\gamma_b(-t)\ln\left(1-\frac{(1-\varphi^+(t))(1-\varphi^-(t))}{1-\varphi(t)}\right)dt\right| < \infty \tag{3.1}$$

then it will be possible to conclude that

$$\textstyle\sum_{k=1}^{\infty} k^{-1}P(\sum_{h=1}^{k}(U_n+V_n) > 0) < \infty \tag{3.2}$$

where $U_1, V_1, U_2, V_2, \cdots$ are mutually independent with $E(e^{itU_n}) = \varphi^+(t)$ and $E(e^{itV_n}) = \varphi^-(t)$ for all n. The fact that $|(1-\varphi^+(t))(1-\varphi^-(t))/(1-\varphi(t))| \leqq |(1-\varphi^+(t))(1-\varphi^-(t))/(\mathscr{R}(1-\varphi^-(t)))|$ means all we need actually look at is

$$\limsup_{b\to\infty}\textstyle\int_{-h}^{h}|\gamma_b(-t)||(1-\varphi^+(t))(1-\varphi^-(t))/(1-\varphi(t))|\,dt$$

where $h > 0$ is fixed. Before attacking this last expression we need to establish that

$E((X_1^+)^{\alpha/2}) < \infty$. We have $E(|X_1^-|^{\alpha/2}) < \infty$ as a direct consequence of our main assumption. Hence from Theorem 1 of [1] for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} k^{-1} P(\sum_{n=1}^{k} X_n^- > \varepsilon k^{2/\alpha}) < \infty.$$

Then
$$\sum_{k=1}^{\infty} k^{-1} P(|S_k| > \varepsilon k^{2/\alpha}) \leqq \sum_{k=1}^{\infty} k^{-1}[P(S_k > 0) + P(\sum_{n=1}^{k} X_n^- > \varepsilon k^{2/\alpha}/2)] < \infty,$$

and again applying Theorem 1 of [1] we get that $E(|X_1|^{\alpha/2}) < \infty$. In particular, Lemma 2 is now applicable and hence $|1-\varphi^+(t)|/|t|^{1+\alpha/2} \in \mathscr{L}_1(-h, h)$ and we have (3.1). From (3.2),

$$\infty > \sum_{k=1}^{\infty} k^{-1} P(\sum_{n=1}^{k} (U_n + V_n) > 0) \geqq \sum_{k=1}^{\infty} k^{-1} P(\sum_{n=1}^{k} V_n > -AB(k)) P(\sum_{k=1}^{n} U_n > AB(k)) \tag{3.3}$$

where A is to be determined. The distribution of $(B(k))^{-1}\sum_{n=1}^{k} V_n$ converges to a stable law of index α and hence $A > 0$ can be chosen so that $P(\sum_{n=1}^{k} V_n > -AB(k)) \geqq \frac{1}{2}$ for all k. Then (3.3) implies that

$$\sum_{k=1}^{\infty} k^{-1} P(\sum_{n=1}^{k} U_n > AB(k)) < \infty. \tag{3.4}$$

The fact that $B(k)$ is regularly varying with exponent $1/\alpha$ allows us to use the techniques of Theorem 1 of [1]. It must be the case that $\lim_{k\to\infty} P(|\sum_{n=1}^{k} U_n^s| > AB(k)) = 0$ and hence that $\lim_{k\to\infty} kP(|U_1^s| > AB(k)) = 0$ where U_n^s is the symmetrized random variable obtained from U_n. This is enough to get from (3.4) that $\sum_{k=1}^{\infty} P(|U_1^s| > AB(k)) < \infty$ which implies $E((|U_1^s|)^\alpha / L(U_1^s)) < \infty$. Then $E(U_1^\alpha / L(U_1)) < \infty$ and the proof of (i) is complete.

Next we turn to (ii) of the theorem. (ii) is equivalent to the statement that $\sum_{k=0}^{\infty} P(S_k \geqq 0) < \infty$ if and only if $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$ where $\alpha \neq \frac{1}{2}$. First we assume that the series converges. Then

$$\infty > \sum_{k=1}^{\infty} P(S_k \geqq 0) \geqq \sum_{k=1}^{\infty} \sum_{n=1}^{k} P(X_n \geqq AB(k), \quad X_j < AB(k) \quad \text{for all } j \neq n, \ j \leqq k,$$
$$\sum_{j=1, j\neq n}^{k} X_j \geqq -AB(k) \geqq \sum_{k=1}^{\infty} kP(X_1 \geqq AB(k)) \cdot [P(S_{k-1} \geqq -AB(k)) - kP(X_1 \geqq AB(k))].$$

From (i) we have $E((X_1^+)^\alpha / L(X_1^+)) < \infty$ and hence $\lim_{k\to\infty} kP(X_1 > AB(k)) = 0$. By an argument like that used in the proof of (i), for any fixed large A, $P(S_{k-1} \geqq -AB(k))$ is bounded away from 0 uniformly in k. Therefore $E(N) < \infty$ implies $\sum_{k=1}^{\infty} kP(X_1 \geqq AB(k)) < \infty$, which in turn implies $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$ by Lemma 4. The above argument is valid when $\alpha = \frac{1}{2}$, so that (ii) of Proposition 2 has now also been proven.

Now assume $E((X_1^+)^{2\alpha}/L^2(X_1^+)) < \infty$. From an argument like that used in (i) with $0 < r < 1$ and $r \uparrow 1$ we find that we will have proven (ii) if we can show that

$$\lim_{b\to\infty} (2\pi)^{-1} \int \gamma_b(-t)(1-\varphi(t))^{-1}\, dt < \infty. \tag{3.5}$$

However (3.5) does hold if we replace $\varphi(t)$ by $\tilde{\varphi}(t)$, so that to establish (3.5) it is sufficient to show

$$\limsup_{b\to\infty} \left|\int \gamma_b(-t)[(1-\varphi(t))^{-1}-(1-\tilde{\varphi}(t))^{-1}]\,dt\right| < \infty. \tag{3.6}$$

From Lemma 3, (2.6), the Garsia–Lamperti lemma, and the identities $1-\varphi(t) = 1-\varphi^+(t)+1-\varphi^-(t)$ and $1-\tilde{\varphi}(t) = 1-\varphi^+(-t)+1-\varphi^-(t)$ we get that for some small $h > 0$, $|\gamma_b(-t)||(1-\varphi(t))^{-1}-(1-\tilde{\varphi}(t))^{-1}| \leqq 4\bar{C}|\mathscr{I}(\varphi^+(t))|/[|I|^2L^2(1/t)|t|^{2\alpha+1}$ whenever $|t| \leqq h$. Lemma 2 with (2.7) then gives (3.6) provided $2\alpha \neq 1$. This gives (3.5) and completes the proof of (ii).

In (iii) we begin by assuming $E(M) < \infty$. From part B of P19.2 of [9] we then have $\sum_{k=1}^{\infty} k^{-1}\int_0^{\infty} x\,dP(S_k \leqq x) < \infty$. Therefore

$$\begin{aligned}\infty > \sum_{k=1}^{\infty} k^{-1}AB(k)P(S_k \geqq AB(k)) \geqq \sum_{k=1}^{\infty} Ak^{-1}B(k)\sum_{n=1}^{k} P(X_n \geqq 2AB(k),\\ X_j < 2AB(k) \qquad \text{for all } j\neq n,\ j\leqq k,\\ \textstyle\sum_{j=1,\,j\neq n}^{k} X_j \geqq -AB(k)) \geqq \sum_{k=1}^{\infty} B(k)P(X_1 \geqq 2AB(k))\\ \cdot[P(S_{k-1} \geqq -AB(k)) - kP(X_1 \geqq 2AB(k))]\end{aligned}$$

where $A \geqq 1$ is to be determined. $E(M) < \infty$ implies $M < \infty$ a.s. and hence by (i) $E((X_1{}^+)^{\alpha}/L(X_1{}^+)) < \infty$, which in turn implies $\lim_{k\to\infty} kP(X_1 \geqq 2AB(k)) = 0$ for each fixed $A > 0$. An argument involving a limit law then gives $P(S_{k-1} \geqq -AB(k))$ bounded away from 0 uniformly in k for some large fixed $A > 0$. Therefore $L(M) < \infty$ implies $\sum_{k=1}^{\infty} B(k)P(X_1 \geqq 2AB(k)) < \infty$ and this together with Lemma 4 yields $E((X_1{}^+)^{\alpha+1}/L(X_1{}^+)) < \infty$.

Finally we assume that $E((X_1{}^+)^{\alpha+1}/L(X_1{}^+)) < \infty$. To show $\sum_{k=1}^{\infty} k^{-1}\cdot \int_0^{\infty} x\,dP(S_k \leqq x) < \infty$ it is enough to show that

$$\limsup_{b\to+\infty} \left|\lim_{r\uparrow 1}\sum_{k=1}^{\infty} k^{-1}\int \mu_b(-t)r^k(\varphi^k(t)-\tilde{\varphi}^k(t))\,dt\right| < \infty. \tag{3.7}$$

To accomplish this, an integration by parts is required. We will need that $(d/dt)\varphi^-(t)$ exists for each $t \neq 0$ and that

$$\left|\frac{d}{dt}\varphi^-(t)\right| \leqq \rho|t|^{\alpha-1}L(1/t) \tag{3.8}$$

where ρ is a constant independent of t. Let $d(x) = c|x|^{-(1+\alpha)}\exp\{\int_0^{|x|}(\varepsilon(y)/y)\,dy\}$ where $L(x) = A(|x|)\exp\{\int_0^{|x|}(\varepsilon(y)/y)\,dy\}$ and $\lim_{|x|\to\infty} A(|x|) = C$, $0 < C < \infty$. By Theorem 9.1 of Chapter 8, [4], $\lim_{r\to-\infty}(\int_{-\infty}^{r} d(x)\,dx/F(r) = 1/\alpha$ so as a consequence of Remark 1 we can assume that for all x smaller than some r_0, $F(x)$ has a density, $d(x)$, with the property that $d(x)$ itself is absolutely continuous and monotone. Then

$$\begin{aligned}\left|\int_{-M}^{0} e^{itx}|x|\,dF(x)\right| \leqq \left|\int_{-M}^{0} e^{itx}\int_{-|x|}^{0} dy\,dF(x)\right| \leqq \left|\int_{-|t|^{-1}}^{0} e^{itx}|x|\,dF(x)\right|\\ +\left|\int_{-M}^{-|t|^{-1}} e^{itx}\int_{-|t|^{-1}}^{0} dy\,dF(x)\right| + \left|\int_{-M}^{-|t|^{-1}} e^{itx}\int_{-|x|}^{-|t|^{-1}} dy\,d(x)\,dx\right|.\end{aligned}$$

The first two terms on the extreme right in the above inequality can be easily bounded in the manner required in (3.8). To handle the third term we observe that $\int_{-M}^{y} d'(x)\,dx \int_{x}^{y} e^{itv}\,dv = \int_{-M}^{y} e^{itv} \int_{-M}^{v} d'(x)\,dx\,dv = \int_{-M}^{y} e^{itv}(d(v)-d(M))\,dv$ and hence

$$\left|\int_{-M}^{y} e^{itv} d(v)\,dv\right| \leqq 2\,|t|^{-1} d(M) + 2\,|t|^{-1} \int_{-M}^{y} d'(x)\,dx = 2\,|t|^{-1} d(y).$$

Then

$$\left|\int_{-M}^{-|t|^{-1}} e^{itx} \int_{-|x|}^{-|t|^{-1}} dy\, d(x)\,dx\right| \leqq \int_{-M}^{-|t|^{-1}} \left|\int_{-M}^{y} e^{itx} d(x)\,dx\right| dy \leqq 2\,|t|^{-1} \int_{-M}^{-|t|^{-1}} d(y)\,dy.$$

This last estimate shows that for fixed $\delta > 0$

$$\lim_{M\to\infty} \int_{-M}^{0} e^{itx} x\,dF(x) = \int_{-\infty}^{0} e^{itx} x\,dF(x) \quad \text{uniformly in} \quad t \quad \text{for} \quad |t| \geqq \delta$$

and hence justifies the differentiation under the integral sign as well as proves (3.8). Integrating by parts in (3.7) shows that (3.7) will be satisfied if

$$\limsup_{b\to+\infty} \left|\lim_{r\uparrow 1} \sum_{k=1}^{\infty} \int \bar{\mu}_b(-t) r^k (\varphi^{k-1}(t)\varphi'(t) \dot{-} \tilde{\varphi}^{k-1}(t)\tilde{\varphi}'(t))\,dt\right|$$

$$(3.9) \qquad = \limsup_{b\to+\infty} \left|\lim_{r\uparrow 1} \int \bar{\mu}_b(-t) r(\varphi'(t)(1-r\varphi(t))^{-1} - \tilde{\varphi}'(t)(1-r\tilde{\varphi}(t))^{-1})\,dt\right| < \infty$$

where $\bar{\mu}_b(-t)$ is such that $(d/dt)\bar{\mu}_b = \mu_b$. $\bar{\mu}_b$ is bounded in t for fixed b and

$$|\varphi'(1-r\varphi)^{-1} - \tilde{\varphi}'(1-r\tilde{\varphi})^{-1}| = \left|\frac{2(\mathscr{I}(\varphi^+))' - r(\varphi'\tilde{\varphi} - \tilde{\varphi}'\varphi)}{(1-r\varphi)(1-r\tilde{\varphi})}\right|$$

$$= 2\left|\frac{(1-r)(\mathscr{I}(\varphi^+))' + r[(\mathscr{I}(\varphi^+))'(1-\varphi) + \mathscr{I}(\varphi^+)\varphi']}{(1-r\varphi)(1-r\varphi)}\right|$$

so that we may take the limit on r under the integral. Applying (2.6) and the lemmas of Section 2 together with the estimate of (3.8) yields (3.7) and finishes the proof of the theorem.

4. Examples.

Example 4.1. We define a sequence of independent identically distributed random variables, $\{X_n\}$, by specifying $\varphi^+(t)$ and $\varphi^-(t)$. $2\varphi^-(t) = \int_{-\infty}^{-4} e^{itx}|x|^{-\frac{3}{2}}\,dx + 1$. $2\varphi^+(t) = \int_{e}^{\infty} e^{itx}\,d(1-1/x\ln^2 x) + (1-e^{-1})\exp[-2\,eit/(e-1)] + 1$. Then

$$\mathscr{I}(2\varphi^+(t)) = \int_{-|t|^{-1}}^{|t|^{-1}} (\sin(xt) - xt)\,dF^+(x) - \int_{|t|^{-1}}^{\infty} xt\,d(1-1/x\ln^2 x)\,dx + \int_{|t|^{-1}}^{\infty} \sin(xt)\,d(1-1/x\ln^2 x)$$

$$= h(t) + t/\ln t + O(t/\ln^2 t)$$

where $0 < t < 1$ and $t^{-2}h(t) \in L_1(0, \varepsilon)$ for every $\varepsilon > 0$. Also for $t > 0$,

$$1-\varphi^-(t) = (\tfrac{1}{2})t^{\frac{1}{2}}\left[\int_{-\infty}^{0} (1-e^{ix})\,|x|^{-\frac{3}{2}}\,dx - \int_{-rt}^{0} (1-e^{ix})\,|x|^{-\frac{3}{2}}\,dx\right] = t^{\frac{1}{2}}[K + O(t^{\frac{1}{2}})].$$

Combining the above relations and (3.6) with the fact that for $0 < \varepsilon < 1$

$$\lim_{N\to\infty} \int_0^{\varepsilon} \frac{\cos Nt - 1}{t\ln t}\,dt = +\infty \quad \text{and} \quad \limsup_{N\to\infty} \left|\int_0^{\varepsilon} \frac{\sin Nt}{t\ln t}\,dt\right| < +\infty$$

gives, for this example $E(N) = +\infty$ despite the fact that for $x \leqq -4$,

$$F(x) = |x|^{-\alpha} \quad \text{and} \quad E((X_1^{\ +})^{2\alpha}) < \infty \quad \text{where} \quad \alpha = \tfrac{1}{2}.$$

Example 4.2. Let $\{X_n\}$ be a sequence of independent identically distributed random variables with $P(X_1 \leqq x) = |x|^{-\alpha} \ln |x|$ for all sufficiently negative x and with $P(X_1 \leqq x) = 1 - x^{-\alpha}$ for all sufficiently large x where $0 < \alpha < 1$. $E((X_1^{\ +})^\alpha / L(X_1^{\ +})) = E((X_1^{\ +})^\alpha / \ln (X_1^{\ +})) = +\infty$ so that $M = +\infty$ and $N = +\infty$ a.s. Hence $P(\limsup_{n\to\infty} S_n = +\infty) = 1$. However, $\lim_{n\to\infty} P(S_n \geqq 0) = 0$. To see this let $B(n) = (n/\alpha)^{1/\alpha}(\ln(n))^{1/\alpha}$. $\lim_{x\to+\infty} P(X_1 \leqq -x)/P(|X_1| \geqq x) = 1$ while $\lim_{x\to+\infty} P(X_1 \geqq x)/P(|X_1| \geqq x) = 0$ and therefore by Theorem 1 of [10], $(B(n))^{-1} \sum_{k=1}^n X_k$ converges in distribution to a stable law of index α with support $(-\infty, 0]$. From this comes $\lim_{n\to\infty} P((B(n))^{-1} S_n \geqq 0) = \lim_{n\to\infty} P(S_n \geqq 0) = 0$.

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