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RECIPROCAL PROCESSES: THE STATIONARY GAUSSIAN CASE

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0. Introduction. Let $\{X_t, a < t < b\}$ be a stochastic process. Suppose that, for each $t \in (a, b)$, the σ-field generated by $\{X_s; a < s < t\}$ is conditionally independent, given X_t, of each event in the σ-field generated by $\{X_s: t < s < b\}$. Then $\{X_t, a < t < b\}$ is called a *Markov process*. Suppose instead that for each s, t in (a, b) with $s < t$ the following holds: each event in the σ-field generated by $\{X_r: s < r < t\}$ is conditionally independent, given X_s and X_t, of each event in the σ-field generated by $\{X_r: a < r < s\} \cup \{X_r: t < r < b\}$. We then call $\{X_t, a < t < b\}$ a *reciprocal process*. The use of the word "reciprocal" to describe this property is due to S. Bernstein [1]. In 1961 Slepian [7] noticed that the stationary Gaussian process with covariance function triangular on $[-1, 1]$ and zero outside $[-1, 1]$ has the reciprocal property on $[0, 1]$, and exploited this property to compute explicitly the first passage time probability density for the restriction of the process to an interval of length 1. We address ourselves to the task of finding other real-valued stationary Gaussian processes having the reciprocal property on a finite or infinite interval. The natural approach to take seems that of Doob (see [2], pages 90–91 and 233–234). Exploiting the fact that in a Gaussian process conditioning is projection, Doob geometrizes the problem, and shows that a stationary Gaussian process is Markov if and only if its covariance function satisfies the functional equation for the exponential function. We geometrize our problem in a similar way, and succeed in showing that if a stationary Gaussian process is reciprocal then its covariance function satisfies a functional equation of a type satisfied by the cosine function. (I have L. A. Shepp to thank for the observation that the functional equation (11) is of this type, and for his reference to current literature on such functional equations.) The continuous functions which satisfy such functional equations on the whole real line were found by Cauchy. These were shown to be all the measurable solutions by Kacmarz [4]. We adapt the argument of Kacmarz to show that these functions ((15), (16), and (17) of our paper) are the only ones which satisfy such functional equations on an open interval. From these functions we select the covariances and obtain the following result. Suppose $\{X_t, 0 < t < T\}$ is a real-valued stationary Gaussian process with continuous covariance. It is reciprocal if and only if one of the following holds: (A) $\{X_t, 0 < t < T\}$ is Markovian, (B) $\{X_t, 0 < t < T\}$ is the restriction to $(0, T)$ of a sine wave of random phase and amplitude but of fixed period no less than $2T$, or (C) $\{X_t, 0 < t < T\}$ is any of a multitude of processes whose covariance function has a graph which is linear on $[0, T]$.

1. Main results. Suppose $\{X_t, a < t < b\}$ is a stochastic process on the open interval (a, b), where $-\infty \leqq a < b \leqq \infty$. Let $(\Omega, \mathscr{F}, P)$ be the underlying prob-

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ability space. For each s and t with $a < s < t < b$ let $\mathscr{E}(s, t)$ be the σ-field generated by the random variables X_r, $r \in (a, b) - (s, t)$, and $\mathscr{I}(s, t)$ that generated by the X_r's with $r \in (s, t)$.

DEFINITION. The process $\{X_t, a < t < b\}$ is called a *reciprocal process* if for each $a < s < t < b$, $\mathscr{E}(s, t)$ and $\mathscr{I}(s, t)$ are conditionally independent given X_s and X_t.

The notion of conditional independence is defined on page 351 of [5].

LEMMA. 1. *The process $\{X_t, a < t < b\}$ is reciprocal if and only if*

$$E\{f(X_u) \mid X_{s_1}, \cdots, X_{s_n}, X_t, X_v\} = E\{f(X_u) \mid X_t, X_v\} \tag{1}$$

for each $a < t < u < v < b$, $\{s_1, \cdots, s_n\} \subset (a, b) - (t, v)$, and bounded Borel-measurable f.

PROOF. First, assume that $\{X_t, a < t < b\}$ is reciprocal. To show that (1) holds it suffices to show that

$$E\{YZE\{f(X_u) \mid X_t, X_v\}\} = E\{YZf(X_u)\},$$

whenever $Y = g(X_{s_1}, \cdots, X_{s_n})$, $Z = h(X_t, X_v)$ with g and h bounded Borel-measurable functions. But

$$\begin{aligned} E\{YZE\{f(X_u) \mid X_t, X_v\}\} &= E\{ZE\{Y \mid X_t, X_v\}E\{f(X_u) \mid X_t, X_v\}\} \\ &= E\{ZE\{Yf(X_u) \mid X_t, X_v\}\} \\ &= E\{E\{YZf(X_u) \mid X_t, X_v\}\} \\ &= E\{YZf(X_u)\}. \end{aligned}$$

Now assume that (1) holds. It is easily seen that the reciprocal property amounts to showing that

$$\begin{aligned} &E\{g(X_{u_1}, \cdots, X_{u_m})h(X_{s_1}, \cdots, X_{s_n}) \mid X_t, X_v\} \\ &\quad = E\{g(X_{u_1}, \cdots, X_{u_m}) \mid X_t, X_v\}E\{h(X_{s_1}, \cdots, X_{s_n}) \mid X_t, X_v\} \end{aligned} \tag{2}$$

for $t, v, s_1, \cdots, s_n$ as above, $t < u_1 < \cdots < u_m < v$, and g and h bounded Borel-measurable functions. The argument just made (as well as Proposition A on page 351 of [5]) makes it clear that (2) holds as specified if and only if

$$E\{g(X_{u_1}, \cdots, X_{u_m}) \mid X_{s_1}, \cdots, X_{s_n}, X_t, X_v\} = E\{g(X_{u_1}, \cdots, X_{u_m}) \mid X_t, X_v\} \tag{3}$$

for all bounded Borel-measurable g. We establish (3) by induction on m. It holds for $m = 1$ by assumption. Assume that (3) holds (for *any* $g, n, s_1, \cdots, s_n$) if m is replaced by $m-1$. It suffices to prove (3) for $g(X_{u_1}, \cdots, X_{u_m}) = g_1(X_{u_1}, \cdots, X_{u_{m-1}})g_2(X_{u_m})$, g_1, g_2 bounded Borel-measurable. Let $W = (X_{s_1}, \cdots, X_{s_n})$, and let $Y = g_1(X_{u_1}, \cdots, X_{u_{m-1}})$, $Z = g_2(X_{u_m})$. Then

$$\begin{aligned} E\{g(X_{u_1}, \cdots, X_{u_m}) \mid W, X_t, X_v\} &= E\{YZ \mid W, X_t, X_v\} \\ &= E\{ZE\{Y \mid W, X_{u_m}, X_t, X_v\} \mid W, X_t, X_v\} \\ &= E\{ZE\{Y \mid X_t, X_{u_m}\} \mid W, X_t, X_v\}. \end{aligned}$$

Since $E\{f(X_{u_m}) \mid W, X_t, X_v\} = E\{f(X_{u_m} \mid X_t, X_v\}$ for any f, $E\{h(X_{u_m}, X_t \mid W, X_t, X_v\} = E\{h(X_{u_m}, X_t) \mid X_t, X_v\}$ for any h (to see this, first consider the case where $h(x, y) = h_1(x)h_2(y)$). But $ZE\{Y \mid X_t, X_{u_m}\} = h(X_t, X_{u_m})$ for some h, so

$$\begin{aligned} E\{ZE\{Y \mid X_t, X_{u_m}\} \mid W, X_t, X_v\} &= E\{ZE\{Y \mid X_t, X_{u_m}\} \mid X_t, X_v\} \\ &= E\{ZE\{Y \mid X_t, X_{u_m}, X_v\} \mid X_t, X_v\} \\ &= E\{E\{YZ \mid X_t, X_{u_m}, X_v\} \mid X_t, X_v\} \\ &= E\{YZ \mid X_t, X_v\}. \end{aligned}$$

Thus (3) holds as it stands, completing the induction.

LEMMA 2. *If $\{X_t, a < t < b\}$ is a Markov process, then it is reciprocal.*

PROOF. We use the criterion established in Lemma 1. Let

$$a < s_1 < \cdots < s_m < t < u < v < w_1 < \cdots < w_n < b.$$

We must show that if $\{X_t, a < t < b\}$ is Markov, then

$$\begin{aligned} &E\{f(X_u)g(X_{s_1}, \cdots, X_{s_m})h(X_{w_1}, \cdots, X_{w_n}) \mid X_t, X_v\} \\ &\quad = E\{f(X_u) \mid X_t, X_v\}E\{g(X_{s_1}, \cdots, X_{s_m})h(X_{w_1}, \cdots, X_{w_n}) \mid X_t, X_v\}, \end{aligned}$$

where f, g, and h are bounded Borel functions. But, letting $Y = g(X_{s_1}, \cdots, X_{s_n})$, $Z = h(X_{w_1}, \cdots, X_{w_n})$, we have

$$E\{f(X_u)YZ \mid X_t, X_v\} = E\{f(X_u)E\{YZ \mid X_t, X_u, X_v\} \mid X_t, X_v\}.$$

The Markov property of $\{X_t, a < t < b\}$ implies that

$$\begin{aligned} E\{YZ \mid X_t, X_u, X_v\} &= E\{YE\{Z \mid Y, X_t, X_u, X_v\} \mid X_t, X_u, X_v\} \\ &= E\{YE\{Z \mid X_v\} \mid X_t, X_u, X_v\} \\ &= E\{Z \mid X_v\}E\{Y \mid X_t, X_u, X_v\} \\ &= E\{Z \mid X_v\}E\{Y \mid X_t\} \\ &= E\{Y \mid X_t, X_v\}E\{Z \mid X_v\} \\ &= E\{YE\{Z \mid X_v\} \mid X_t, X_v\} \\ &= E\{YE\{Z \mid X_t, X_v, Y\} \mid X_t, X_v\} \\ &= E\{YZ \mid X_t, X_v\}, \end{aligned}$$

so that

$$\begin{aligned} E\{f(X_u)YZ \mid X_t, X_v\} &= E\{f(X_u)E\{YZ \mid X_t, X_v\} \mid X_t, X_v\} \\ &= E\{f(X_u) \mid X_t, X_v\}E\{YZ \mid X_t, X_v\}, \end{aligned}$$

as was to be proved.

Suppose $\{X_t, -\infty < t < \infty\}$ is a real, stationary Gaussian process. We assume that $E\{X_t\} = 0$ and $E\{X_t^2\} = 1$. Then $\{X_t\} \subset L_2(\Omega, \mathscr{F}, P)$, which is a Hilbert space with respect to the inner product $(Y, Z) = E\{YZ\}$ (we are considering real L_2). Because it is stationary, $(X_s, X_t) = R(s-t)$, where R is the covariance function. We assume that R is continuous. Then R is an even positive-definite function, with $R(0) = 1$ and $|R(t)| \leqq 1$ for all t. The condition that $\{X_t, a < t < b\}$ be reciprocal for some interval (a, b) of length T is a condition on $R(t)$, $0 \leqq t \leqq T$.

THEOREM. *Suppose $\{X_t, -\infty < t < \infty\}$ is a real stationary Gaussian process with $E\{X_t\} = 0$, $E\{X_t^2\} = 1$, and continuous covariance function R. Then $\{X_t, 0 < t < T\}$ is a reciprocal process if and only if one of the following holds*:

(i) $R(t) = e^{-at}$, $0 \leqq t \leqq T$, *where* $a > 0$.

(ii) $R(t) = \cos at$, $0 \leqq t \leqq T$, *where* $a > 0$, *and* $T \leqq \pi/a$.

(iii) $R(t) = 1-at$, $0 \leqq t \leqq T$, *where* $0 \leqq a \leqq 2/T$.

We require the following lemma.

LEMMA 3. *Suppose $\{X_t, -\infty < t < \infty\}$ is a real stationary Gaussian process with $E\{X_t\} = 0$, $E\{X_t^2\} = 1$. Suppose that $-1 < R(t) < 1$ for $0 < t < T$. Then $\{X_t, 0 < t < T\}$ is a reciprocal process if and only if the following holds*:

for each σ, τ, γ with $\sigma \geqq 0, \tau \geqq 0, \gamma \geqq 0, \sigma+\tau+\gamma \leqq T$,

$$\alpha R(\sigma) + \beta R(\sigma+\tau+\gamma) = R(\sigma+\tau), \tag{4}$$

where

$$\alpha = \frac{R(\tau)-R(\gamma)R(\tau+\gamma)}{1-R^2(\tau+\gamma)}, \qquad \beta = \frac{R(\gamma)-R(\tau)R(\tau+\gamma)}{1-R^2(\tau+\gamma)}. \tag{5}$$

PROOF. Suppose first that $\{X_t, 0 < t < T\}$ is a reciprocal process. Let $0 < s < t < u < v < T$. We infer from Lemma 1 that

$$E\{X_u \mid X_s, X_t, X_v\} = E\{X_u \mid X_t, X_v\}. \tag{6}$$

Because $\{X_t\}$ is Gaussian, conditioning is projection; that is, the left-hand side is the projection of X_u onto the linear manifold spanned by X_s, X_t, and X_v, while the right-hand side is the projection of X_u onto the linear span of X_t and X_v. Thus (6) is equivalent to

$$X_u - E\{X_u \mid X_t, X_v\} \perp X_s. \tag{7}$$

An easy computation shows that

$$E\{X_u \mid X_t, X_v\} = \alpha X_t + \beta X_v,$$

where α and β are given by (5). Thus (7) holds if and only if

$$(X_u - \alpha X_t - \beta X_v, X_s) = 0. \tag{8}$$

But (8) is equivalent to (4), where $\sigma = t-s$, $\tau = u-t$, and $\gamma = v-u$.

Now suppose that (4) holds as specified in the statement of the lemma. Let $0 < t < v < T$. Then (6) holds if $s \in (0, t)$. Suppose that $t < v < s < T$. Let $\gamma = u - t$, $\tau = v - u$, and $\sigma = s - v$. It is easily checked that (7) is again equivalent to (4). Thus (7) holds whether $s \in (0, t)$ or $s \in (v, T)$. Suppose $\{s_1, \cdots, s_n\} \subset (0, T) - (t, v)$. Then

$$X_u - E\{X_u \mid X_t, X_v\} \perp X_{s_i}, \qquad i = 1, \cdots, n, \tag{9}$$

whence

$$E\{X_u \mid X_{s_1}, \cdots, X_{s_n}, X_t, X_v\} = E\{X_u \mid X_t, X_v\}. \tag{10}$$

Even without (9) holding, $X_u - E\{X_u \mid X_{s_1}, \cdots, X_{s_n}, X_t, X_v\}$ is orthogonal hence by normality independent of $X_{s_1}, \cdots, X_{s_n}, X_t, X_v$, so the conditional variance of X_u given $X_{s_1}, \cdots, X_{s_n}, X_t, X_v$ is equal to

$$E(E^2\{X_u \mid X_{s_1}, \cdots, X_{s_n}, X_t, X_v\}) = E(E^2\{X_u \mid X_t, X_v\}).$$

Thus the conditional densities of X_u given X_t, X_v on the one hand, and $X_{s_1}, \cdots, X_{s_n}, X_t, X_v$ on the other, have the same mean, $E\{X_u \mid X_t, X_v\}$, and the same variance, $E(E^2\{X_u \mid X_t, X_v\}$. Since both conditional densities are normal, they are almost surely identical, so (1) holds almost surely. (For the prototype of this argument see pages 90–91 of [2].) Since $\{s_1, \cdots, s_n\}$ is an arbitrary finite subset of $(0, T) - (t, v)$, and since t and v are arbitrary points with $0 < t < v < T$, $\{X_t, 0 < t < T\}$ is reciprocal by virtue of Lemma 1. This completes the proof of Lemma 3.

We proceed with the proof of the theorem. $\{X_t, 0 < t < T\}$ is a stationary Gaussian reciprocal process with $E\{X_t\} \equiv 0$, $E\{X_t^2\} \equiv 1$, and continuous covariance function R. Assume first that $-1 < R(t) < 1$ for all $0 < t < T$. Set $s = \sigma + \tau$ and $t = \gamma = \tau$ in (4) and (5), obtaining

$$R(s) = \left(\frac{R(t)}{1 + R(2t)}\right)(R(s-t) + R(s+t)) \tag{11}$$

for $0 \leqq t \leqq s$, $s + t < T$. If we set $\varphi(2t) = (1 + R(2t))R^{-1}(t)$, we have

$$R(s+t) + R(s-t) = \varphi(2t)R(s) \tag{12}$$

for $0 \leqq t \leqq s$, $s + t < T$. Let $0 < s < T$. Let $\delta' > 0$ be chosen so that R is positive in the interval $(0, \delta')$ and also so that $0 < s - \delta' < s + \delta' < T$. Then (12) holds for $0 \leqq t \leqq \delta'$. We now follow the argument of Kacmarz [4]. Let $0 < \delta < \delta'$, and integrate (12) with respect to t over the integral $(0, \delta)$, getting

$$\int_s^{s+\delta} R(t)\, dt + \int_{s-\delta}^{s} R(t)\, dt = R(s) \int_0^\delta \varphi(2t)\, dt.$$

Differentiating with respect to s, we obtain

$$R(s+\delta) - R(s-\delta) = R'(s) \int_0^\delta \varphi(2t)\, dt.$$

This shows that $R'(s)$ exists for $0 < s < T$; we see in fact that R has derivatives of all orders in $(0, T)$. Substituting $t = 0$ in (12), and subtracting the result from (12), we obtain

$$R(s+t)-R(s-t)-2R(s) = (\varphi(2t)-\varphi(0))R(s). \tag{13}$$

Dividing (13) by t^2 and letting $\delta' > t\downarrow 0$, we obtain

$$R''(s) = kR(s), \qquad 0 < s < T, \tag{14}$$

where $k = 2\varphi''(0)$. Solutions to (13) are well known:

$$R(t) = A\cos at + B\sin at \qquad k > 0, \tag{15}$$

$$R(t) = Ae^{-at}+Be^{at} \qquad k < 0, \tag{16}$$

$$R(t) = A+Bt, \qquad k = 0, \tag{17}$$

where $a = |k|^{\frac{1}{2}}$, the equations holding for $0 \leqq t \leqq T$. In cases (15), (16), however, equation (11) is not satisfied if both $A \neq 0$ and $B \neq 0$ (unless $A = B$ in (16)). This together with the assumption that R is a covariance with $R(0) = 1$ leads to the alternatives (i), (ii) and (iii) of the theorem (but with the assertion in (ii) that $T \leqq \pi/a$ still unproved).

So far we have supposed that $-1 < R(t) < 1$ for $0 < t < T$. Assume, however, that $|R(s)| = 1$ for some $0 < s < T$. Then, if $t-u = s$

$$\begin{vmatrix} (X_t, X_t)(X_t, X_u) \\ (X_u, X_t)(X_u, X_u) \end{vmatrix} = 1-R^2(s) = 0,$$

so $\{X_t, X_u\}$ is a linearly dependent set, whence

$$E\{X_r \mid X_t, X_u\} = E\{X_r \mid X_t\} = R(t-r)X_t \tag{18}$$

for any r and t, with $u = t+s$. Now suppose $0 < r < t < T-s$, and that R does not vanish in $[0, t]$. Then, by the reciprocal property,

$$X_r - R(t-r)X_t \perp X_{r'} \tag{19}$$

for all r' with $r < r' < t$. That is,

$$R(r'-r) = R(t-r)R(t-r').$$

Now let $r'\downarrow r$. We obtain $R^2(t-r) = 1$. Thus $R(v) = 1$ for all $0 \leqq v \leqq t$, which is impossible unless R is constantly equal to 1. This falls under case (iii), with $a = 0$.

Suppose now that (i) holds, that is, $R(t) = e^{-at}$ for $0 \leqq t \leqq T$. Then $\{X_t, 0 < t < T\}$ is a Markov process ([2], 233–234), hence reciprocal by virtue of Lemma 2. Any even continuous function which is equal to $e^{-a|t|}$ on $[-T, T]$ and which is convex and non-decreasing on $[0, \infty)$ is a covariance function, so (i) does not define a unique Gaussian process. The situation is different in case (ii). Here $R(t) = \cos at$, $0 < t < T$. Since $\cos at$ is an analytic function of t, $R(t) = \cos at$ for all t. It follows from [2], page 524, that

$$X(t) = Y\cos at + Z\sin at,$$

where Y and Z are independent, each with mean zero and variance 1. It is easy to verify that unless t and v differ by an integral multiple of π/a, $X(t)$ and $X(v)$ together determine $X(s)$ for all s, but that if $r-t=\pi/a$ and if $s<t<u<v$, then $E\{X_s \mid X_t, X_u, X_v\} \neq E\{X_s \mid X_t, X_v\}$. Thus $\{X_t, 0<t<T\}$ is reciprocal if $T \leqq \pi/a$, but not if $T>\pi/a$.

Suppose now that (iii) holds, so $R(t)=1-at$ if $0 \leqq t \leqq T$. The reciprocality of $\{X_t, 0<t<T\}$ is most easily established by a direct verification of (4); we leave this to the reader. This completes the proof of the theorem. It should be observed at this point, however, that a multitude of very different processes fall under case (iii). Perhaps the most interesting one is the one with the triangular covariance function

$$\begin{aligned} R(t) &= 1-|t|/T \qquad && |t| \leqq T \\ &= 0 && |t| > T; \end{aligned}$$

this is the process studied by Slepian [7] (see also page 349 of [6]). One of the saw-tooth covariances in figure 2 on page 480 of [3] defines a process which is reciprocal in any interval of length 1; the other defines one reciprocal in any interval of length 2. Suppose that f is any real characteristic function, and let R be the piecewise linear function with vertices $(nT, f(nT))$, $n=0, \pm 1, \pm 2, \cdots$. Then R is a covariance function ([3], page 610) and the corresponding stationary Gaussian process is reciprocal on $[0, T]$.

A corresponding classification for n-dimensional stationary Gaussian processes which are reciprocal on an interval would be of interest even from a purely functional-equation-theoretic viewpoint.

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