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INADMISSIBILITY OF VARIOUS "GOOD" STATISTICAL PROCEDURES WHICH ARE TRANSLATION INVARIANT[1]

BY S. K. PERNG

Kansas State University

1. Summary and introduction. The purpose of this paper is to try to show that certain moment conditions are essential for the admissibility of various "good" translation invariant statistical procedures. In 1951 Blackwell [1] first gave an example in which he proved that a best invariant estimate may be inadmissible. Since then many papers dealing with the admissibility of the best invariant procedures have been published.

The problem of admissibility in the case of point estimation of a location parameter was treated by Blyth [2], Blackwell [1], Stein [9], [10], [11], Fox and Rubin [6], Farrell [4], Brown [3]. The problem of admissibility of certain confidence intervals was treated by Joshi [7]. The problem of admissibility in the case of a best invariant test involving a location parameter was treated by Lehmann and Stein [8]. In each paper cited above, admissibility requires the existence of one more moment than what is needed for finite risk. The first two examples of this paper indicate that, without this extra moment, inadmissibility may result.

In Section 3, we show, by example, a unique best translation invariant estimate may be inadmissible if a certain moment condition fails to be satisfied. In Section 4 we prove a theorem which gives a set of sufficient conditions for the admissibility of certain translation invariant confidence interval procedures and we also give an example which shows that a certain translation invariant confidence interval procedure may be inadmissible if a certain moment condition fails to hold. In Section 5 we show by example a best translation invariant test may be inadmissible if the test is non-unique.

2. Notation and assumptions. In this section we shall state the notation and assumptions which we will use later.

Let $\mathscr{B}$ be the σ-field of all Borel subsets of the real line $\mathscr{X}$ and $\mathscr{C}$ be a σ-field of subsets of a set $\mathscr{Y}$. Consider the following estimation problem. Let G be a probability measure on $\mathscr{C}$ and F be a $\mathscr{B}\times\mathscr{C}$ measurable function on $\mathscr{X}\times\mathscr{Y}$ such that $F(\cdot-\theta, y)$ is a distribution function for each $y\in\mathscr{Y}$ where θ is an unknown real-valued parameter. We observe (X, Y) and try to estimate θ with loss $L(\theta,\hat\theta) = W(\theta-\hat\theta)$ where $\hat\theta$ is the estimate of θ and W is a fixed, nonnegative, Borel function from E_1 to $E_1{}^*$ (where E_1 is one dimension Euclidean space and $E_1{}^* = E_1\cup\{\infty\}$). A non-randomized estimate of θ is of the form $\delta(x, y) = x+\gamma(y)$.

Secondly, consider the confidence interval problem. We assume that X, Y, have joint distribution given by $H(dx, dy) = f(x-\theta\,|\,y)\,G(dy)$ where $f(x-\theta\,|\,y)$ is the

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conditional density function of x given y, and G be a probability measure on $\mathscr{C}$. We observe (X, Y) and try to set up a confidence interval for θ. Then every non-randomized invariant confidence interval has the form $[x+a(y), x+b(y)]$ where a, b are real-valued functions of y and $a(y) \leqq b(y)$.

Finally, for each $i = 1, 2$ and $y \in \mathscr{Y}$ let $F_i(\cdot, y)$ be distribution functions on X such that $F_i(\cdot, \cdot)$ is $\mathscr{B} \times \mathscr{C}$ measurable. For every θ and for $i = 1, 2$ let the joint distribution of $(X, Y) \in \mathscr{X} \times \mathscr{Y}$ be given by usual extension of $P_{i\theta}((X, Y) \in C \times D) = \int_D dG_i(y) \int_C F_i(dx-\theta, y)$ to measurable subsets of $\mathscr{X} \times \mathscr{Y}$. Now consider the problem of testing $H_1 : i = 1$ versus $H_2 : i = 2$. If we apply the invariance principle to the testing hypothesis problem then the maximal invariant statistic is Y and, hence, a test is invariant if and only if it depends on y only.

3. Inadmissibility of the best invariant estimate of a location parameter. We shall follow the notation and assumptions stated in Section 2. Brown (Theorem 2.1.1 in [3]) presented a set of sufficient conditions for the admissibility of the best invariant estimate of a real location parameter in the sequential case under quite general assumptions on the loss function. Now let us consider the fixed sample size case with loss function of the form $W(x) = |x|^k$ $k \geqq 1$. By applying Brown's theorem just mentioned, a unique best invariant estimate is admissible if the following moment condition is satisfied.

$$E|X|^\alpha W(X) < \infty \qquad \text{for } \alpha = 1. \tag{3.1}$$

It is interesting to see whether this is the weakest moment condition we can have. Brown (Theorem 2.4.1 in [3]) gave a partial answer to this question by giving an example. He gave a probability density function such that (3.1) is valid for $0 < \alpha < k/(2^k - 1)$, yet the unique best invariant estimate is inadmissible. Is a unique best invariant estimate admissible if the moment condition (3.1) is satisfied for $k/(2^k - 1) \leqq \alpha < 1$? We answer this question by the following theorem.

THEOREM 3.1. *In the fixed-sample size case, if the loss function is $W(t) = |t|^k$ for $k > 1$ then for every $\alpha (0 \leqq \alpha < 1)$ there exists a family of probability densities such that $E|X|^\alpha W(X) < \infty$ and the best invariant estimate of the real location parameter is unique but it is inadmissible.*

PROOF. Let θ be an unknown real parameter $-\infty < \theta < \infty$, Y be a random variable according to the known distribution G such that

$$\begin{aligned} dG(y) &= \frac{c}{y^{k+2-\eta}}\,dy && \text{for } y > 1, \\ &= 0 && \text{otherwise;} \end{aligned} \tag{3.2}$$

where η, c, are positive constants and $\eta < 1$. Assume that X given y is distributed according to $F(x-\theta \mid y)$ where

$$\begin{aligned} dF(x-\theta \mid y) &= \frac{1}{y}\frac{1}{2b}\,dx && \text{for } \left|\frac{x-\theta}{y}\right| \leqq b, \\ &= 0 && \text{otherwise;} \end{aligned} \tag{3.3}$$

and b is a positive constant.

The unique best invariant estimate of θ is X (unique a.e. with respect to Lebesgue measure). By elementary integrations, we can show that $E|X|^{\alpha} W(X) < \infty$ for $0 \leqq \alpha < 1-\eta$. Next, we shall show that X is inadmissible.

Let us consider the estimate of the form

$$\varphi(x, y) = y\psi(x/y)$$

$$\text{where} \quad \psi(x/y) = x/y + f(x/y) \tag{3.4}$$

$$f(z) = -\varepsilon\delta z \qquad \text{if } |z| \leqq 1/\varepsilon,$$

$$= 0 \qquad \text{otherwise;}$$

and ε, δ are constants such that $b^{-1} > \varepsilon > \delta > 0$. Then the risk of φ is

$$R(\varphi, \theta) = \int_1^{\infty} \int_{-by+\theta}^{by+\theta} |\varphi-\theta|^k \frac{c}{y^{k+2-\eta}} \frac{1}{y} \frac{1}{2b}\, dx\, dy \tag{3.5}$$

$$= \int_1^{\infty} \int_{-by+\theta}^{by+\theta} y^k \left|\psi\left(\frac{x}{y}\right) - \frac{\theta}{y}\right|^k \frac{c}{2by^{k+3-\eta}}\, dx\, dy.$$

Let $x/y = z$ and $\theta/y = \tau$, then (3.5) becomes

$$R(\varphi, \theta) = \frac{c\theta^{\eta-1}}{2b} \int_0^{\theta} \left[\int_{\tau-b}^{\tau+b} |\psi(z) - \tau|^k\, dz \right] \tau^{-\eta}\, d\tau. \tag{3.6}$$

By (3.4) replace $\psi(z)$ by $z + f(z)$ and $z - \tau = w$. We have

$$R(\varphi, \theta) = \frac{c\theta^{\eta-1}}{2b} \int_0^{\theta} \left[\int_{-b}^{b} |f(w+\tau) + w|^k\, dw \right] \tau^{-\eta}\, d\tau. \tag{3.7}$$

Now we will evaluate the inner integral in (3.7). For $k > 1$, $|w| \leqq b$, and $|f| \leqq \delta < \infty$,

$$|w + f(w+\tau)|^k = |w|^k + f(w+\tau)k|w|^{k-1} \operatorname{sgn} w + o(f(w+\tau)). \tag{3.8}$$

Hence

$$\int_{-b}^{b} |f(w+\tau) + w|^k\, dw \tag{3.9}$$

$$= \int_{-b}^{b} |w|^k\, dw + k \int_{-b}^{b} f(w+\tau)\, |w|^{k-1} (\operatorname{sgn} w)\, dw + o(\sup_{|w| \leqq b} |f(w+\tau)|).$$

Thus

$$R(\varphi, \theta) = \frac{cb^k}{(k+1)(1-\eta)} \tag{3.10}$$

$$+ \frac{c\theta^{\eta-1}}{2b} \left\{ k \int_0^{\theta} \tau^{-\eta} \int_{-b}^{b} f(w+\tau)\, |w|^{k-1} (\operatorname{sgn} w)\, dw\, d\tau \right.$$

$$\left. + \int_0^{\theta} o(\sup_{|w| \leqq b} |f(w+\tau)|)\, \tau^{-\eta}\, d\tau \right\}.$$

The second term on the right of (3.10) equals

$$(3.11)\quad \frac{c\theta^{\eta-1}}{2b}k\int_{-b}^{b}|w|^{k-1}(\operatorname{sgn} w)\int_0^\theta f(w+\tau)\tau^{-\eta}\,d\tau\,dw$$
$$=\frac{c\theta^{\eta-1}}{2b}k\int_0^b\left[\int_0^\theta (f(w+\tau)-f(-w+\tau))\tau^{-\eta}\,d\tau\right]|w|^{k-1}\,dw.$$

For $\theta<\varepsilon^{-1}-b$ and $\varepsilon^{-1}>b$, the inner integral on the right-hand side of (3.11) becomes

$$(3.12)\quad \int_0^\theta (f(w+\tau)-f(-w+\tau))\tau^{-\eta}\,d\tau=\int_0^\theta -2\varepsilon\delta w\tau^{-\eta}\,d\tau=-2\varepsilon\delta w\left[\frac{\theta^{1-\eta}}{1-\eta}\right].$$

For $\theta\geqq\varepsilon^{-1}-b$ and $\varepsilon^{-1}>b$, the inner integral on the right-hand side of (3.11) becomes

$$(3.13)\quad \textstyle\int_0^\theta [f(w+\tau)-f(-w+\tau)]\tau^{-\eta}\,d\tau=\int_0^{\min(\varepsilon^{-1}-w,\theta)}-2\varepsilon\delta w\tau^{-\eta}\,d\tau$$
$$\textstyle+\int_{\min(\varepsilon^{-1}-w,\theta)}^{\min(\varepsilon^{-1}+w,\theta)}\varepsilon\delta(\tau-w)\tau^{-\eta}\,d\tau$$
$$\leqq\frac{-2\varepsilon\delta w}{1-\eta}[\min(\varepsilon^{-1}-w,\theta)]^{1-\eta}+\varepsilon\delta O(\varepsilon^\eta).$$

Substituting (3.12), (3.13) in (3.10) and evaluating the error term gives

$$(3.14)\quad R(\varphi,\theta)\leqq\frac{cb^k}{(k+1)(1-\eta)}+\frac{c\theta^{\eta-1}}{2b}\left\{\frac{-2\varepsilon\delta\theta^{1-\eta}b^{k+1}}{(1-\eta)}+o(\varepsilon\delta\theta)\right\},\quad \text{if}\quad 0<\theta<\varepsilon^{-1}-b,$$

and

$$(3.15)\quad R(\varphi,\theta)\leqq\frac{cb^k}{(k+1)(1-\eta)}+\frac{c\theta^{\eta-1}}{2b}\left\{\frac{-2\varepsilon\delta\eta b^k}{(1-\eta)}[\min(\varepsilon^{-1}-b,\theta)]^{1-\eta}+O(\varepsilon^\eta)\varepsilon\delta+o(\min(\delta/\varepsilon,\varepsilon\delta\theta))\right\},\quad \text{if}\quad \theta\geqq\varepsilon^{-1}-b.$$

Hence by choosing ε sufficiently small and then choosing δ so that δ/ε is sufficiently small then we have

$$(3.16)\quad R(\varphi,\theta)<\frac{cb^k}{(k+1)(1-\eta)}$$

for all $\theta>0$.

For the case $\theta<0$, by the symmetry of the problem, we can show (3.16) still holds. For $\theta=0$ we can prove (3.16) by direct computation. Summarizing, for fixed $\eta\in(0,1)$, $k>1$, $0<\delta<\varepsilon<b^{-1}$, and ε, δ/ε sufficiently small we have

$$(3.17)\quad R(\varphi,\theta)<\frac{cb^k}{(k+1)(1-\eta)}$$

for $-\infty < \theta < \infty$. On the other hand, we know the risk of the best invariant estimate X of θ is equal to $cb^k[(k+1)(1-\eta)]^{-1}$. Therefore we have

$$R(\varphi, \theta) < R(X, \theta) \tag{3.18}$$

for $-\infty < \theta < \infty$. This completes the proof of the theorem.

REMARK. The form of dF in the proof is not critical. In fact one can certainly take $dF(x-\theta \mid y) = y^{-1} h(|y^{-1}(x-\theta)|)\, dx$ so long as $h(w) = 0$ for $|w| > b$. Furthermore, any loss function which satisfies an appropriate version of (3.8) can be used in the example.

4. Inadmissibility of confidence intervals for a location parameter. We shall use the notation and assumptions stated in Section 2. We define the admissibility of confidence intervals as follows.

DEFINITION 4.1. A set of confidence intervals $[a(x, y), b(x, y)]$ is said to be *admissible* if, and only if, there exists no other set of confidence intervals $[a_1(x, y), b_1(x, y)]$ satisfying

(i) $b_1(x, y) - a_1(x, y) \leqq b(x, y) - a(x, y)$ for almost all $(x, y) \in \mathscr{X} \times \mathscr{Y}$ with respect to Lebesgue measure and

(ii) $P_\theta(a_1(x, y) \leqq \theta \leqq b_1(x, y)) \geqq P_\theta(a(x, y) \leqq \theta \leqq b(x, y))$

for all $\theta \in \Omega$ where Ω is the parameter space, the strict inequality holding for at least one $\theta \in \Omega$.

In 1966 Joshi [7] proved a theorem which gave a set of sufficient conditions for the admissibility of certain confidence interval procedures for a location parameter. Instead of stating Joshi's theorem, we shall state and prove a simpler theorem given by a referee.

THEOREM 4.1. *Suppose X, Y have joint distribution given by $H(dx, dy) = f(x-\theta) \mid y)\, G(dy)$. Suppose $f(-v_2(y) \mid y) = f(v_1(y) \mid y)$, where $v_1(Y)$, $v_2(Y)$ are two nonnegative statistics, and $f(t \mid y)$ is strictly decreasing in $|t|$ on the set $f(t \mid y) > 0$. Suppose*

$$\int G(dy) \int |t| f(t \mid y)\, dt < \infty. \tag{4.1}$$

Then the confidence interval procedure given by $x - v_1(y) \leqq \theta \leqq x + v_2(y)$ is admissible according to Definition 4.1.

PROOF. Consider the problem of estimating θ with loss function

$$\begin{aligned} L(\delta-\theta, y) &= 0 \quad \text{if } -v_2(y) \leqq \delta - \theta \leqq v_1(y); \\ &= 1 \quad \text{otherwise.} \end{aligned} \tag{4.2}$$

Then the best invariant estimator is $\delta(x, y) = x$. The risk of this estimator is $R(\theta, \delta) = 1 - P\{x - v_1(y) \leqq \theta \leqq x + v_2(y)\}$.

Now, suppose the confidence interval procedure $a(x, y) \leqq \theta \leqq b(x, y)$ is better than the procedure of the theorem in the sense of the Definition 4.1. Define the estimator $\delta_1(x, y) = a(x, y) + v_1(y)$. By (i), (ii) in Definition 4.1 we have $R(\theta, \delta_1) \leqq R(\theta, \delta)$ with strict inequality for some θ. Hence $\delta(x, y) = x$ is an inadmissible estimator. However (4.1) guarantees that $\delta(x, y) = x$ is admissible by Brown [3]. This contradiction proves the theorem.

Now we notice that the moment condition in Theorem 4.1 is quite similar to the moment conditions in the estimation problem. It is interesting to check whether the moment condition here is also essential for the admissibility of the specified confidence interval procedure mentioned in Theorem 4.1.

THEOREM 4.2. *For every* $\alpha (0 \leqq \alpha < 1)$ *there are a family of probability density functions such that* $E|X|^\alpha < \infty$, *and a confidence interval procedure I which satisfy all but the moment condition in Theorem* 4.1, *but I is inadmissible.*

PROOF. Let θ be an unknown real-valued parameter $-\infty < \theta < \infty$, Y be a random variable according to the known density function

$$g(y) = c_1/y^{2-\eta} \quad \text{for } y > 1, \qquad = 0 \quad \text{otherwise}; \tag{4.3}$$

where $0 < \eta < 1$, $c_1 > 0$ are constants. Let x given y have density function

$$p(x-\theta \mid y) = \frac{c_2}{y}\left(b - l\left|\frac{x-\theta}{y}\right|\right) \quad \text{if } \left|\frac{x-\theta}{y}\right| \leqq b, \qquad = 0 \quad \text{otherwise}; \tag{4.4}$$

where c_2, b, l, are proper positive constants and $b > 2$. We define

$$I(x, y) = [x-y, x+y] \tag{4.5}$$

and

$$I^*(x, y) = I(x, y) \quad \text{if } y < \varepsilon|x|+1, \qquad = [x(1-\varepsilon\delta)-y, x(1-\varepsilon\delta)+y] \quad \text{if } y \geqq \varepsilon|x|+1, \tag{4.6}$$

where ε, δ are constants such that $b^{-1} > \varepsilon > \delta > 0$. By elementary integrations, we can show that $E|X|^\alpha < \infty$ for $0 \leqq \alpha < 1-\eta$ and diverges if $\alpha = 1-\eta$. Clearly all but the moment conditions in Theorem 4.1 are satisfied.

Now we shall show that $I^*(x, y)$ dominates $I(x, y)$ in the sense of Definition 4.1. for sufficient small ε, δ and l. Clearly the length of $I(x, y)$ is equal to the length of $I^*(x, y)$ for every $(x, y) \in \mathscr{X} \times \mathscr{Y}$. Hence we need only to show that

$$P_\theta(\theta \in I(x, y)) \leqq P_\theta(\theta \in I^*(x, y)) \tag{4.7}$$

for all θ and strict inequality holds for some θ. For $\theta = 0$, clearly (4.7) holds. By the symmetry of the problem, we need only to consider the case $\theta > 0$. That is, we wish to show

$$P_\theta(\theta \in I(x, y)) < P_\theta(\theta \in I^*(x, y)) \tag{4.8}$$

for all $\theta > 0$. Showing (4.8) is equivalent to showing

$$P_\theta(\theta \in I(x, y) \quad \text{and} \quad y \geqq \varepsilon|x| + 1) < P_\theta(\theta \in I^*(x, y) \quad \text{and} \quad y \geqq \varepsilon|x| + 1) \tag{4.9}$$

for all $\theta > 0$.

Now we shall evaluate the two probabilities in (4.9).

$$\begin{aligned} &P_\theta(\theta \in I^*(x, y) \quad \text{and} \quad y \geqq \varepsilon|x| + 1) \\ &= \int_{\frac{1+\varepsilon\theta-\varepsilon\delta}{1+\varepsilon-\varepsilon\delta}}^{\frac{1+\varepsilon\theta-\varepsilon\delta}{1-\varepsilon-\varepsilon\delta}} \int_{\frac{\theta-y}{1-\varepsilon\delta}}^{\frac{y-1}{\varepsilon}} \frac{c_1 c_2}{y^{2-\eta}} \frac{1}{y}\left(b - l\left|\frac{x-\theta}{y}\right|\right) dx\, dy \\ &+ \int_{\frac{1+\varepsilon\theta-\varepsilon\delta}{1-\varepsilon\delta-\varepsilon}}^{\infty} \int_{\frac{\theta-y}{1-\varepsilon\delta}}^{\frac{\theta+y}{1-\varepsilon\delta}} \frac{c_1 c_2}{y^{2-\eta}} \frac{1}{y}\left(b - l\left|\frac{x-\theta}{y}\right|\right) dx\, dy, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &P_\theta(\theta \in I(x, y) \quad \text{and} \quad y \geqq \varepsilon|x| + 1) \\ &= \int_{\frac{1+\varepsilon\theta}{1+\varepsilon}}^{\frac{1+\varepsilon\theta}{1-\varepsilon}} \int_{\theta-y}^{\frac{y-1}{\varepsilon}} \frac{c_1 c_2}{y^{2-\eta}} \frac{1}{y}\left(b - l\left|\frac{x-\theta}{y}\right|\right) dx\, dy \\ &+ \int_{\frac{1+\varepsilon\theta}{1-\varepsilon}}^{\infty} \int_{\theta-y}^{\theta+y} \frac{c_1 c_2}{y^{2-\eta}} \frac{1}{y}\left(b - l\left|\frac{x-\theta}{y}\right|\right) dx\, dy. \end{aligned} \tag{4.11}$$

We note that the regions of the integrations are subsets of the support of the joint distribution.

Using the dominated convergence theorem and by elementary computation we have

$$\begin{aligned} &\lim_{l \to 0} P_\theta(\theta \in I^*(x, y) \quad \text{and} \quad y \geqq \varepsilon|x| + 1) \\ &= \int_{\frac{1+\varepsilon\theta-\varepsilon\delta}{1+\varepsilon-\varepsilon\delta}}^{\frac{1+\varepsilon\theta-\varepsilon\delta}{1-\varepsilon\delta-\varepsilon}} \int_{\frac{\theta-y}{1-\varepsilon\delta}}^{\frac{y-1}{\varepsilon}} \frac{c_1 c_2 b}{y^{3-\eta}} dx\, dy \\ &+ \int_{\frac{1+\varepsilon\theta-\varepsilon\delta}{1-\varepsilon\delta-\varepsilon}}^{\infty} \int_{\frac{\theta-y}{1-\varepsilon\delta}}^{\frac{\theta+y}{1-\varepsilon\delta}} \frac{c_1 c_2 b}{y^{3-\eta}} dx\, dy \\ &= \frac{c_1 c_2 b[(1+\varepsilon-\varepsilon\delta)^{2-\eta} - (1-\varepsilon\delta-\varepsilon)^{2-\eta}]}{\varepsilon(1-\varepsilon\delta)(1-\eta)(2-\eta)(1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}}, \end{aligned} \tag{4.12}$$

and

$$\lim_{l\to 0} P_\theta(\theta \in I(x,y) \text{ and } y \geqq \varepsilon|x|+1) \tag{4.13}$$

$$= \int_{\frac{1+\varepsilon\theta}{1+\varepsilon}}^{\frac{1+\varepsilon\theta}{1-\varepsilon}} \int_{\theta-y}^{\frac{y-1}{\varepsilon}} \frac{c_1 c_2 b}{y^{3-\eta}}\, dx\, dy$$

$$+ \int_{\frac{1+\varepsilon\theta}{1-\varepsilon}}^{\infty} \int_{\theta-y}^{y+\theta} \frac{c_1 c_2 b}{y^{3-\eta}}\, dx\, dy$$

$$= \frac{c_1 c_2 b[(1+\varepsilon)^{2-\eta}-(1-\varepsilon)^{2-\eta}]}{\varepsilon(1-\eta)(1+\varepsilon\theta)^{1-\eta}(2-\eta)}.$$

From (4.12) and (4.13) we have

$$\lim_{l\to 0}[P_\theta(\theta \in I^*(x,y) \text{ and } y \geqq \varepsilon|x|+1) - P_\theta(\theta \in I(x,y) \text{ and } y \geqq \varepsilon|x|+1)] \tag{4.14}$$

$$= \frac{\begin{array}{r} c_1 c_2 b\{[(1+\varepsilon-\varepsilon\delta)^{2-\eta}-(1-\varepsilon-\varepsilon\delta)^{2-\eta}](1+\varepsilon\theta)^{1-\eta} \\ -[(1+\varepsilon)^{2-\eta}-(1-\varepsilon)^{2-\eta}](1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}(1-\varepsilon\delta)\} \end{array}}{\varepsilon(1-\varepsilon\delta)(1-\eta)(2-\eta)(1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}(1+\varepsilon\theta)^{1-\eta}}.$$

To show that the right-hand side of (4.14) is positive for all $\theta > 0$, it is sufficient to show that the term in the braces of the numerator is positive for all $\theta > 0$. Expand $(1+\varepsilon-\varepsilon\delta)^{2-\eta}$, $(1-\varepsilon-\varepsilon\delta)^{2-\eta}$ and $(1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}$ by Taylor's expansion then we have

$$\begin{aligned} (1+\varepsilon-\varepsilon\delta)^{2-\eta} &= (1+\varepsilon)^{2-\eta}-(2-\eta)(1+\varepsilon)^{1-\eta}\varepsilon\delta+o(\varepsilon\delta), \\ (1-\varepsilon-\varepsilon\delta)^{2-\eta} &= (1-\varepsilon)^{2-\eta}-(2-\eta)(1-\varepsilon)^{1-\eta}\varepsilon\delta+o(\varepsilon\delta), \\ (1+\varepsilon\theta-\varepsilon\delta)^{1-\eta} &= (1+\varepsilon\theta)^{1-\eta}-(1-\eta)(1+\varepsilon\theta)^{-\eta}\varepsilon\delta+o(\varepsilon\delta). \end{aligned} \tag{4.15}$$

Using (4.15), we have

$$\begin{aligned} &[(1+\varepsilon-\varepsilon\delta)^{2-\eta}-(1-\varepsilon-\varepsilon\delta)^{2-\eta}](1+\varepsilon\theta)^{1-\eta} \\ &\qquad -[(1+\varepsilon)^{2-\eta}-(1-\varepsilon)^{2-\eta}](1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}(1-\varepsilon\delta) \\ &\geqq [-(2-\eta)(1+\varepsilon)^{1-\eta}\varepsilon\delta+(2-\eta)(1-\varepsilon-\varepsilon\delta)^{1-\eta}\varepsilon\delta](1+\varepsilon\theta)^{1-\eta} \\ &\qquad +[(1+\varepsilon)^{2-\eta}-(1-\varepsilon)^{2-\eta}](1+\varepsilon\theta)^{1-\eta}\varepsilon\delta-o(\varepsilon\delta). \end{aligned} \tag{4.16}$$

Expand $(1+\varepsilon)^{1-\eta}$, $(1-\varepsilon-\varepsilon\delta)^{1-\eta}$, $(1+\varepsilon)^{2-\eta}$ and $(1-\varepsilon)^{2-\eta}$ by Taylor's expansion then we have

$$\begin{aligned} (1+\varepsilon)^{1-\eta} &= 1+(1-\eta)\varepsilon+o(\varepsilon), \\ (1-\varepsilon-\varepsilon\delta)^{1-\eta} &= 1-(1-\eta)\varepsilon(1-\delta)+o(\varepsilon), \\ (1+\varepsilon)^{2-\eta} &= 1+(2-\eta)\varepsilon+o(\varepsilon), \\ (1-\varepsilon)^{2-\eta} &= 1-(2-\eta)\varepsilon+o(\varepsilon). \end{aligned} \tag{4.17}$$

Substituting (4.17) into (4.16), we have

$$(4.18)\quad [(1+\varepsilon-\varepsilon\delta)^{2-\eta}-(1-\varepsilon-\varepsilon\delta)^{2-\eta}](1+\varepsilon\theta)^{1-\eta} - [(1+\varepsilon)^{2-\eta}-(1-\varepsilon)^{2-\eta}](1+\varepsilon\theta-\varepsilon\delta)^{1-\eta}(1-\varepsilon\delta) \geqq \varepsilon\delta(1+\varepsilon\theta)^{1-\eta}(2-\eta)\varepsilon[2\eta-\delta(1-\eta)-o(\varepsilon)].$$

For given $\eta > 0$, it is possible to choose ε, δ, sufficiently small so that the right-hand side of (4.18) is positive for all $\theta > 0$. Equivalently, for sufficiently small ε, δ,

$$(4.19)\quad \lim_{l\to 0}[P(\theta\in I^*(x,y) \text{ and } y \geqq \varepsilon|x|+1)-P(\theta\in I(x,y) \text{ and } y\geqq \varepsilon|x|+1)] > 0 \quad \text{for all} \quad \theta > 0.$$

(4.19) implies that there exists a positive l_0 such that

$$P_\theta(\theta\in I^*(x,y) \quad \text{and} \quad y\geqq\varepsilon|x|+1) > P_\theta(\theta\in I(x,y) \quad \text{and} \quad y\geqq \varepsilon|x|+1)$$

for sufficiently small ε, δ and all $\theta > 0$. This completes the proof of the theorem.

5. Inadmissibility of a non-unique best invariant test involving a location parameter. We shall use the notation and assumptions stated in Section 2. Consider the problem of testing $H_1: i = 1$ against $H_2: i = 2$. For any level of significance a best invariant test φ_0 is of the form

$$(5.1)\quad \varphi_0(x,y) = 1 \quad \text{if } \frac{dG_2}{d(G_1+G_2)}(y) > c; \qquad = 0 \quad \text{if } \frac{dG_2}{d(G_1+G_2)}(y) < c.$$

Lehmann and Stein [8] have shown that if $F_i(\cdot, y)$ are absolutely continuous with respect to μ where μ is Lebesgue measure on the real line, $E_{i0}|X| < \infty$ for $i = 1, 2$, and if

$$(5.2)\quad G_1\left\{y \,\middle|\, \frac{dG_2}{d(G_1+G_2)}(y) = c\right\} = 0$$

then φ_0 is admissible.

It is interesting to see whether condition (5.2) and the moment conditions are essential for the admissibility of the best invariant test. Fox and Perng [5] have given an example showing that, with the moment condition violated, φ_0 may not be admissible. In this section we give an example which shows (5.2) is essential.

THEOREM 5.1. *There are distributions $P_{i\theta}$ with $F_i(\cdot\,|\,y)$ absolutely continuous with respect to μ, $E_{i0}|X| < \infty$ for $i = 1, 2$ and $G_1\{y\,|\,dG_2[d(G_1+G_2)]^{-1}(y) = c\} = \delta$ where $\delta > 0$, $0 < c < 1$ such that a best invariant test is inadmissible.*

PROOF. Without loss of generality, we assume $\delta = 1$ and $c = \frac{1}{2}$. Let

$$(5.3)\quad G_1(\{y\}) = G_2(\{y\}) = \tfrac{1}{2} \quad \text{if } y = 2, 3; \qquad = 0 \quad \text{otherwise.}$$

Let $f_i(x-\theta \mid y)$ be the conditional densities of $F_i(x-\theta \mid y)$ with respect to Lebesgue measure on real line for $i = 1, 2$ and define

$$(5.4) \quad \begin{aligned} f_1(x-\theta \mid y) &= 1 \quad \text{if } y = 2 \text{ and } \theta \leqq x \leqq \theta+1 \text{ or } y = 3 \text{ and } \theta-1 \leqq x \leqq \theta; \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

It is easy to see that

$$(5.5) \qquad G_1\left\{y \,\middle|\, \frac{dG_2}{d(G_1+G_2)}(y) = \frac{1}{2}\right\} = 1,$$

hence the best invariant test with $c = \frac{1}{2}$ is not unique. One version of the best invariant tests at significance level .5 is

$$(5.6) \qquad \begin{aligned} \varphi_0(x, y) &= 1 \quad \text{if } y = 3; \\ &= 0 \quad \text{if } y = 2. \end{aligned}$$

Define

$$(5.7) \qquad \begin{aligned} \varphi^*(x, y) &= 1 \quad \text{if } y = 2 \text{ and } x \geqq 0 \text{ or } y = 2 \text{ and } x \leqq 0; \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then, clearly,

$$(5.8) \qquad \begin{aligned} E_{1\theta}\,\varphi_0(X, Y) &= \tfrac{1}{2} \\ E_{2\theta}\,\varphi_0(X, Y) &= \tfrac{1}{2} \end{aligned}$$

for all θ, and

$$(5.9) \qquad \begin{aligned} E_{1\theta}\,\varphi^*(X, Y) &< \tfrac{1}{2} \quad \text{for } |\theta| < 1 \\ &= \tfrac{1}{2} \qquad\quad |\theta| \geqq 1; \\ E_{2\theta}\,\varphi^*(X, Y) &> \tfrac{1}{2} \quad \text{for } |\theta| < 1 \\ &= \tfrac{1}{2} \qquad\quad |\theta| \geqq 1. \end{aligned}$$

Therefore φ^* is better than φ_0. This completes the proof.

REMARK. Consider the sample space $\mathscr{X} \times \mathscr{Y}$, where $\mathscr{Y} = \mathscr{X}$. The sample space is divided into four quadrants by the axis $x = \theta$ and $y = 0$. Then any distributions $P_{i\theta}$ on $\mathscr{X} \times \mathscr{Y}$ such that the support of $P_{1\theta}$ is the upper right corner and lower left corner, while the support of $P_{2\theta}$ is the upper left corner and lower right corner can be used to provide an example to show that inadmissibility of a best invariant test when the test is not unique.

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