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MARKOV RENEWAL PROCESSES WITH AUXILIARY PATHS

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1. Introduction. A question of general interest in the theory of probability is that concerning the asymptotic behavior of a stochastic process. The purpose of the present paper is to investigate the ergodic properties of a class of stochastic processes characterized by the fact that the Markov property holds at an increasing sequence of stopping times $\{S_n\}$ called regeneration points.

In [9], [10] D. G. Kendall developed the analysis of such stochastic processes, which frequently occur in the theory of queues, by the method of the embedded Markov chain (MC). Replacing the continuous time parameter by the discrete parameter of an MC, he obtains results about the ergodic properties of these processes. In general, however, the limiting distribution of the embedded MC is not the limiting distribution of the original process. Therefore, the question of the relationship between these limiting distributions arises. After being solved in some special cases, this problem is now examined from a general point of view. An application of the present theory to the theory of queues will be given in a separate paper [15], where the system $M/G/1$ with state-dependent service times and the system $GI/M/1$ with state-dependent input will be dealt with.

A simple example for a stochastic process with regeneration points is a semi-Markov process (SMP) including as special cases Markov chains in discrete and continuous time. It is a fundamental structure of the class of stochastic processes studied in this paper that they are associated with an SMP. Therefore, results for SMP's by Pyke and Schaufele ([12]) turn out to be very useful.

By the same authors the concept of a Markov renewal process with auxiliary paths (MRPAP) was introduced, which is likely to include all stochastic processes with regeneration points arising from any practical situation. The name MRPAP is also used for the class of stochastic processes $(X_t, t \geqq 0)$ studied in the present paper although a slightly different definition is given. Pyke and Schaufele allow the sequence of regeneration points to have several accumulation points. This case will be excluded. On the other hand, for the most part of this paper we only require a weakened Markov property: If the value of the embedded SMP at a regeneration point is known, a statement of the past history of the SMP loses all its predictive value for the subsequent development of the MRPAP. However, for some purposes this class of processes is too large and a stronger Markov property is needed, which insures that the whole history of the MRPAP up to time S_n becomes irrelevant to its future development. MRPAP's enjoying this property are called MRPAP's in the strict sense.

2. Summary. Section 3 contains the definition of an MRPAP and some important remarks. In Section 4, a system of shift operators $\{\theta_n\}$ is introduced characterized

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by the property $\theta_n\{X_h \in \Gamma\} = \{X_{S_n+h} \in \Gamma\}$. In Section 5, some notation concerning the embedded SMP is listed and a solidarity theorem for the SMP is proved. Section 7 and Section 8 contain preliminaries of the main limit theorem. This theorem evaluates the limit of $P\{X_t \in \Gamma\}$ as $t \to \infty$ and is the subject of Section 9 where the SMP is assumed to be recurrent-positive. Section 10 contains some results in the recurrent-null and transient case. The results in Section 9 interrelate the limiting distribution of an MRPAP and that of the embedded MC. In a sequel [14] to this paper, the rate of convergence of the solution of a renewal equation will be investigated. The results are applied to $P\{X_t \in \Gamma\}$ where $(X_t, t \geqq 0)$ is an MRPAP. Section 11 contains two ratio limit theorems. In Section 12, the concept of an MRPAP in the strict sense is introduced. For functionals of these processes, a strong law of large numbers and a central limit theorem is given in the last two sections.

3. Definition of an MRPAP and notations. Suppose we are given:

(a) a probability space $(\Omega, \hat{\mathfrak{A}}, P)$;

(b) a sequence of random variables $\{S_n, n \in N\}$[1] defined on $(\Omega, \hat{\mathfrak{A}}, P)$ such that

$$0 \equiv S_0 \leqq S_1 < S_2 < \cdots < S_n < S_{n+1} < \cdots < \infty \equiv S_\infty;$$

(c) a family of random variables Y_0, Y_n $(n \in N)$, Y_∞ defined on $(\Omega, \hat{\mathfrak{A}}, P)$ with countable state spaces I_0, I, I_∞, respectively, such that $P(\bigcup_{0<n<\infty}\{Y_n = i\}) > 0$ $(i \in I)$;

(d) for each $i \in I$ a probability measure P_i defined on $(\Omega, \hat{\mathfrak{A}})$;

(e) a stochastic process $(Z_t, t \geqq 0)$ defined on $(\Omega, \hat{\mathfrak{A}}, P)$ with a state space $(\mathscr{Z}, \mathfrak{C})$. Let $\mathscr{Z}$ be a metric space and $\mathfrak{C}$ the Borel-field on $\mathscr{Z}$ and let $(Z_t, t \geqq 0)$ have right-continuous trajectories.

The random variables $S_n (n \in N)$ will be the regeneration points of the MRPAP and $(Y_n, n \in N)$ will be the embedded MC. Y_0, Y_∞, S_0, and S_∞ are introduced for technical reasons. $(Z_t, t \geqq 0)$ will be called the auxiliary path process. The value of $P_i(A)$ can be interpreted as the probability of the event A under the condition that $t = 0$ is a regeneration point and the embedded MC starts in i.

Let $n(t)$ count the number of regeneration points in $[0, t]$, i.e. $n(t) = \sup\{n \geqq 0; S_n \leqq t\}$.

Suppose that a strictly positive and finite random variable U_0' on $(\Omega, \hat{\mathfrak{A}}, P)$ is given and define, for every $t \geqq 0$,

$$\begin{aligned} U_t &= U_0' + t && \text{if } n(t) = 0; \\ &= t - S_{n(t)} && \text{if } 0 < n(t) < \infty; \\ &= t - \lim S_n && \text{if } n(t) = \infty. \end{aligned}$$

Set $X_t \equiv (Z_t, U_t, Y_{n(t)}) (t \geqq 0)$, $(\mathfrak{X}, \mathfrak{B}) \equiv (\mathscr{Z} \times R_1 \times I, \mathfrak{C} \otimes \mathfrak{B}_1 \otimes \mathfrak{P}(I))$,[2] $\mathfrak{A} \equiv \sigma(X_s, s \geqq 0)$[3] and $\sigma_n \equiv \sigma(Y_1, S_1, \cdots, Y_n, S_n)$.

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[1] Let N denote the set of the positive integers.

[2] We write R_1 for the set of real numbers and $\mathfrak{B}_1$ for the set of all Borel sets on R_1. $\mathfrak{P}(\cdots)$ denotes the set of all subsets on $\cdots$

[3] Let $\sigma(\cdots)$ be the σ-algebra generated by $\cdots$

DEFINITION 3.1. $X \equiv (X_t, t \geqq 0)$ is said to be an MRPAP, if

(E1) $P\{X_{S_n+h_1} \in B_1, \cdots, X_{S_n+h_m} \in B_m \mid \sigma_n\} = P_{Y_n}\{X_{h_1} \in B_1, \cdots, X_{h_m} \in B_m\}[\sigma_n, P]$ for all $h_i \geqq 0$, $B_i \in \mathfrak{B}$, $1 \leqq i \leqq m$, $m, n \in N$,

(E2) for each $\omega \in \Omega$ and each $n \in N$, an $\omega' \in \Omega$ can be found such that $X_u(\omega') = X_{S_{n(\omega)}+u}(\omega)(u \geqq 0)$.

REMARKS.

(i) (cf. Dynkin [6] page 79) (E2) only has a technical character. If (E2) does not already hold, it can be satisfied by enlarging the space Ω. Indeed, to each pair $n \in N$, $\omega \in \Omega$, there corresponds a function $\phi_{n,\omega}$ with values in $\mathfrak{X}$, given by the formula

$$\phi_{n,\omega}(u) = X_{S_{n(\omega)}+u}(\omega) \qquad (u \geqq 0).$$

Let Ω_1 denote the set of all such functions and put $\tilde{\Omega} = \Omega \cup \Omega_1$. Extend the function $X_u(\omega)$ to Ω by setting

$$X_u(\phi_{n,\omega}) = \phi_{n,\omega}(u) \qquad (u \geqq 0).$$

Set $\tilde{\mathfrak{A}} \equiv \{A \subset \tilde{\Omega}; A\Omega \subseteq \mathfrak{A}\}$ and define $\tilde{P}$, $\tilde{P}_i(i \in I)$ by the equality $\tilde{P}(A) = P(A\Omega)$, $\tilde{P}_i(A) = P_i(A\Omega)$ $(A \in \tilde{\mathfrak{A}})$. It is easy to see that $(X_t, t \geqq 0)$ now defines an MRPAP. (E2) ensures the existence of the operators θ_n introduced in the next section.

(ii) $S_n(n \in N)$ is a stopping time relative to X, i.e.

$$\{S_n \leqq u\} \in \mathfrak{A}_u \equiv \sigma(X_s, 0 \leqq s \leqq u), \qquad (u \geqq 0).$$

The proof is straightforward and therefore omitted.

(iii) X has right-continuous trajectories and is, therefore, strongly measurable (cf. Dynkin [6] page 98, [5] page 18). Hence, $X_{S_n+h}(h \geqq 0, n \in N)$, especially Y_n, is $\mathfrak{A}$-measurable (cf. Dynkin [5] page 89). Since I is countable, $P_{Y_n}(A)$ $(A \in \mathfrak{A})$ is σ_n-measurable. Later on we will be concerned with sets of the form $\{X_{S_\tau+h} \in B\}$, where τ is an $\mathfrak{A}$-measurable function with values in $N \cup \{\infty\}$. Until now $X_{S_\tau+h}$ is only defined on $\Omega_\tau = \{\tau < \infty\}$. To define $X_{S_\tau+h}$ everywhere, let X_∞ be a constant not contained in $\mathfrak{X}$. (Note that X_∞ is excluded from the stochastic process X.) This convention implies

$$\{X_{S_\tau+h} \in B\} \subset \Omega_\tau \quad \text{and} \quad \{X_{S_\tau+h} \in B\} \in \mathfrak{A} \qquad (h \geqq 0, B \in \mathfrak{B}).$$

(iv) (E1) still holds if P is replaced by P_i $(i \in I)$. The condition $P(\bigcup_{0<n<\infty}\{Y_n = i\}) > 0$ $(i \in I)$ will only be used to prove this fact. Therefore, the theorems proved for P also hold for P_i instead of P $(i \in I)$. The probability measures P_i have the characteristic properties:

$$P_i\{S_1 = 0\} = 1 \quad \text{and} \quad P_i\{Y_1 = i\} = 1 \qquad (i \in I).$$

(v) Upon setting $Z_t' = (Z_t, Y_{n(t)+1}, S_{n(t)+1} - t)$ one obtains an MRPAP $((Z_t', U_t, Y_{n(t)}), t \geqq 0)$ with the same given probability measures P_i. Hence, there is no loss of generality in assuming Z_t to contain $(Y_{n(t)+1}, S_{n(t)+1} - t)$.

The proofs of (4) and (5) are given in Section 5.

(vi) For any fixed $l \in N$, $h_i \geqq 0$, $1 \leqq i \leqq l$, set $Z_t'' \equiv (X_{t+h_1}, \cdots, X_{t+h_l})$. It is clear that $((Z_t'', U_t, Y_{n(t)}), t \geqq 0)$ is an MRPAP with the same probability measures P_i.

4. The operators θ_n. Most considerations collected in this section parallel those of Dynkin in [6] page 81, where a stronger property than (E2) is required.

In the formulation of (E1) a transition takes place from $\{X_{S_n+h} \in B\}$ to $\{X_h \in B\}$. Condition (E2) allows us to introduce a system of operators describing this transition. Set $\Phi \equiv \mathfrak{X}^{R+4}$ and $\mathfrak{A}^* \equiv X^{-1}(\mathfrak{P}(\Omega))$. One immediately sees that $\mathfrak{A}^*$ is the minimal system of subsets of Ω that contains all the sets $\{X_t \in \Gamma\}(t \geqq 0, \Gamma \subset \mathfrak{X})$ and is closed under the union and intersection of any number of sets, and under the operation of taking complements. Clearly, $\mathfrak{A}^*$ is a σ-algebra containing $\mathfrak{A}$. Put $\theta^t A = X^{-1} c_t^{-1} X(A)$ $(A \in \mathfrak{A}^*)$, where c_t is the shift of the functions $\phi \in \Phi$ defined by $(c_t\phi)(u) = \phi(t+u)$ $(u \geqq 0)$. θ^t is a mapping of $\mathfrak{A}^*$ in $\mathfrak{A}^*$ satisfying the following conditions.

(4.1) $$\theta^t(A-A') = \theta^t A - \theta^t A'; \qquad \theta^t \bigcup_\alpha A_\alpha = \bigcup_\alpha \theta^t A_\alpha, \qquad \theta^t \bigcap_\alpha A_\alpha = \bigcap_\alpha \theta^t A_\alpha.$$

$(A, A', A_\alpha \in \mathfrak{A}^*, A' \subset A$; α runs through an arbitrary set of values).

(4.2) $$\{S_n = t\} \cap \theta^t \{X_h \in \Gamma\} = \{S_n = t\} \cap \{X_{t+h} \in \Gamma\} \qquad (\Gamma \subset \mathfrak{X})$$

(4.3) $$\{S_n = s\} \cap \theta^s \theta^t A = \{S_n = s\} \cap \theta^{s+t} A \qquad (A \in \mathfrak{A}^*).$$

(4.1) is a consequence of the fact that for each $A \in \mathfrak{A}^*$ some $D \subset X(\Omega)$ can be found such that $A = X^{-1}(D)$ and that $D = X(X^{-1}D))$ $(D \subset X(\Omega))$. (4.2 and (4.3) follow by means of (E2).

Let θ_n be defined by $\theta_n A \equiv \bigcup_{t \geqq 0} \{S_n = t\} \cap \theta^t A$ $(A \in \mathfrak{A}^*)$. θ_n can not only be defined for any fixed $n \in N$ but also for any function τ on Ω that takes values in $N \cup \{\infty\}$: $\theta_\tau A = \bigcup_{n \in N} \{\tau = n\} \theta_n A$. From (4.1) and (4.2), the following properties of the operator θ_τ are easily derived, where $\Omega_\tau \equiv \{\tau < \infty\}$.

(4.4) $$\theta_\tau \Omega = \Omega_\tau, \qquad \theta_\tau(A - A') = \theta_\tau A - \theta_\tau A'$$
$$\theta_\tau(\bigcup_\alpha A_\alpha) = \bigcup_\alpha \theta_\tau A_\alpha, \qquad \theta_\tau(\bigcap_\alpha A_\alpha) = \bigcap_\alpha \theta_\tau A_\alpha$$

$(A, A', A_\alpha \in \mathfrak{A}^*, A' \subset A$; α runs through an arbitrary set of values.)

(4.5) $$\theta_\tau \{X_h \in \Gamma\} = \{X_{S_\tau + h} \in \Gamma\} \qquad (\Gamma \subset \mathfrak{X})$$

It is not difficult to see that (4.5) implies

(4.6) $$\theta_n S_p = S_{n+p-1} - S_n, \qquad \theta_n Y_p = Y_{n+p-1}$$

When (4.6) is combined with (4.3), one obtains

(4.7) $$\theta_n \circ \theta_p = \theta_{n+p-1}$$

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[4] Let R_+ denote the set of the nonnegative real numbers.

Let $f(\omega)$ be an $\mathfrak{A}^*$-measurable function. We define $\theta_\tau f$ by $\{\theta_\tau f = a\} = \theta_\tau\{f = a\}$. Thus, to every $\mathfrak{A}^*$-measurable function f, there corresponds a function $\theta_\tau f$ defined on Ω_τ, where the operator θ_τ satisfies the following conditions.

(4.8) $$\theta_\tau \mathrm{I}_A = \mathrm{I}_{\theta_\tau A} \qquad (A \in \mathfrak{A}^*)$$

(4.9) $$\theta_\tau A_m \uparrow \theta_\tau A \quad \text{if} \quad A_m \uparrow A \qquad (A_m, A \in \mathfrak{A}^*)$$

(4.10) $$\theta_\tau f \equiv 1 \quad \text{if} \quad f \equiv 1;$$
$$\theta_\tau(cf + dg) = c\theta_\tau f + d\theta_\tau g$$
$$\theta_\tau f \leqq \theta_\tau f \quad \text{if} \quad f \leqq g;$$
$$\theta_\tau f_m \to \theta_\tau f \quad \text{if} \quad f_m \to f;$$

(c, d are constants; f, g, f_m are $\mathfrak{A}^*$-measurable functions.)

5. Characteristic properties of an MRPAP. Throughout this paper, it is assumed that we are working with an MRPAP $X = (X_t, t \geqq 0)$.

LEMMA 5.1. *Let τ be a stopping time relative to $((Y_n, S_n), n \geqq 1)$ such that $P(\Omega_\tau) > 0$, where $\Omega_\tau \equiv \{\tau < \infty\}$.*
Then, for all $h_i \geqq 0$, $B_i \in \mathfrak{B}$, $1 \leqq i \leqq m$, $m, n \in N$,

$$P\{X_{S_\tau + h_1} \in B_1, \cdots, X_{S_\tau + h_m} \in B_m \mid \Omega_\tau \sigma_\tau\} = P_{Y_\tau}\{X_{h_1} \in B_1, \cdots, X_{h_m} \in B_m\} \qquad [\Omega_\tau \sigma_\tau, P]$$

where $\sigma_\tau \equiv \{A;\ A \in \sigma((Y_m, S_m), m \geqq 1), A\{\tau = n\} \in \sigma_n\ (n \in N)\}$.

PROOF. On making use of the fact that $\Omega_\tau \in \sigma_\tau$ and $\{X_{S_\tau + h} \in B\} \subset \Omega_\tau$ one obtains by (E1) for every $A \in \Omega_\tau \sigma_\tau$

$$\begin{aligned} P(A\{X_{S_\tau + h_1} \in B_1, \cdots, X_{S_\tau + h_m} \in B_m\}) &= \textstyle\sum_p P(A\{\tau = p\}\{X_{S_p + h_1} \in B_1, \cdots\}) \\ &= \textstyle\sum_p \int_{A\{\tau = p\}} P_{Y_p}\{X_{h_1} \in B_1, \cdots\}\, dP \\ &= \textstyle\sum_p \int_{A\{\tau = p\}} P_{Y_\tau}\{X_{h_1} \in B_1, \cdots\}\, dP = \int_A P_{Y_\tau}\{X_{h_1} \in B_1, \cdots\}\, dP \end{aligned}$$

and the proof is complete.

In the following theorem, the concept of the post-σ-algebra ${}_{S_\tau}\mathfrak{A}$ of S_τ defined by ${}_{S_\tau}\mathfrak{A} \equiv \sigma(X_{S_\tau + t}, t \geqq 0)$ is needed.

THEOREM 5.1. *Under the assumptions of Lemma 5.1, for every $\mathfrak{A}$-measurable function f, $\theta_\tau f$ is $\Omega_\tau \cap {}_{S_\tau}\mathfrak{A}$-measurable and $E(\theta_\tau f \mid \Omega_\tau \sigma_\tau) = E_{Y_\tau} f \equiv \int f\, dP_{Y_\tau} [\Omega_\tau \sigma_\tau, P]$ if, in addition, f is P-integrable.*

The proof of this theorem by means of (4.5) and Lemma 5.1 is standard and therefore omitted.

Now, the proofs of Remark 4 and Remark 5 can be given. To prove Remark 4, we need the following relationship which is a consequence of Theorem 5.1 and (4.6).

$$\theta_n D \in {}_{S_n}\mathfrak{A} \cap \sigma_{n+p-1} \quad \text{if} \quad D \in \sigma_p.$$

We now show that

$$P_i(\theta_p A \mid \sigma_p) = P_{Y_p}(A)[\sigma_p, P_i] \qquad (A \in \mathfrak{A}).$$

Choose a fixed n such that $P\{Y_n = i\} > 0$; then, for every $D \in \sigma_p$, using (4.6) and (4.7),

$$\begin{aligned} &P_i(D\theta_p A)P\{Y_n = i\} \\ &= \int_{\{Y_n=i\}} P_{Y_n}(D\theta_p A)\,dP = \int_{\{Y_n=i\}} P(\theta_n(D\theta_p A) \mid \sigma_n)\,dP \\ &= P(\theta_n(\{Y_1 = i\}D)\theta_{n+p-1}A) = \int_{\theta_n\{Y_1=i\}D} P_{Y_{n+p-1}}(A)\,dP \\ &= \int_{\{Y_n=i\}} I_{\theta_n D} P_{\theta_n Y_p}(A)\,dP = \int_{\{Y_n=i\}} E(\theta_n(I_D P_{A_p}(A)) \mid \sigma_n)\,dP \\ &= \int_{\{Y_n=i\}} \int I_D P_{Y_p}(A)\,dP_{Y_n}\,dP = P\{Y_n = i\} \int_D P_{Y_p}(A)\,dP_i \end{aligned}$$

which completes the proof. The remaining statements $P_i\{Y_1 = i\} = 1$ and $P_i\{S_1 = 0\} = 1$ are proved by a similar argument. Choosing n such that $P\{Y_n = i\} > 0$, we have $P\{Y_n = i\}P_i\{S_1 = 0, Y_1 = i\} = P\{Y_n = i\}P_i\{U_0 = 0, Y_1 = i\} = \int_{\{Y_n=i\}} P(\theta_n\{U_0 = 0, Y_1 = i\} \mid \sigma_n)\,dP = P\{Y_n = i, U_{S_n} = 0\} = P\{Y_n = i\}$.

To prove Remark 5, it will be enough to prove (E1) with respect to $(X_t' \equiv (X_t, Y_{n(t)+1}, S_{n(t)+1} - t), t \geqq 0)$. On making use of the following relationship, which is easily verified, $(\theta_n(S_{n(h)+1} - h),\ \Theta_n Y_{n(h)+1}) = (S_{n(S_n+h)+1} - S_n + h,\ Y_{n(S_n+h)+1})$ we have for all $h_i \geqq 0$, $B_i \in \mathfrak{B} \times \mathfrak{B}(I) \times \mathfrak{P}_1$ $(1 \leqq i \leqq m,\ m, n \in N)$

$$\{X'_{S_n+h_1} \in B_1, \cdots, X'_{S_n+h_m} \in B_m\} = \theta_n\{X'_{h_1} \in B_1, \cdots, X'_{h_m} \in B_m\}.$$

Now (E1) follows from Theorem 5.1.

The next theorem shows an important structure of an MRPAP, namely that $Y_{n(t)}$ is an SMP. Contrary to the definition of Pyke in [11], we allow S_1 to be positive on a set having positive measure.

THEOREM 5.2. *$Y_{n(t)}$ is an* SMP *and, for all $n \in N$, $k \in I$, $u \geqq 0$,*

$$P\{Y_{n+1} = k, S_{n+1} - S_n \leqq u \mid \sigma_n\} = P_{Y_n}\{Y_2 = k, S_2 \leqq u\} \equiv Q_{Y_n k}(u)\,[\sigma_n, P].$$

The proof immediately follows from (4.6) and Theorem 5.1. A simple, but important consequence of Theorem 5.2 is the following

COROLLARY. *$(Y_n, n \geqq 1)$ is a Markov chain with the stationary transition probabilities $p_{ij} = Q_{ij}(+\infty)$. $(Y_n, n \geqq 1)$ is said to be an embedded* MC.

6. Notations and a theorem on SMP's. The notation used here is similar to Pyke and Schaufele's in [11] and [12] to whose papers we refer the reader for a more detailed treatment of the quantities defined below.

For every $i, j \in I$, $C \subset I$, set $\tau_{C1} \equiv \inf\{m \geqq 1; Y_m \in C\}$, $\tau_{Cn+1} = \inf\{m; m > \tau_{Cn}, Y_m \in C\}$ where $\inf \emptyset = \infty$, $T_{Cn} \equiv S_{\tau_{Cn}}$, $T_{C0} = 0$, $G_{ii}(t) \equiv P_i\{T_{i2} \leqq t\}$ (note that $P_i\{T_{i1} = 0\} = 1$) $K_C(t) \equiv P\{T_{C1} \leqq t\}$, $K_{jC}(t) \equiv P_j\{T_{C1} \leqq t\}$, (if $C = \{i\}$, C is to be replaced by i), $G_{ji} \equiv K_{ji}$ $(i \neq j)$, (note that $K_{ii}(t) = 1$ $(t \geqq 0)$). Thus, G_{ii} is the distribution function of the first return time to i of the SMP and G_{ji} $(i \neq j)$ is the distribution function of the first entrance time from j to i of the SMP.

Further, set $n_i(t) \equiv \sup\{n \geqq 0, T_{in} \leqq t\}$, $M_i(t) \equiv En_i(t)$, $M_{ji}(t) \equiv E_j n_i(t)$, ${}_kG_{ji}(t) \equiv P_j\{T_{i(1+\delta_{ij})} \leqq t,\ T_{i(1+\delta_{ij})} < T_{k(1+\delta_{kj})}\}$. Consequently ${}_kG_{ji}(t)$ is the distribution function of the first entrance (or return time, if $i = j$) under the taboo k.

Finally set

$$
\begin{aligned}
{}_kM_{ji}(t) &\equiv \textstyle\sum_{n=1}^{\infty} P_j\{T_{in} \leqq t,\ T_{in} < T_{k(1+\delta_{kj})}\} \\
F_{ij}(t) &\equiv P_i\{S_2 \leqq t \mid Y_2 = j\} \qquad \text{if} \qquad p_{ij} > 0; \\
&\equiv I_{[1,\infty)}(t) \qquad\qquad \text{otherwise.} \\
H_i(t) &\equiv P_i\{S_2 \leqq t\} = \textstyle\sum_j p_{ij} F_{ij}(t) \\
Q_{ij}(t) &\equiv p_{ij} F_{ij}(t) \\
b_{ij} &\equiv \textstyle\int t\, dF_{ij}(t), \qquad \eta_i \equiv \int t\, dH_i(t) \\
\mu_{ij} &\equiv \textstyle\int t\, dG_{ij}(t) \qquad \text{if} \quad G_{ij}(+\infty) = 1 \\
&\equiv \infty \qquad\qquad \text{otherwise.}
\end{aligned}
$$

The notations concerning the embedded MC are those of Chung in [4]. A state $i \in I$ is said to enjoy some property if it does so with respect to the embedded MC. The following relationships are obvious:

$$f_{ij}^* = G_{ij}(+\infty), \qquad {}_jf_{ii}^* = {}_jG_{ii}(+\infty) \quad \text{and} \quad {}_if_{ij}^* = {}_iG_{ij}(+\infty).$$

If $P\{Y_n = i \text{ infinitely often}\} = 1$, $\{T_{in} - T_{in-1},\ n \in N\}$ forms a general renewal process (cf. Pyke [11] page 1240). For the definition of a general renewal process we refer the reader to [16] page 20). The property $P\{Y_n = i \text{ infinitely often}\} = 1$ is equivalent to $P(\bigcap_n \{T_{in} < \infty\}) = 1$. If $P\{T_{im} = \infty\} > 0$ for some $m \in N$, $\{T_{in} - T_{in-1},\ n \in N\}$ no longer forms a sequence of independent random variables, since $\{T_{in} = \infty\} \subset \{T_{in+1} = \infty\}$. However, the following statements are always valid.

LEMMA 6.1. *For arbitrary* $n \in N$, $B_m \in [0, \infty)\mathfrak{B}_1$ $(1 \leqq m \leqq n)$, $i \in I$,

$$P\Big(\bigcap_{m=1}^{n} \{T_{im} - T_{im-1} \in B_m\}\Big) = \prod_{m=1}^{n} P\{T_{im} - T_{im-1} \in B_m\}. \tag{6.1}$$

$$P\{T_{in} \leqq t\} = K_i * G_{ii}^{(n-1)}(t), \qquad M_i(t) = \textstyle\sum_{n=1}^{\infty} K_i * G_{ii}^{(n-1)}(t), \tag{6.2}$$

$$M_i(t) < \infty \ (t \geqq 0), \qquad \sup\,(M_i(t+1) - M_i(t)) < \infty. \tag{6.3}$$

Let $*$ denote the convolution symbol and let $G^{(n)}$ be the n-fold convolution of G with itself. As usual, $G^{(0)}$ is defined as the atomic distribution concentrated at the origin.

PROOF OF LEMMA 6.1. The following relationship is easily verified.

$$\theta_{\tau_{in}} T_{ip} = T_{in+p-1} - T_{in} \qquad \text{on} \quad \{T_{in} < \infty\} \tag{6.4}$$

Since $\{T_{in} - T_{in-1} \in B_n\} = \theta_{\tau_{in-1}}\{T_{i2} \in B_n\}$ and $\bigcap_{m=1}^{n-1} \{T_{im} - T_{im-1} \in B_m\} \in \Omega_{\tau_{in-1}} \sigma_{\tau_{in-1}}$, Theorem 5.1 entails (6.1). (6.2) is a consequence of (6.1) and (6.3) follows by (6.2) from the Renewal theory.

Now we want to prove a solidarity theorem for the SMP, which is an extension of Theorem 2 in [4] page 13). For that purpose, we need the concept of an arithmetic distribution function with span λ, which is used here in the sense of Feller [8] page 136. Feller's definition is extended to arbitrary mass functions (i.e. non-decreasing right-continuous functions) in an obvious way. For any function G, let G^* be its Laplace–Stieltjes transform, whenever G^* is well defined. If G is a mass function inducing a finite measure, $G^*(-i\cdot 2\pi/\lambda)=G^*(0)$ $(\lambda\neq 0)$ is equivalent to the property that all points of increase of G are among $0, \pm\lambda, \pm 2\lambda, \cdots$ (cf. Feller [8] page 475). The span of a non-arithmetic mass function is defined to be zero.

THEOREM 6.1. *For two states $i, j\in I$ in the same class, G_{ii} and G_{jj} have the same span. Especially, if G_{ii} is non-arithmetic, then this is true for all states in the class of i.*

PROOF. Suppose that G_{ii} has span λ $(0<\lambda<\infty)$. We shall show that, for every state $j\neq i$ in the class of i, G_{jj} has the same span λ. Since $G_{jj}(0)<1$ and $G_{jj}(+\infty)>0$, there exist only two possibilities: either G_{jj} is non-arithmetic or G_{jj} is arithmetic with a finite span. The following identities are used (cf. Cheong [3] page 123, Pyke, Schaufele [12] page 1752)

$$G_{ii}^*(s) = {}_jG_{ii}^*(s)+{}_iG_{ij}^*(s)G_{ji}^*(s) \tag{6.5}$$

$$G_{ji}^*(s) = {}_jG_{ji}^*(s)+{}_iG_{jj}^*(s)G_{ji}^*(s). \tag{6.6}$$

Now suppose that $G_{ii}^*(s)=G_{ii}^*(0)$, where $s=-i\cdot 2\pi/\lambda$. (6.5) implies that

$$\begin{aligned}G_{ii}^*(0) &= {}_jG_{ii}^*(0)+{}_iG_{ij}^*(0)G_{ji}^*(0)\\ &= {}_jG_{ii}^*(s)+{}_iG_{ij}^*(s)G_{ji}^*(s).\end{aligned}$$

Since for every mass function G inducing a finite measure the inequality $|G^*(s)|\leqq G^*(0)$ on $\operatorname{Re}(s)=0$ is valid, we have ${}_jG_{ii}^*(s) = {}_jG_{ii}^*(0)$ and ${}_iG_{ij}^*(s)G_{ji}^*(s) = {}_iG_{ij}^*(0)G_{ji}^*(0)$

hence

$${}_iG_{ij}^*(s) = {}_iG_{ij}^*(0)\,e^{-i\theta},\qquad G_{ji}^*(s) = G_{ji}^*(0)\,e^{i\theta}\qquad\text{for some}\quad \theta\in[0,2\pi).$$

(6.6) implies that

$$G_{ji}^*(0) = {}_jG_{ji}^*(s)\,e^{-i\theta}+{}_iG_{jj}^*(s)G_{ji}^*(0).$$

The same sort of argument as was used above will show that

$${}_jG_{ji}^*(s) = {}_jG_{ji}^*(0)\,e^{i\theta}\quad\text{and}\quad {}_iG_{jj}^*(s) = {}_iG_{jj}^*(0).$$

An interchange of i and j in (6.6) yields

$$G_{ij}^*(s) = \frac{{}_iG_{ij}^*(s)}{1-{}_jG_{ii}^*(s)} = \frac{{}_iG_{ij}^*(0)\,e^{-i\theta}}{1-{}_jG_{ii}^*(0)} = G_{ij}^*(0)\,e^{-i\theta},$$

because ${}_jG_{ii}^*(0) = {}_jf_{ii}^*<1$. Finally, by (6.5),

$$G_{jj}^*(s) = {}_iG_{jj}^*(0)+{}_jG_{ji}^*(0)\,e^{i\theta}G_{ij}^*(s)\,e^{-i\theta} = G_{jj}^*(0).$$

Therefore, the span of G_{jj} is an integral multiple of λ. An interchange of i and j in the above argument completes the demonstration.

7. Preliminaries of the main limit theorem. We introduce the following notations.

$$_iP_i(B, u) \equiv P_i\{X_u \in B, T_{i2} > u\}, \qquad \psi_i(B, u) \equiv P_i\{X_u \in B, S_2 > u\},$$
$$\eta_i(B) \equiv E_i(\int I_{\{X_s B, 0 \leqq s < S_2\}} ds).$$

By the following Lemma 7.1, $I_{\{X_s \in B, 0 \leqq s < S_2\}}$ is an $\mathfrak{A} \times [0, \infty)\mathfrak{B}_1$-measurable function of (ω, s). Hence, $\int I_{\{X_s \in B, 0 \leqq s < S_2\}} ds$ is an $\mathfrak{A}$-measurable random-variable. $\eta_i(B)$ is the expected amount of time spent in B by X from time S_1 until time S_2, given $S_1 = 0$, $Y_1 = i$.

LEMMA 7.1. *For every* $n \in N$, $i \in I$, $B \in \mathfrak{B}$

$$P\{X_t \in B, T_{in+1} > t \mid \Omega_{\tau_{in}} \sigma_{\tau_{in}}\} = {}_iP_i(B, t - T_{in})[\Omega_{\tau_{in}} \sigma_{\tau_{in}}, P] \tag{7.1}$$

on the set where $T_{in} \leqq t$ *and*

$$P\{X_t \in B, S_{n+1} > t \mid \sigma_n\} = \psi_{Y_n}(B, t - S_n)[\sigma_n, P] \tag{7.2}$$

on the set where $S_n \leqq t$. $I_{\{X_t(\omega) \in B, T_{i2} > t\}}$ *and* $I_{\{X_t(\omega) \in B, S_2 > t\}}$ *are* $\mathfrak{A} \times [0, \infty)\mathfrak{B}_1$-*measurable functions of* (ω, t).

Since the stochastic processes $((X_t, I_{\{T_{in+1} > t\}}), t \geqq 0)$ and $((X_t, I_{\{S_{n+1} > t\}}), t \geqq 0)$ are right-continuous, (7.1) and (7.2) can be verified by the same sort of argument as is used in the proof of Theorem 5.7 from Dynkin in [5] page 107. The measurability of the functions specified above can be proved simultaneously with no additional labor.

If i is a recurrent state, $\{T_{in+1} - T_{in}, n \in N\}$ is a Renewal process under the probability measure P_i. Therefore, $P_i\{\sum (T_{in+1} - T_{in}) = \infty\} = P_i\{\lim S_n = \infty\} = 1$ (cf. Pyke [11] page 1240). This result can be slightly generalized.

LEMMA 7.2. *For every recurrent class* C,

$$P\{T_{C1} < \infty, \lim S_n < \infty\} = 0.$$

PROOF. Because of $\{T_{C1} < \infty\} = \{\tau_{C1} < \infty\}$ it suffices to show $P\{\tau_{C1} < \infty, \lim S_n = \infty\} = P\{\tau_{C1} < \infty\}$. Now τ_{C1} is a stopping time relative to $(Y_n, n \in N)$. Hence, on making use of Theorem 5.1,

$$\begin{aligned} &P\{\tau_{C1} < \infty, \lim_{n\to\infty} S_n = \infty\} \\ &\quad = P\{\tau_{C1} < \infty, \lim_{n\to\infty}(S_{n+\tau_{C1}-1} - S_{\tau_C}) = \infty\} \\ &\quad = P(\{\tau_{C1} < \infty\}\theta_{\tau_{C1}}\{\lim S_n = \infty\}) = \int_{\{\tau_{C1} < \infty\}} P_{Y_{\tau_{C1}}}\{\lim S_n = \infty\} \, dP \\ &\quad = \int_{\{\tau_{C1} < \infty\}} 1 \, dP. \end{aligned}$$

Now we want to summarize some relationships used for the proof of the main limit theorem.

$$M_i(t) = \sum_{n=1}^{\infty} P\{S_n \leqq t, Y_n = i\}. \tag{7.3}$$

$$_iM_{ij}(t) = \sum_{n=1}^{\infty} P_i\{S_n \leqq t, Y_n = j, Y_v = i \, (1 < v \leqq n)\}. \tag{7.4}$$

$$\int \psi_i(B, u) \, du = \eta_i(B). \tag{7.5}$$

$$_iP_i(B, t) = \sum_j \int_0^t \psi_j(B, t-u) \, d_i M_{ij}(u). \tag{7.6}$$

If $P_i\{\lim S_n = \infty\} = 1$ then

(7.7) $$\int {}_iP_i(B, u)\, du = \sum_j {}_iM_{ij}(+\infty)\eta_j(B).$$

If i is a recurrent state then

(7.8) $${}_iM_{ij}(+\infty) = {}_ip_{ij}^* \equiv e_{ij}.$$

For every state $j \in I$

(7.9) $$P\{X_t \in B\} = P\{X_t \in B,\ T_{j1} > t\} + \int_0^t {}_jP_j(B, t-u)\, dM_j(u).$$

If C is a recurrent class then

(7.10) $$P\{X_t \in B\} = P\{X_t \in B,\ T_{C1} > t\} + \sum_{j \in C} \int_0^t \psi_j(B, t-u)\, dM_j(u).$$

(7.3) follows from

$$M_i(t) = \sum_{n=1}^\infty EI_{[0,t]}(T_{in}) = E\sum_{n=1}^\infty I_{[0,t]}(S_n) I_{\{i\}}(Y_n)$$
$$= \sum_{n=1}^\infty P\{S_n \leqq t,\ Y_n = i\}.$$

A similar argument proves (7.4), which implies (7.8). (7.5) can be deduced from Lemma 7.1 and Fubini's theorem.

Now suppose that $P_i\{\lim S_n = \infty\} = 1$. By (7.2) and (7.4) we have

$${}_iP_i(B, t) = P_i\{X_t \in B,\ T_{i2} > t\}$$
$$= \sum_{n=1}^\infty \sum_j P_i\{X_t \in B,\ S_n \leqq t < S_{n+1},\ Y_n = j,\ T_{i2} > t\}$$
$$= \sum_n \sum_j P_i\{X_t \in B,\ S_n \leqq t < S_{n+1},\ Y_n = j,\ Y_v \neq i\ (1 < v \leqq n)\}$$
$$= \sum_n \sum_j \int_{\{S_n \leqq t,\ Y_n = j,\ Y_v \neq i (1 < v \leqq n)\}} P_i\{X_t \in B,\ t < S_{n+1} \mid \sigma_n\}\, dP$$
$$= \sum_n \sum_j \int_0^t \psi_j(B, t-u)\, d_u P_i\{S_n \leqq u,\ Y_n = j,\ Y_v \neq i\ (1 < v \leqq n)\}$$
$$= \sum_j \int_0^t \psi_j(B, t-u)\, d_i M_{ij}(u).$$

(7.7) is a consequence of (7.6). Making use of (7.1), we obtain

$$P\{X_t \in B,\ T_{j1} \leqq t\} = \sum_{n=1}^\infty P\{X_t \in B,\ T_{jn} \leqq t < T_{jn+1}\}$$
$$= \sum_n \int_{\{T_{jn} \leqq t\}} P\{X_t \in B,\ T_{jn+1} > t \mid \Omega_{\tau_{jn}} \sigma_{\tau_{jn}}\}\, dP$$
$$= \sum_n \int_0^t {}_jP_j(B, t-u)\, d_u P\{T_{jn} \leqq u\} = \int_0^t {}_jP_j(B, t-u)\, dM_j(u),$$

this is (7.9). Finally suppose that C is a recurrent class. Then, by (7.2) and (7.3), $P\{X_t \in B,\ T_{C1} \leqq t\} = \sum_{j \in C} \sum_{n=1}^\infty P\{X_t \in B,\ S_n \leqq t < S_{n+1},\ Y_n = j\}$. The rest of the proof of (7.10) goes through as in the proof of (7.6).

8. The Key Renewal theorem. In this section, the known results about the Key Renewal theorem are summarized. The version of this theorem quoted here is obtained by an application of the extension of Wiener's Tauberian theorem for positive measures due to Benes [1] to the Renewal functions $M_i(t)$. By use of (6.5) and

(8.1) $$K_i(t) = \int_0^t (1 - G_{ii}(t-u))\, dM_i(u)$$

it is easily shown that $M_i(t)$ $(i \in I)$ satisfies the hypothesis of Wiener's theorem ([1] page 4) if G_{ii} is non-arithmetic and $\mu_{ii} < \infty$. Assume throughout this section that these conditions are satisfied for some fixed $i \in I$. We shall need the concept of strong regularity ([1] pages 13, 17).

A Borel set $B \subset [0, \infty)$ is said to be strongly regular with respect to M_i if for every $\varepsilon > 0$ there exist a compact $C \subset B$ and an open $U \supset B$ such that for all sufficiently large $t \int_0^t I_{U-C}(t-u)\, dM_i(u) < \varepsilon$. Let K_{M_i} denote the set of all Borel measurable functions f such that, for every $k > 0$ and $\varepsilon > 0$, there exist f^+ and f^- defined on $[0, k]$, with the properties

(i) $f^- \leqq f \leqq f^+$
(ii) $\int_0^k (f^+ - f^-)\, du < \varepsilon$
(iii) f^+ and f^- are of the form $\sum_{j=1}^n b_j I_{B_j}$, where the B_j are strongly regular with respect to M_i. There exist two simple criterions for strong regularity.

LEMMA 8.1. *Every bounded Borel set whose frontier has Lebesgue measure zero is strongly regular with respect to M_i. Hence, every Riemann integrable function defined on* $[0, \infty)$ *is an element of K_{M_i}. If G_{ii} possesses an absolutely continuous component, every Borel set $B \subset [0, \infty)$ is strongly regular with respect to M_i. Therefore, every Borel measurable function bounded in each finite interval is an element of K_{M_i}.*

The first part of Lemma 8.1 is proved in [1] by Benes and the second part is a consequence of Theorem 1 in [16] of Smith.

The definition of K_{M_i} is justified by the following theorem of Benes ([1] page 18).

THEOREM 8.1. *If $f \in K_{M_i}$ satisfies the condition $\sum_n \sup_{n \leqq u < n+1} |f(u)| < \infty$ then*

$$\lim_{t\to\infty} \int_0^t f(t-u)\, dM_i(u) = \frac{K_i(+\infty)}{\mu_{ii}} \int_0^\infty f(u)\, du.$$

9. The main limit theorem. Define $P(B, t) = P\{X_t \in B\}$, $P_i(B, t) = P_i\{X_t \in B\}$ and $\pi(B) = \lim_{t\to\infty} P(B, t)\, (B \in \mathfrak{B}, i \in I)$.

THEOREM 9.1. *If*
(i) *$C \subset I$ is a recurrent class,*
(ii) *$\mu_{jj} < \infty$ for some $j \in C$,*
(iii) *G_{jj} is non-arithmetic for some $j \in C$,*
(iv) *$\lim_{t\to\infty} P\{X_t \in B, T_{C_1} = \infty\} = 0$ for some fixed $B \in \mathfrak{B}$, and if either*
(v.a) *$\psi_j(B, \cdot) \in K_{M_j}$ for all $j \in C$ or*
(v.b) *${}_jP_j(\mathscr{B}, \cdot) \in K_{M_j}$ for some $j \in C$,* then $\pi(B)$ exists and

$$\pi(B) = K_C(+\infty) \frac{\sum_{j\in C} e_{kj} \eta_j(B)}{\sum_{j \in C} e_{kj} \eta_j}$$

where $$\sum_{j\in C} e_{kj}\eta_j < \infty \qquad (k \in C).$$

Evidently, the form of the limit $\pi(B)$ is that of limits of functionals of the embedded MC (cf. Chung [4] **1** Section 15). In applications of this theorem, it is much easier to compute $\psi_j(B, \cdot)$ than to compute ${}_jP_j(B, \cdot)$. Hence, in most cases, one will appeal to condition (v.a). But in certain processes, it may be known that G_{jj} has an absolutely continuous component and, therefore, the alternative condition (v.b) is always satisfied.

Pyke and Schaufele prove in [12] page 1459 under different conditions that $\pi(B)$ is the unique positive stationary measure for X. On the one hand Pyke and Schaufele assume $(Y_n, n \in N)$ to be an irreducible MC (that implies $K_C(+\infty) = 1$) and require a more restrictive condition than (E1). On the other hand no regularity condition has to be imposed and the arithmetic case and the non-arithmetic case need not to be distinguished.

Theorem 9.1 yields as special case the result of Fabens [7] in an example of queuing theory.

It is easily seen that an MRPAP is an equilibrium process in the sense of Smith ([16] page 14). Hence, the proof of the existence of $\pi(B)$ under condition (v.b) is similar to that given by Smith [16] and Benes [1] when dealing with the analogous problem of equilibrium processes.

PROOF OF THEOREM 9.1. By (7.7) and (7.8), where $B = \mathfrak{X}$, we have

$$\mu_{kk} = \textstyle\sum_{j \in C} e_{kj} \eta_j = e_{kl} \mu_{ll} \qquad (l, k \in C) \tag{9.1}$$

(cf. Pyke, Schaufele [12] page 1756), since $e_{kj} = 0$ if $j \in C$, $k \in C$,

$$0 < e_{kl} < \infty \quad \text{and} \quad e_{kj} = e_{kl} e_{lj} \qquad (j, k, l \in C) \tag{9.2}$$

(cf. Chung [4] pages 47, 51). For that reason and with reference to Theorem 6.1, conditions (ii) and (iii) are satisfied for all $j \in C$.

(a) At first suppose that (v.a) is valid. By assumption, it suffices to consider $P\{X_t \in B, T_{C1} \leqq t\}$. Now

$$P\{X_t \in B, T_{C1} \leqq t\} = \textstyle\sum_{j \in C} \int_0^t \psi_j(B, t-u)\, dM_j(u) \tag{7.10}$$

especially

$$P\{Y_{n(t)} = j, T_{C1} \leqq t\} = \textstyle\int_0^t \psi_j(\mathfrak{X}, t-u)\, dM_j(u) = \int_0^t (1 - H_j(t-u))\, dM_j(u).$$

From (9.1) and condition (ii), it follows that

$$\eta_j = \textstyle\int_0^\infty (1 - H_j(u))\, du \qquad (j \in C).$$

Considering that

$$K_j(+\infty) = K_C(+\infty) \qquad (j \in C)$$

$$\psi_j(B, t) \leqq \psi_j(\mathfrak{X}, t) \qquad (t \geqq 0)$$

and making use of Lemma 8.1 and Theorem 8.1 of Benes, we have $\lim_{t \to \infty} P\{Y_{n(t)} = j, T_{C1} \leqq t\} = K_C(+\infty)\eta_j/\mu_{jj}$ (cf. [16] page 20) and $\lim_{t \to \infty} \int_0^t \psi_j(B, t-u)\, dM_j(u) = K_C(+\infty)\eta_j(B)/\mu_{jj}$.

Then, by use of (9.1) and (9.2),

$$\begin{aligned}\lim_{t \to \infty} \textstyle\sum_{j \in C} P\{Y_{n(t)} = j, T_{C1} \leqq t\} &= K_C(+\infty) \\ &= K_C(+\infty) \cdot \textstyle\sum_{j \in C} \eta_j/\mu_{jj} = \sum_{j \in C} \lim_{t \to \infty} P\{Y_{n(t)} = j, T_{C1} \leqq t\}.\end{aligned}$$

Hence, letting $t \to \infty$ in (7.10), the limit may be passed over the summation sign, and the proof is complete.

(b) Secondly suppose that condition (v.b) is satisfied. Since $P\{T_{C1} < \infty, T_{j1} = \infty\} = 0$, it is again sufficient to consider $P\{X_t \in B, T_{j1} \leqq t\}$. Now we recall

$$(7.9) \qquad P\{X_t \in B, T_{j1} \leqq t\} = \int_0^t {}_jP_j(B, t-u)\, dM_j(u).$$

On collecting the statements of Lemma 8.1, Theorem 8.1, (7.7), (7.8) and using the fact that ${}_iP_i(B, u) \leqq (1-G_{ii}(u))$ $(u \geqq 0)$, we obtain for every $k \in C$

$$\lim_{t\to\infty} P\{X_t \in B, T_{j1} \leqq t\} = K_C(+\infty) \textstyle\sum_{i \in C} e_{ji}\eta_i(B)/\mu_{jj}.$$

The relationships (9.1) and (9.2) complete the demonstration.

If class C is recurrent-positive, the taboo probabilities e_{ji} can be replaced by the stationary probabilities $\pi_j = 1/m_{jj}$. If in addition $\sup_{j \in C} \eta_j < \infty$, one has $\mu_{jj} < \infty$ $(j \in C)$. Suppose $\{B_n\}$ to be a set of disjoint elements of $\mathfrak{B}$ satisfying $\sum B_n = \mathfrak{X}$. Then $\sum \eta_j(B_n) = \eta_j$ $(j \in C)$. The following corollary summarizes these considerations.

COROLLARY 9.1. *If*

(i) $C \subset I$ *is a recurrent-positive class,*
(ii) $\sup_{j \in C} \eta_j < \infty$ *or* $\mu_{jj} < \infty$ *for some* $j \in C$,
(iii) G_{jj} *is non-arithmetic for some* $j \in C$,
(iv) $K_C(+\infty) = 1$,

and if $\{B_n\}$ *is a countable set of disjoint elements of* $\mathfrak{B}$ *such that* $\sum B_n = \mathfrak{X}$ *and for each n either*

(v.a) $\psi_j(B_n, \cdot) \in K_{M_j}$ *for all* $j \in C$, *or*
(v.b) ${}_jP_j(B_n, \cdot) \in K_{M_j}$ *for some* $j \in C$, *then*

$$\pi(B_n) = \frac{\sum_{j \in C} \pi_j \eta_j(B_n)}{\sum_{j \in C} \pi_j \eta_j} \qquad \textit{for every} \quad n$$

and $\sum \pi(B_n) = 1$.

There is a result for the arithmetic case which is analogous to Theorem 9.1. The proof runs on very similar lines and will be omitted. If G_{ii} is arithmetic with span λ', so is G_{jj} for all states j of the class of i, as has been proved in Theorem 6.1. This case is likely to arise in any application of the theory only if the values of the regeneration points are restricted to integral multiples of a positive number λ. For convenience, only this case will be considered. Therefore, if C is a class, there exists an integer ν such that G_{ii} has span $\nu\lambda$ $(i \in C)$. In order to formulate the theorem, we set

$$K_j(\infty, p) \equiv \textstyle\sum_{m=0}^{\infty} P\{T_{j1} = (m\nu+p)\lambda\} \qquad (0 \leqq p \leqq \nu-1),$$

$$\eta_j(B, \xi) \equiv E_j \textstyle\sum_{q=\xi \bmod \nu\lambda} I_{\{X_q \in B, 0 \leqq q < S_2\}} \qquad (\xi \in [0, \nu\lambda)).$$

THEOREM 9.2. *Suppose there exists a positive* λ *such that* $P(\bigcap_n \sum_m \{S_n = m\lambda\}) = 1$. *If*

(i) $C \subset I$ *is a recurrent class,*

(ii) $\mu_{jj} < \infty$ *for some* $j \in C$,

(iii) G_{jj} *is arithmetic with span* $\nu\lambda$ *for some* $j \in C$,

(iv) $\lim_{n\to\infty} P\{X_{n\nu\lambda+\xi} \in B,\ T_{C1} = \infty\} = 0$ *for some* $\xi \in [0, \nu\lambda)$, *then* $\pi(B, \xi) \equiv \lim_{n\to\infty} P\{X_{n\nu\lambda+\xi} \in B\}$ *exists and*

$$\pi(B, \xi) = \nu\lambda \frac{\sum_{j\in C} e_{kj} \sum_{p=0}^{\nu-1} K_j(\infty, p)\eta_j(B, \xi - p\lambda)}{\sum_{j\in C} e_{kj}\eta_j}$$

where $\sum_{j\in C} e_{kj}\eta_j < \infty$ $(k \in C)$.

An additional corollary can be formulated without difficulty as in the non-arithmetic case.

10. The recurrent-null and the transient case. In this section some results are obtained in case that $\mu_{jj} = \infty$ for some $j \in I$. The following lemma is easily derived from renewal theory and probably very well known.

LEMMA 10.1. *Suppose that* $\mu_{jj} = \infty$.

(a) *If* G_{ii} *is non-arithmetic, then* $\lim_{t\to\infty} \int_0^t k(t-u)\,dM_j(u) = 0$ *provided that* $\sum_n \sup_{n \leqq u \leqq n+1} |k(u)| < \infty$.

(b) *If* $M_j(t)$ *is a step-function such that all the jumps of* $M_j(t)$ *occur on the sequence* $\{n\lambda, n \in N\}$ *for some* $\lambda > 0$, *then* $\lim_{n\to\infty} \int_0^{n\lambda} k(t-u)\,dM_j(u) = 0$ *provided that* $\sum_n |k(n\lambda)| < \infty$.

On making use of (7.9), (8.1), and Lemma 10.1 we have

THEOREM 10.1. *Suppose that*

(i) $\mu_{jj} = \infty$ *for some fixed* $j \in I$,

(ii) $\lim_{t\to\infty} P\{X_t \in B,\ T_{j1} = \infty\} = 0$. *If*

(iii.a) G_{jj} *is non-arithmetic,*

(iv.a) $\sum_n \sup_{n\leqq u \leqq n+1} |{}_jP_j(B, u) - \rho(1 - G_{jj}(u))| < \infty$ *for some* $B \in \mathfrak{B}$ *and some* $\rho \in [0, 1]$, *then* $\lim_{t\to\infty} P\{X_t \in B\} = \rho K_j(+\infty)$. *If*

(iii.b) $P\{\bigcap_n \sum_m \{S_n = m\lambda\}) = 1$ *for some* $\lambda > 0$,

(iv.b) $\sum_n |{}_jP_j(B, n\lambda + \xi) - \rho(1 - G_{jj}(n\lambda))| < \infty$ *for some* $B \in \mathfrak{B}$ *and some* $\rho \in [0, 1]$, *then* $\lim_{n\to\infty} P\{X_{n\lambda+\xi} \in B\} = \rho K_j(+\infty)$.

Generally, it will be difficult to verify condition (iv) of Theorem 10.1. Therefore, a simple but useful criterion is given, the proof of which is trivial.

LEMMA 10.2. *Suppose that, for a fixed* $B \in \mathfrak{B}$, *a finite subset* J_B *of* I *exists such that either*

(i.a) $\{X_t \in B, n(t) > 0\} \subset \{Y_{n(t)} \in J_B, 0 < n(t) < \infty\}$

(ii.a) $\lim_{t\to\infty} P\{Y_{n(t)} = k,\ n(t) < \infty\} = 0$ *for each* $k \in J_B$ *or*

(i.b) $\{X_t \in B, n(t) > 0\} \subset \{Y_{n(t)+1} \in J_B, 0 < n(t) < \infty\}$

(ii.b) $\lim_{t\to\infty} P\{Y_{n(t)+1} = k,\ n(t) < \infty\} = 0$ *for each* $k \in J_B$ *then* $\lim_{t\to\infty} P\{X_t \in B\} = 0$.

This lemma has many applications in queuing theory. We will now enumerate some conditions implying (ii.a) or (ii.b).

THEOREM 10.2 (1) *If j is a transient state, then* $\lim_{t\to\infty} P\{Y_{n(t)} = j, n(t) < \infty\} = \lim_{t\to\infty} P\{Y_{n(t)+1} = j, n(t) < \infty\} = 0$. *Suppose that* (i) *$j$ is a recurrent state,* $\mu_{jj} = \infty$, *and either* (ii.a) *G_{jj} is non-arithmetic or* (ii.b) $P(\bigcap_n \sum_m \{S_n = m\lambda\}) = 1$ *for some* $\lambda > 0$.

(2) (cf. Smith [16] page 20). *If* (iii) $\eta_j < \infty$ *then* $\lim_{t\to\infty} P\{Y_{n(t)} = j, n(t) < \infty\} = 0$.

(3) *If* (iii) $\sum_k e_{jk} p_{kj} b_{kj} < \infty$ (*especially if* $\sup_k b_{kj} < \infty$), *then* $\lim_{t\to\infty} P\{Y_{n(t)+1} = j, n(t) < \infty\} = 0$.

PROOF. (1) Since $P\{Y_{n(t)(+1)} = j, n(t) < \infty\} \leqq P\{Y_n = j$ for some $n \geqq m\} + P\{S_m > t\}$, this term can be made arbitrarily small by choosing m large enough and by letting $t \to \infty$.

(2) is proved in [16] by Smith under condition (ii.a). If (ii.b) holds, then an analogous argument will show that $\lim_{m\to\infty} P\{Y_{n(m\lambda)} = j,\ n(m_\lambda) < \infty\} = 0$. The relationship $Y_{n(m\lambda+\xi)} = Y_{n(m\lambda)}$ (a.s.) ($\xi \in [0, \lambda)$) completes the demonstration of (2).

(3) Since $\{Y_{n(t)+1} = j, T_{j1} > t, n(t) < \infty\} \subset \{t < T_{j1} < \infty\}$ and, by Lemma 7.2, $P\{T_{j1} \leqq t, n(t) < \infty\} = P\{T_{j1} \leqq t\}$, it suffices to consider $P\{Y_{n(t)+1} = j, T_{j1} \leqq t\} = \int_0^t P_j\{Y_{n(t-u)+1} = j,\ T_{j2} > t-u\}\, dM_j(u)$. This identity is a consequence of (7.9). Since, by use of (7.6), $P_j\{Y_{n(t)+1} = j,\ T_{j2} > t\} = \sum_k \int_0^t p_{kj}(1-F_{kj}(t-u))\, d_j\, M_{jk}(u)$, an appeal to Lemma 10.1 and the same sort of argument used in the proof of (2) will prove statement (3).

11. Ratio limit theorems. In this section, we consider the expected amount of time which an MRPAP spends in certain sets. We will use the convention that a bar over a function of t will denote the integral over $[0, t]$ of the same function without bar. For example, $\overline{P(B, t)} = \int_0^t P(B, u)\, du$. The methods of proof used in this paper are similar to those used by Pyke and Schaufele when dealing with the analogous problems of MRP's. At first we quote here a lemma of Pyke and Schaufele ([12] page 1752).

LEMMA 11.1. *Let K be a mass function for which $K(t) = 0$ if $t < 0$ and* $\sup(K(t+1) - K(t)) < \infty$. *Then for any mass function F satisfying $F(t) = 0$ if $t < 0$ one has* $\lim_{t\to\infty} [K * F(t)]/K(t) = F(+\infty)$.

THEOREM 11.1. (1) *If C is a recurrent class such that $\mu_{jj} < \infty$ ($j \in C$) and* $\lim_{t\to\infty} P\{X_t \in B, T_{C1} = \infty\} = 0$, *then*

$$\lim_{t\to\infty} \overline{P(B, t)}/t = K_C(+\infty)\frac{\sum_{j\in C} e_{kj}\eta_j(B)}{\sum_{j\in C} e_{kj}\eta_j} \qquad (k \in C)$$

where $\sum_{j\in C} e_{kj}\eta_j < \infty$.

(2) *If* $\mu_{jj} = \infty$, $\lim_{t\to\infty} P\{X_t \in B,\ T_{j1} = \infty\} = 0$ *and* $\int_0^\infty |{}_iP_i(B, u) - \rho(1 - G_{ii}(u))| < \infty$ *for some* $\rho \in [0, 1]$, *then* $\lim_{t\to\infty} \overline{P(B, t)}/t = \rho K_j(+\infty)$.

PROOF. We recall from renewal theory that $\lim_{t\to\infty} M_j(t)/t = K_j(+\infty)/\mu_{jj}$. By assumption, it suffices to consider $P\{X_t \in B,\ T_{j1} \leqq t\}$. Making use of (7.9) we have

$$\int_0^t P\{X_u \in B,\ T_{j1} \leqq u\}\, du = [M_j *_j \overline{P_j(B,\cdot)}(t)].$$

By Lemma 11.1,

$$\lim_{t\to\infty} [M_j *_j \overline{P_j(B,\cdot)}(t)]/M_j(t) = \textstyle\sum_k e_{jk}\eta_k(B)$$

and the proof is complete, if $\mu_{jj} < \infty$. If $\mu_{jj} = \infty$ the assertion is proved by a similar argument and by (8.1).

We extend now the definition of e_{ij} including the case of nonrecurrence:

$$\begin{aligned} e_{ij} &\equiv {}_iM_{ij}(+\infty)/K_{ij}(+\infty) && \text{if } K_{ij}(+\infty) > 0, \\ &\equiv 0 && \text{otherwise,} \end{aligned}$$

especially $e_{ii} = 1$. If i, k are elements of the same class, we have

$$e_{ij} = e_{ik}\, e_{kj} \qquad (j \in I). \tag{11.1}$$

This identity is an immediate consequence of the following relationship

$$\lim_{t\to\infty} M_{jj}(t)/M_{ii}(t) = {}_iM_{ij}(+\infty)/K_{ij}(+\infty) \quad \text{if } K_{ij}(+\infty) > 0,\ (i, j \in I), \tag{11.2}$$

which has been proved by Pyke and Schaufele in [12] page 1754.

THEOREM 11.2. *If* $P_a\{S_n \to \infty\} = 1$ *for some* a *of the class* C, $\overline{P_i(B, +\infty)} > 0$, $\overline{P_j(A, +\infty)} > 0$, *for some* $i, j \in C$, $B, A \in \mathfrak{B}$, *and*

$$\left(\textstyle\sum_l {}_iM_{il}(+\infty)\eta_l(B),\ \sum_l {}_jM_{jl}(+\infty)\eta_l(A)\right) \neq (+\infty, +\infty),$$

then

$$\lim_{t\to\infty} \frac{\overline{P_i(B, t)}}{\overline{P_j(A, t)}} = \frac{\sum_l e_{kl} K_{i1}(+\infty)\eta_1(B)}{\sum_l e_{kl} K_{j1}(+\infty)\eta_1(A)} \qquad (k \in C)$$

PROOF. It can be easily verified that $P_k\{S_n \to \infty\} = 1$ for all $k \in C$. We can, therefore, show exactly as in the proof of (7.10) that $P_k(B, t) = \sum_l \int_0^t \psi_l(B, t-u)\, dM_{k1}(u)$ and hence $P_k(B, +\infty) = \sum_l M_{kl}(+\infty)\eta_l(B)$ $(k \in C, B \in \mathfrak{B})$. $\overline{P_i(B, +\infty)} > 0$ implies that $\sum_1 {}_iM_{i1}(+\infty)\eta_1(B) > 0$ where i and B can be replaced with j and A. By (7.9) $\overline{P_k(B, t)} = M_{kk} *_k \overline{P_k(B, \cdot)}(t)$ $(k \in C, B \in \mathfrak{B})$ and by (7.7) ${}_k\overline{P_k(B, +\infty)} = \sum_1 {}_kM_{k1}(+\infty)\eta_1(B)$ $(k \in C, B \in \mathfrak{B})$. Now, Theorem 11.2 is a consequence of Lemma 11.1, (11.1) and (11.2) applied to the identity

$$\frac{\overline{P_i(B, t)}}{\overline{P_j(A, t)}} = \frac{M_{ii} *_i \overline{P_i(B, \cdot)}(t)}{M_{ii}(t)} \frac{M_{ii}(t)}{M_{jj}(t)} \frac{M_{jj}(t)}{M_{jj} *_j \overline{P_j(A, \cdot)}(t)}.$$

12. MRPAP's in the strict sense. The class of processes defined in Section 3 is too large to yield a strong law of large numbers and a central limit theorem. Throughout the remainder of this paper we shall work with a subclass of the class of MRPAP's. For every stopping time T relative to X, let us put $A \in \mathfrak{A}_T$ if $A \in \mathfrak{A}$ and, for any $t \geqq 0$, $A\{T \leqq t\} \in \mathfrak{A}_t$. $\mathfrak{A}_T$ is clearly a σ-algebra on Ω.

DEFINITION 12.1. If in Definition 3.1 (E1) is replaced with

(E3) $$P\{X_{S_n+h_1}\in B_1,\cdots,X_{S_n+h_m}\in B_m\mid \mathfrak{A}_{S_n}\} = P_{Y_n}\{X_{h_1}\in B_1,\cdots,X_{h_m}\in B_m\}[\mathfrak{A}_{S_n},P]$$
$$(h_i \geqq 0,\ B_i\in\mathfrak{B},\ 1\leqq i\leqq m,\ m,\ n\in N),$$

X is said to be an MRPAP in the strict sense.

REMARKS. (1). Since X is strongly measurable, X_{S_n} and hence $Y_n = Y_{n(S_n)}$ is $\mathfrak{A}_{S_n}$-measurable (cf. Dynkin [5] Section 69). Therefore, we have $\sigma_n \subset \mathfrak{A}_{S_n}$ $(n\in N)$. Thus (E3) implies (E1).

(2). Again, (E2) has only a technical character and can be satisfied by enlarging the space Ω.

(3). (E3) is satisfied if supplementary variables W_t $(t\geqq 0)$ and a measurable function f can be found such that $((X_t, W_t), t\geqq 0)$ is a strong Markov process and $(X_{S_n}, W_{S_n}) = f(Y_n)$ $(n\in N)$. Assume throughout the remainder of this paper that X is an MRPAP in the strict sense.

THEOREM 12.1. *Let τ be a stopping time relative to $((Y_n, S_n), n\geqq 1)$ such that $P(\Omega_\tau) > 0$ where $\Omega_\tau \equiv \{\tau < \infty\}$. Then S_τ is a stopping time relative to X and for every $\mathfrak{A}$-measurable, P-integrable function f*

$$E(\theta_\tau f\mid \Omega_\tau \mathfrak{A}_{S_\tau}) = E_{Y_\tau} f = \int f\, dP_{Y_\tau}[\Omega_\tau \mathfrak{A}_{S_\tau}, P].$$

PROOF. The proof of the second part is analogous to the proof of Lemma 5.1 and Theorem 5.1. Since $\{\tau = p\}\in\sigma_p\subset\mathfrak{A}_{S_p}$ we have

$$\{S_\tau \leqq t\} = \textstyle\sum_{p=1}^{\infty}\{\tau = p\}\{S_p\leqq t\}\in\mathfrak{A}_t$$

which completes the proof.

13. Strong law of large numbers. Throughout Section 13 and Section 14 of this paper, we will be concerned with sums of a functional of an MRPAP satisfying hypothesis (E3). It is assumed that the embedded MC enters a.s. some fixed recurrent class C, i.e. $K_C(+\infty) = 1$. Let $\{f_n, n\in N\}$ be a sequence of ${}_{S_n}\mathfrak{A}\cap\mathfrak{A}_{S_{n+1}}$-measurable functions with values in $R_1\cup\{+\infty\}$ such that $f_n = \theta_n f$ where f is $\mathfrak{A}$-measurable and finite almost everywhere with respect to P. For all $t\geqq 0$, define

$$W_f(t) \equiv \textstyle\sum_{n=1}^{n(t)-1} f_n.$$

We shall study limit theorems for $W_f(t)$ in these two sections. Most of the results and the methods of proof are analogous to those given by Chung [4] and, in the main, by Pyke and Schaufele [12] when dealing with the corresponding problems of MC's and MRP's, respectively. Therefore, the proofs of the following statements will for the most part be omitted. We will focus on some fixed state $j\in C$, say $j = 0$, and we put

$$W_{0n}(f) \equiv W_f(T_{0n+1}) - W_f(T_{0n}) = \textstyle\sum_{\nu=\tau_{0n}}^{\tau_{0n+1}} f_\nu].$$

(Set $W_{0n}(f) = \infty$ on the null set $\{T_{0n} = \infty\}\cup\{T_{0n+1} = \infty\}$). It is not difficult to deduce from Theorem 12.1 the following

LEMMA 13.1. *$W_{0n}(f)$ is ${}_{T_{0n}}\mathfrak{A} \cap \mathfrak{A}_{T_{0n+1}}$-measurable ($n \in N$) where ${}_{T_{0n}}\mathfrak{A} = \sigma(X_{T_{0n}+t}, t \geqq 0)$. Hence, $\{W_{0n}(f), n \in N\}$ forms a sequence of independent, identically distributed random variables.*

At first we shall give explicit formulas for the first two moments of $W_{0n}(f)$ (cf. [4] pages 87, 88; [12] page 1756). We write

$$\zeta_{ij}(f) \equiv E(f_n \mid Y_n = i,\ Y_{n+1} = j)$$

$$\zeta_i(f) \equiv \textstyle\sum_k p_{ik}\zeta_{ik}(f) = E(f_n \mid Y_n = i)$$

if $P\{Y_n = i\} > 0$ and $p_{ij} > 0$.

LEMMA 13.2. *If $f \geqq 0$ or $EW_{0n}(|f|) < \infty$ then*

$$EW_{0n}(f) = \textstyle\sum_{k \in c} e_{0k}\zeta_k(f). \tag{13.1}$$

If g satisfies the same conditions as f and if $f \geqq 0$, $g \geqq 0$ or $EW_{0n}(|f|)W_{0n}(|g|) < \infty$, then

$$EW_{0n}(f)W_{0n}(g) = \textstyle\sum_i e_{0i}[\zeta_i(fg) + \sum_{l \neq 0}\sum_{k \neq 0} p_{ik}\,{}_iM_{kl}(+\infty)(\zeta_{ik}(f)\zeta_l(g) + \zeta_{ik}(g)\zeta_l(f))]. \tag{13.2}$$

On making use of (9.1) and (9.2) we have

COROLLARY 13.1. *If (13.1) is valid and $(|EW_{0n}(f)|, \mu_{00}) \neq (\infty, \infty)$ then $EW_{0n}(f)/\mu_{00}$ does not depend on the choice $j = 0$.*

The following lemma may be proved by paralleling the analogous result for MC's in [4] page 84.

LEMMA 13.3. *If $E|W_{in}(f)|^q$ is finite for any $i \in C$, then it is finite for every $i \in C$ ($q > 0$).*

Set

$$W(t) \equiv W_f(t), \qquad W_{0n} \equiv W_{0n}(f), \qquad m(0) \equiv EW_{0n}, \qquad \tau_{00} = 0, \qquad f_0 \equiv 0,$$

$$R_1(t) \equiv I_{\{T_{01} \leqq t\}}\textstyle\sum_{n=1}^{\tau_{01}-1} f_n, \qquad V(t) \equiv \sum_{n=1}^{n_0(t)-1} W_{0n}, \qquad R_2(t) \equiv \sum_{n=\tau_{0n_0(t)}}^{n(t)-1} f_n$$

which yields the decomposition $W(t) = R_1(t) + V(t) + R_2(t)$.

Compare the following results with Lemma 5.1, Theorem 5.1 and Theorem 5.2 in [12].

LEMMA 13.4. *$R_1(t)/t^{\frac{1}{2}} \to 0$ (a.s.). If $E|W_{0n}| < \infty$ or if $W_{0n} \geqq 0$ and $\mu_{00} < \infty$, then $V(t)/t \to m(0)/\mu_{00}$ (a.s.)*

THEOREM 13.1. *If $EW_{0n}(|f|) < \infty$ or if $f \geqq 0$ and $\mu_{00} < \infty$, then $W(t)/t \to m(0)/\mu_{00}$ (a.s.)*

The next theorem gives a much weaker condition for the strong law of large numbers as well as a necessary and sufficient condition. Define for each $j \in C$

$$D_j \equiv \sup_{m \geqq 0}\left|\textstyle\sum_{v=\tau_{j1}}^{\tau_{j1}+m-1} f_v I_{\{\tau_{j2}-\tau_{j1} > m\}}\right|.$$

THEOREM 13.2. (a) *If* $ED_0 < \infty$, $E|W_{01}(f)| < \infty$ *then* $W(t)/t \to m(0)/\mu_{00}$ (a.s.)
(b) *If* $E|W_{01}| < \infty$, $\mu_{00} < \infty$ *then the following statements are equivalent*:
(1). $W(t)/t \to m(0)/\mu_{00}$ (a.s.).
(2). $R_2(t)/t \to 0$ (a.s.).
(3). $ED_0 < \infty$.

14. Central limit theorem.

LEMMA 14.1. *If* $m_{00} < \infty$, $\mu_{00} < \infty$ *and if* G_{00} *is non-arithmetic or* $P(\bigcap_n \sum_m \{S_n = m\lambda\}) = 1$ *for some* $\lambda > 0$, *then* $R_2(t)/t^{\frac{1}{2}} \to_p 0$.

PROOF. In the non-arithmetic case, $R_2(t)$ even converges in distribution. The proof parallels that of Theorem 6.1 in [12]. The arithmetic assumption implies that the span of G_{00} is an integer multiple of λ, say $v\lambda$. The same sort of argument as is used in the non-arithmetic case will show that $R_2((nv+p)\lambda)$ $(0 \leqq p < v)$ converges in distribution as $n \to \infty$. The statement of Lemma 14.1 is now a consequence of the fact that

$$R_2((nv+p)\lambda+\xi) = R_2((nv+p)\lambda)\,(\text{a.s.}) \qquad \text{if} \quad \xi \in [0, \lambda).$$

Throughout the remainder of this section it is assumed that the assumptions of Lemma 14.1 are satisfied. By Lemma 13.3 and Lemma 14.1 it suffices to consider the asymptotic behavior of $V(t)/t^{\frac{1}{2}}$ when studying that of $W(t)/t^{\frac{1}{2}}$. Define $g = f-m(0)/m_{00}$, $g_n = \theta_n g = f_n - m(0)/m_{00}$, $h = f-(S_2-S_1)m(0)/\mu_{00}$, $h_n = \theta_n h = f_n-(S_{n+1}-S_n)m(0)/\mu_{00}$, $B_f = \operatorname{Var} W_{01}(f)/\mu_{00}$ where f can be replaced by g or h, and suppose that $B_f < \infty$. Then $\operatorname{Var}(\tau_{02}-\tau_{01}) < \infty$ implies that $B_g < \infty$ and $\operatorname{Var}(T_{02}-T_{01}) < \infty$ implies that $B_h < \infty$.

THEOREM 14.1. *Under the assumptions of Lemma* 14.1 *and the assumption that* $B_f < \infty$,
(a) $(W(t)-n_0(t)m(0))/t^{\frac{1}{2}} \to_L aN(0, B_f)$ rv,
(b) $(W(t)-n(t)m(0)/m_{00})/t^{\frac{1}{2}} \to_L aN(0, B_g)$ rv, *if* $\operatorname{Var}(\tau_{02}-\tau_{01}) < \infty$,
(c) $(W(t)-tm(0)/\mu_{00})/t^{\frac{1}{2}} \to_L aN(0, B_h)$ rv, *if* $\operatorname{Var}(T_{02}-T_{01}) < \infty$.

The proof is analogous to the proof of Lemma 7.1 and Theorem 7.1 in [12]. In the proof of Theorem 14.1(c) the following fact is needed. $(t-S_{n(t)})/t^{\frac{1}{2}} \to 0$ (a.s.). This is a consequence of the inequality $t-S_{n(t)} \leqq T_{0n_0(t)+1}-T_{0n_0(t)}$ and of the fact that $(T_{0n+1}-T_{0n})/n^{\frac{1}{2}} \to 0$ (a.s.). If moreover $E(W_{01}(|f|) < \infty$, it is easily verified, by use of the results in Section 13, that neither the left hands in Theorem 14.1 nor the assumptions used depend on the choice $j = 0$. Thus neither do the quantities B_f, B_g, B_h.

Suppose that ϕ is a measurable and, for convenience, bounded function defined on $\mathfrak{X}$. In defining $f_n = \int_{S_n}^{S_{n+1}} \phi(X_u)\,du = \theta_n \int_{S_1}^{S_2} \phi(X_u)\,du$ the above results apply to $W_f(t) = \int_{S_1}^{S_{n(t)}} \phi(X_u)\,du$. If one is interested in

$$\begin{aligned}\tilde{W}(t) &= \int_0^t \phi(X_u)\,du \\ &= \int_0^{S_1} \phi(X_u)\,du + W_f(t) + \int_{S_{n(t)}}^t \phi(X_u)\,du,\end{aligned}$$

the analogous results for $\tilde{W}(t)$ may be derived by combining the above results and those of Smith for cumulative processes ([16]).

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REFERENCES

[1] BENES, V. E. (1961). Extensions of Wiener's Tauberian theorem for positive measures. *J. Math. Anal. Appl.* **2** 1–20.

[2] BENES, V. E. (1962). A "Renewal" limit theorem for general stochastic processes. *Ann. Math. Statist.* **33** 98–113.

[3] CHEONG, C. K. (1967). Geometric convergence of semi-Markov transition probabilities. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **9** 122–130.

[4] CHUNG, K. L. (1967). *Markov Chains with Stationary Transition Probabilities.* Springer-Verlag, Berlin.

[5] DYNKIN, E. B. (1961). Die Grundlagen der Theorie der Markoffschen Prozesse. Springer-Verlag, Berlin.

[6] DYNKIN, E. B. (1965). *Markov Processes* **1**. Springer-Verlag, Berlin.

[7] FABENS, A. J. (1961). The solution of queuing and inventory models by semi-Markov-processes. *J. Roy. Statist. Soc.* **23** 113–127.

[8] FELLER, W. (1966) *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

[9] KENDALL, D. G. (1951). Some problems in the theory of queues. *J. Roy. Statist. Soc. Ser. B.* **13** 151–207.

[10] KENDALL, D. G. (1953). Stochastic processes occurring in the theory of queues and their analysis by the method of the embedded Markov chains. *Ann. Math. Statist.* **24** 338–354.

[11] PYKE, R. (1961). Markov Renewal processes: definitions and preliminary properties. *Ann. Math. Statist.* **32** 1231–1242.

[12] PYKE, R. and SCHAUFELE, R. (1964). Limit theorems for Markov Renewal processes. *Ann. Math. Statist.* **35** 1746–1764.

[13] PYKE, R. and SCHAUFELE, R. (1966). Stationary measures for Markov Renewal processes. *Ann. Math. Statist.* **37** 1439–1462.

[14] SCHÄL, M. (1970). Rates of convergence in Markov renewal processes with auxiliary paths. To be published in *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.*

[15] SCHÄL, M. (1970). The analysis of the queuing system $M/G/1$ with state-dependent service times and the queuing system $GI/M/1$ with state-dependent input by Markov Renewal processes with auxiliary paths. To be published.

[16] SMITH, W. L. (1955). Regenerative stochastic processes. *Proc. Roy. Soc. Ser. A.* **232** 6–31.

[17] SMITH, W. L. (1960). Remarks on the paper "Regenerative stochastic processes." *Proc. Roy. Soc. Ser. A.* **256** 496–501.