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A CHARACTERIZATION OF CERTAIN INFINITELY DIVISIBLE LAWS

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1. In the theory of infinitely divisible (i.d.) distribution functions (df's), it is well known that a finite df (i.e. a df whose entire mass is concentrated on a finite interval) cannot be i.d. unless it is degenerate. Different proofs of this result have been given, most of them in connection with the investigation of one-sided df's (see [1], [3], [6], [7]). The purpose of the present note is to generalize the above statement, i.e. the following question will be answered: How "close" can a non-degenerate i.d. df F be to a finite df, or more precisely: How rapidly can the "tail" T of F, given by $T(x) = 1-F(x)+F(-x)$, converge to zero as $x \to \infty$ if F is a non-degenerate i.d. df ?

2. THEOREM 1. *If F is* i.d., *and if there exist constants $a > 0$ and $\alpha > 1$ such that $T(x) = 0[\exp(-ax^{1+\alpha})]$ as $x \to \infty$, then F is degenerate.*

If F is finite, the above hypothesis holds for any positive α; Theorem 1 therefore generalizes the result mentioned in **1**.

THEOREM 2. *If F is* i.d., *non-degenerate, and if there exist constants $a > 0$ and $\alpha(0 < \alpha \leqq 1)$ such that $T(x) = 0[\exp(-ax^{1+\alpha})]$ as $x \to \infty$, then F is normal.*

PROOF OF THEOREMS 1 AND 2. By Theorem 7.2.4. ([4] page 142), the characteristic function (ch.f.) f of F is an entire function of finite order $\rho_f \leqq 1+\alpha^{-1}$. Since F is i.d., f has no zeros ([4] page 187), and therefore $f(z) = \exp(g(z))$, where g denotes the principal determination of $\log f$, vanishing at $z = 0$.

By the definition of ρ_f, we have for every positive ε

$$\max_{|z|=r} \mathscr{R} g(z) = \max_{|z|=r} \log |f(z)|$$
$$= \log \max_{|z|=r} |f(z)| \leqq r^{\rho_f+\varepsilon}$$

for all sufficiently large r, hence by Theorem 1.3.4. ([2] page 3), g is a polynomial and its degree is equal to ρ_f. But a classical result due to Marcinkiewicz ([4] page 147) states that the only ch.f.'s which have the form $\exp(g(z))$, g being a polynomial, are either $\exp(-az^2+ibz)$ (normal law) or $\exp(ibz)$ (degenerate law) with respective orders of 2, 1 or 0, and since $\rho_f \leqq 1+\alpha^{-1}$, the assertions of Theorems 1 and 2 follow immediately.

COROLLARY 1. *The only* i.d. ch.f.'*s which are entire functions of finite order are the normal and the degenerate* ch.f.

3. By using a different and slightly more involved method of proof, the hypothesis of Theorem 2 can be weakened in the following way.

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THEOREM 3. *If F is* i.d., *non-degenerate, and if there exist constants $a > 0$ and $\delta > 1$ such that $T(x) = 0[\exp(-ax(\log x)^{\delta})]$ as $x \to \infty$, then F is normal.*

If f is an entire function, its order ρ_f and, in case ρ_f is positive and finite, its type τ_f are used to characterize its rate of growth ([2] page 8). For an entire function f of infinite order, we will require the following concept of "form" λ_f, which gives a more precise description of the rate of growth of f:

$$\lambda_f = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

$(0 \leqq \lambda_f \leqq \infty)$. Here $M(r,f)$ denotes as usual the maximum of $|f(z)|$ for $|z| = r$.
We first state and prove the following

LEMMA. *Let $f(z)$ be an entire function which has no zeros, and let $g(z) = \log f(z)$. Then, for any positive γ, we have $\rho_g = \gamma$ iff $\rho_f = \infty$ and $\lambda_f = \gamma$.*

PROOF OF LEMMA. If $\rho_g > 0$, then $\rho_f = \infty$, because if f were of finite order, it would follow as in the proof of Theorems 1 and 2 that g is a polynomial, i.e. $\rho_g = 0$.
If $\rho_f = \infty$, then we have by the definition of λ_f for every positive ε

$$\begin{aligned}\max_{|z|=r}\mathscr{R}g(z) &= \max_{|z|=r}\log|f(z)| \\ &= \log\max_{|z|=r}|f(z)| \leqq \exp[r^{\lambda_f+\varepsilon}]\end{aligned}$$

for all sufficiently large r. But since (for $|z| \leqq r$) $\mathscr{R}g(z) \leqq \max_{|z|=r}\mathscr{R}g(z)$ (maximum principle for harmonic functions), Carathéodory's inequality ([2] page 2) can be used to obtain

$$\max_{|z|=r}|g(z)| \leqq \frac{2r}{R-r}\max_{|z|=R}\mathscr{R}g(z) \qquad (0 < r < R).$$

It follows that for $2r = R$ sufficiently large

$$\begin{aligned}\max_{|z|=r}|g(z)| &\leqq 2\max_{|z|=2r}\mathscr{R}g(z) \\ &\leqq 2\exp[(2r)^{\lambda_f+\varepsilon}],\end{aligned}$$

and therefore $\rho_g \leqq \lambda_f+\varepsilon$ for every positive ε, i.e. $\rho_g \leqq \lambda_f$.
On the other hand, we have

$$\begin{aligned}\max_{|z|=r}|g(z)| &\geqq \max_{|z|=r}\mathscr{R}g(z) \\ &= \log\max_{|z|=r}|f(z)|\end{aligned}$$

and therefore $M(r, g) \geqq \log M(r,f)$, which implies that $\rho_g \geqq \lambda_f$, i.e. $\rho_g = \lambda_f$, thereby completing the proof of the lemma.

PROOF OF THEOREM 3. It follows from the hypothesis of Theorem 3 and from Lemma 9.1. ([5] page 1252) that F has an entire ch.f. f either of finite order or of infinite order and form $\lambda_f \leqq \delta^{-1}$, i.e. $\lambda_f < 1$ since $\delta > 1$. Since F is non-degenerate, it follows from Corollary 1 that either f is normal or $\rho_f = \infty$ and $\lambda_f < 1$. We will show that the second possibility cannot occur.

Let us therefore suppose that $\rho_f = \infty$. By Theorem 8.4.2. ([4] page 189), the Kolmogorov canonical representation

$$\log f(z) = g(z) = imz + \int_{-\infty}^{+\infty} (e^{izu} - 1 - izu)u^{-2}\, dK(u)$$

is valid in the whole complex plane. Here m is a real constant, K a non-decreasing bounded function, and the integrand is defined at $u = 0$ by continuity to be equal to $-\frac{1}{2}z^2$. As it follows from the proof of Theorem 8.4.2., we can interchange differentiation and integration, so that

$$g''(z) = -\int_{-\infty}^{+\infty} e^{izu}\, dK(u)$$

holds in the whole complex plane. Since f is non-degenerate, K cannot vanish identically, and since f is non-normal, K cannot concentrate its total mass at $u = 0$. It follows that the entire function g'' is (up to a constant factor) a non-constant ch.f., and therefore its order $\rho_{g''}$ is at least equal to one (Theorem 7.1.3. [4] page 135). But since differentiation does not change the order of an entire function ([2] page 13), we have $\rho_g \geqq 1$, and the Lemma implies that $\lambda_f \geqq 1$, which contradicts the above inequality for λ_f.

COROLLARY 2. *There exist no* i.d. ch.f.'*s of infinite order and of form less than one.*

4. Corollaries 1 and 2 imply that the only entire i.d. ch.f.'s f whose rate of growth is smaller than the one determined by $\rho_f = \infty$ and $\lambda_f = 1$ (the Poisson law gives an example of such a ch.f.) are the normal and the degenerate ch.f.'s. The situation becomes completely different if we consider larger rates of growth.

THEOREM 4. *For every $\lambda \geqq 1$, there exist (infinitely many)* i.d. df'*s F whose* ch.f.'*s f are entire functions of infinite order and of form λ.*

PROOF. Let K be a df whose ch.f. k is an entire function of order $\lambda \geqq 1$. (This is certainly possible because of Theorems 2.2.5 and 6.1 in [5].) It is then easy to verify that

$$g(z) = \int_{-\infty}^{+\infty} (e^{izu} - 1 - izu)u^{-2}\, dK(u)$$

also represents an entire function whose order is, by the same reasoning as above, equal to λ, and it follows from the lemma that $f(z) = \exp(g(z))$ is an entire i.d. ch.f. of infinite order and form λ.

REFERENCES

[1] BAXTER, G. and SHAPIRO, J. M. (1960). On bounded infinitely divisible random variables. *Sankhyā* **22** 253–260.

[2] BOAS, R. P. (1954). Entire Functions. Academic Press, New York.

[3] CHATTERJEE, S. D. and PAKSHIRAJAN, R. P. (1956). On the unboundedness of infinitely divisible laws. *Sankhyā* **17** 349–350.

[4] LUKACS, E. (1960). *Characteristic Functions*. Griffin, London.

[5] RAMACHANDRAN, B. (1962). On the order and the type of entire characteristic functions. *Ann. Math. Statist.* **33** 1238–1255.

[6] RAMACHANDRAN, B. (1966). On one-sided distribution functions. *Sankhyā* **28** 315–318.

[7] TUCKER, H. G. (1961). Best one-sided bounds for infinitely divisible random variables. *Sankhyā* **23** 387–396.