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A SIMPLE PROOF OF AN INEQUALITY FOR MULTIVARIATE NORMAL PROBABILITIES OF RECTANGLES[1]

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Suppose $X_1, \cdots, X_k$ are jointly normally distributed random variables with zero means and correlation matrix $R = \{\rho_{ij}\}$. Then intuitively, it is clear that for a fixed set of positive constants, $c_1, \cdots, c_k$, the probability $P[|X_i| \leqq c_i, i = 1, \cdots, k]$ should increase when the correlation coefficients increase in some fashion. A precise result of this nature was obtained by Šidák (1968): if $R(\lambda)$ denotes the correlation matrix with $\rho_{1j}(\lambda) = \rho_{j1}(\lambda) = \lambda\rho_{1j}$ for $j > 1, 0 \leqq \lambda \leqq 1$; $\rho_{ij}(\lambda) = \rho_{ij}$ for all other i, j and P_λ denotes the probability measure corresponding to $R(\lambda)$ then

$$\frac{d}{d\lambda} P_\lambda[|X_i| \leqq c_i] \geqq 0. \tag{1}$$

This result has several interesting applications and as pointed out by Šidák it presents a partial analog to a result by Slepian (1962), obtained in connection with "one sided barrier" problem:

$$\frac{d}{d\rho_{ij}} P[X_l \leqq c_l, l = 1, \cdots, k] \geqq 0 \tag{2}$$

where $c_1, \cdots, c_k$ are arbitrary real numbers. Slepian (1962) showed that (2) is an immediate consequence of the following equation. (Chartres (1963) gave a geometrical proof of (2)). Assuming (without loss of generality, as far as the statements of theorems in the present paper are concerned) that variances of X_i are unity, it may be verified that

$$\frac{d}{d\rho_{ij}} g(\mathbf{x}, R) = \frac{\partial^2}{\partial x_i \partial x_j} g(\mathbf{x}, R) \tag{3}$$

where g is the multivariate normal density of $X_1, \cdots, X_k$.

Unfortunately, Šidák's proof of (1) is very lengthy and from his remarks it seems as if Slepian's method is not workable for this "two sided" version. In the following it will be shown that Slepian's method when applied to a lemma given in this paper, readily gives the desired inequality. This lemma is a simple corollary to an inequality of Anderson (1955) which also served as the key to Šidák's proof.

As observed by Šidák it suffices to prove (1) under the assumption that R is nonsingular, since for the general case one may obtain a sequence of nonsingular matrices approaching the given one and the inequality would still be preserved. Henceforth, R will be assumed to be nonsingular.

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To motivate the basic idea of the proof, notice that using (3) one obtains

$$\frac{d}{d\rho_{12}} P[|X_1| \leqq c_1, |X_2| \leqq c_2] = 2\{g(c_1, c_2, \rho_{12}) - g(-c_1, c_2, \rho_{12})\} \tag{4}$$

$$\geqq 0 (\leqq 0) \qquad \text{for} \quad \rho_{12} \geqq 0 (\leqq 0),$$

which has an interpretation: *the derivative in* (4) *is nonnegative in the direction of* ρ_{12}, *the correlation coefficient between* X_1 *and* X_2.

With this viewpoint, the inequality (1) is same as the proposition: *the derivative of* $P[|X_i| \leqq c_i, i = 1, \cdots, k]$ *with respect to* $\boldsymbol{\rho}_{12}$, *the correlation vector between* X_1 *and* $\mathbf{X}_2 = (X_2, \cdots, X_k)$ *is nonnegative in the direction of* $\boldsymbol{\rho}_{12}$. We will show that this is a consequence of

LEMMA. *Let E be a convex, symmetric* ($\mathbf{x} \in E => -\mathbf{x} \in E$) *set in m dimensional Euclidean space and f be a nonnegative function satisfying* (i) $f(\mathbf{x}) = f(-\mathbf{x})$, (ii) *f is differentiable,* (iii) *f is unimodal*: $K_u = \{\mathbf{x} \mid f(\mathbf{x}) \geqq u\}$ *is convex for every* $u > 0$, (iv) $\int_E f(\mathbf{x})\, d\mathbf{x} < \infty$. *Then, for any arbitrary but fixed vector* $\mathbf{a}$,

$$\int_E \mathbf{a}' \frac{\mathbf{d}}{\mathbf{dx}} f(\mathbf{x}+\mathbf{a})\, \mathbf{dx} \leqq 0$$

where $\mathbf{a}'$ *denotes the transpose of* $\mathbf{a}$.

PROOF. Under the conditions of the lemma, Anderson (1955) has shown that $\int_E f(\mathbf{x}+k\mathbf{a})\, \mathbf{dx}$ is a nonincreasing function of k, where $k > 0$. Since f is differentiable it follows that $\int_E d/(dk) f(\mathbf{x}+k\mathbf{a})\, \mathbf{dx} \leqq 0$, or equivalently $\int_E \sum a_i D_i f(\mathbf{x}+k\mathbf{a})\, \mathbf{dx} \leqq 0$, where D_i is the partial derivative with respect to the ith component. Since the inequality holds for $k > 0$ it holds for $k = 1$ which gives the desired inequality of the lemma.

PROOF OF INEQUALITY (1). As remarked before, the inequality (1) is equivalent to the directional derivative being nonnegative:

$$\boldsymbol{\rho}'_{12} \frac{\mathbf{d}}{\mathbf{d}\boldsymbol{\rho}_{12}} P[|X_i| \leqq c_i, i = 1, \cdots, k] \geqq 0. \tag{5}$$

Since, $P[|X_i| \leqq c_i, i = 1, \cdots, k] = 2\int_0^{c_1} \int_{\mathbf{c}_2} g(x_1, \mathbf{x}_2, R)\, dx_1\, \mathbf{dx}_2$, where $\mathbf{c}_2 = [-c_2, c_2] \times [-c_3, c_3] \times \cdots \times [-c_k, c_k]$, using (3) it follows that

$$\begin{aligned} \boldsymbol{\rho}'_{12} \frac{\mathbf{d}}{\mathbf{d}\boldsymbol{\rho}_{12}} & P[|X_i| \leqq c_i, i = 1, \cdots, k] \\ &= 2\boldsymbol{\rho}'_{12} \int_0^{c_1} \int_{\mathbf{c}_2} \frac{\partial^2}{\partial x_1\, \partial \mathbf{x}_2} g(x_1, \mathbf{x}_2; R)\, dx_1\, \mathbf{dx}_2 \\ &= 2\boldsymbol{\rho}'_{12} \left[\int_{\mathbf{c}_2} \frac{\partial}{\partial \mathbf{x}_2} g(c_1, \mathbf{x}_2; R)\, \mathbf{dx}_2 - \int_{\mathbf{c}_2} \frac{\partial}{\partial \mathbf{x}_2} g(0, \mathbf{x}_2; R)\, \mathbf{dx}_2 \right]. \end{aligned} \tag{6}$$

It is easy to verify that the second term inside the rectangular brackets on the right side of last equality sign in (6) is zero. Further, the first term can be rewritten by using the conditional density function of $\mathbf{x}_2$ given $X_1 = c_1$. Thus

$$\rho'_{12} \frac{\mathbf{d}}{\mathbf{d}\rho_{12}} P[|X_i| \leqq c_i, i = 1, \cdots, k] = b\rho'_{12} \int_{\mathbf{c}_2} \frac{\partial}{\partial \mathbf{x}_2} \varphi(\mathbf{x}_2 - c_1 \rho_{12}; R_1)\, \mathbf{dx}_2, \tag{7}$$

where b is a positive constant, φ is the conditional density of $\mathbf{x}_2$ given x_1 and R_1 is the conditional variance covariance matrix. Since R_1 is nonsingular, φ satisfies the conditions of the lemma and thus the right side of (7) when multiplied by $-c_1$ is seen to be nonpositive. Hence the left side of (7) must be nonnegative.

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