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NOTE ON THE UNIFORM CONVERGENCE OF DENSITY ESTIMATES

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent identically distributed random variables having a common distribution function F and let $f_n(x) = (na_n)^{-1} \sum_{i=1}^n k((x-X_i)/a_n)$ where $\{a_n\}$ is a sequence of positive numbers converging to zero and k is a probability density function.

If $\sum_{n=1}^\infty \exp(-cna_n^2)$ is finite for all positive c and if k satisfies:

(i) k is continuous and of bounded variation on $(-\infty, \infty)$.
(ii) $uk(u) \to 0$ as $u \to +\infty$ or $-\infty$.
(iii) There exists a δ in (0, 1) such that $u(V_{-\infty}^{-u^\delta}(k) + V_{u^\delta}^{\infty}(k)) \to 0$ as $u \to \infty$.
(iv) $\int |u|\, dk(u)$, the integral of $|u|$ with respect to the signed measure determined by k, is finite.

Then the author [2] has established the following:

THEOREM. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$$

with probability one for a function g is that g be the uniformly continuous derivative of F.

The purpose of this note is to show that this theorem remains true if conditions (i)–(iv) on k are replaced by the condition that k is of bounded variation on $(-\infty, \infty)$.

2. Proof of theorem. The sufficiency of the condition has been established by Nadaraya [1].

Conversely, we can establish Lemmas 3.1, 3.2, 3.3 of [2] assuming only that k is of bounded variation on $(-\infty, \infty)$. We first note that the proofs of Lemmas 3.1 and 3.2 in [2] only use the fact that k is of bounded variation. Next, Lemma 3.8 can be established by observing that the series of inequalities presented in its proof (in [2]) are valid here also and lead to a contradiction of Lemma 3.2. Upon integration by parts, we see that:

$$Ef_n(x) = -\int (a_n)^{-1} F(x - a_n u)\, dk(u) \tag{1}$$

so that the proof of Lemma 3.3 can be completed by a "3-epsilon" proof utilizing (1) above, the uniform continuity of F (implied by Lemma 3.8), and Lemma 3.2.

Let y be an arbitrary but fixed point and let (a, b) be an open interval containing y. If g is such that $\sup_x |f_n(x) - g(x)| \to 0$ with probability one then Lemma 3.2 tells us that $\lim_{n\to\infty} \sup_x |Ef_n(x) - g(x)| = 0$ so that $\lim_{n\to\infty} \int_a^y Ef_n(x)\, dx = \int_a^y g(x)\, dx$. By

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interchanging the order of integration and using Lebesgue's dominated convergence theorem we see that $\lim_{n\to\infty} \int_a^y Ef_n(x)\,dx = F(y)-F(a)$ so that $F(y)-F(a) = \int_a^y g(x)\,dx$.

Since g is uniformly continuous (Lemma 3.3) the (Lebesgue) $\int_a^y g(x)\,dx =$ (Riemann) $\int_a^y g(x)\,dx$ and hence $F'(y) = g(y)$ by the fundamental theorem of calculus for Riemann integrals. Since y was arbitrary the necessity of the condition has been established.

REFERENCES

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