The Annals of Mathematical Statistics
1970, Vol. 41, No. 5, 1568–1581

ADMISSIBILITY OF INVARIANT CONFIDENCE PROCEDURES FOR ESTIMATING A LOCATION PARAMETER

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1. Introduction. Let X be a random variable with a probability density $f(X\text{–}\theta)$ involving the location parameter θ. Let $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ be n independent observations of X and let $g(\theta \mid \mathbf{x})$ be the conditional probability density of θ given $\mathbf{x}$ defined as follows:

$$g(\theta \mid \mathbf{x}) = \textstyle\prod_{r=1}^{n} f(x_r - \theta) \cdot [\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(x_i - \theta)\, d\theta]^{-1} \tag{1}$$

Let C_0 be the confidence procedure which assigns to the observed values $\mathbf{x}$, the confidence set for θ, given by

$$C_0(\mathbf{x}, \cdot) = \{\theta : g(\theta \mid \mathbf{x}) \geqq b\} \tag{2}$$

where $b > 0$ is some fixed constant. The procedure C_0 is translation invariant, i.e., if $x_i' = x_i + k$, $i = 1, 2, \cdots, n$, then the confidence set $C_0(\mathbf{x}', \cdot)$ is obtained by translating each point θ of $C_0(\mathbf{x}, \cdot)$ to $\theta + k$. It is easily verified that the expected Lebesgue measure of the confidence sets of C_0, viz. $E_\theta v C_0(\mathbf{x}, \cdot)$ is equal to some constant v_0 for all θ. Similarly the inclusion probability, i.e., the probability that the "true value" θ is included in the observed confidence set $C_0(\mathbf{x}, \cdot)$ is independent of θ and equal to $(1-\alpha)$ say. Further C_0 has the minimax property that amongst the confidence procedures C with given lower confidence level $(1-\alpha)$, C_0 minimizes the maximum expected Lebesgue measure of the confidence sets viz. $E_\theta v C(\mathbf{x}, \cdot)$. This minimax property has been proved by Kudō (1955) and is also deducible from results proved by Valand (1968).

In the following we investigate the question whether the procedure C_0 is unique in having the minimax property and show that subject to the density $f(x)$ satisfying two conditions, the procedure is essentially unique, i.e. to say, it is unique if we treat as equivalent procedures whose confidence sets for almost all $\mathbf{x}$, differ from each other at most by null subsets of the parameter space. The uniqueness is proved in the extended class of randomized confidence procedures.

Investigating a conjecture of Stein (1958) a similar uniqueness property of the usual confidence sets for univariate and bivariate normal populations was proved previously (1969). The present result contains the previous result for the univariate normal population as a particular case.

2. Preliminaries. X is a random variable with a probability density $f(x-\theta)$ where θ is a location parameter; $x_1, x_2, \cdots, x_n$ denote n independent observations of X: $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ denotes a point in the n-dimensional Euclidean sample space R; θ assumes values in the parameter space $\Omega = (-\infty, \infty)$; on R, Ω and the

Received October 23, 1968; revised March 13, 1970.

Cartesian product space $R \times \Omega$ is defined Lebesgue measure, all sets and functions considered being Lebesgue measurable.

For convenience we make a transformation of variables in R, by putting

$$x = x_1 \tag{3}$$
$$y_i = x_{i+1} - x_1, \qquad i = 1, 2, \cdots, (n-1).$$

y denotes the $(n-1)$-dimensional vector $(y_1, y_2, \cdots, y_{n-1})$; the point $\mathbf{x} \in R$, will hereafter be denoted by (x, y).

A randomized confidence procedure is one in which the confidence set assigned to the point (x, y) instead of being fixed, is selected from a number of sets by an independent random procedure. We obtain the class of all such procedures by taking as the decision space

$$\mathscr{D} = \{\phi(x, y, \theta);\ \phi \text{ jointly measurable in } x, y \text{ and } \theta,\ 0 \leqq \phi(x, y, \theta) \leqq 1\}. \tag{4}$$

A ϕ which for every $(x, y) \in R$, is a simple or elementary function of θ represents a randomized or non-randomized confidence procedure. When ϕ represents a confidence procedure, the following relations hold, viz.

(5) $\phi(x, y, \theta) =$ probability that the point θ is included in the confidence set selected when (x, y) represents the observed values;

(6) $v\phi(x, y, \cdot) = \int_\Omega \phi(x, y, \theta)\, d\theta$, = expected Lebesgue measure of the confidence set selected when (x, y) represents the observed values;

(7) $E_\theta[\phi(\cdot, \cdot, \theta)] =$ Total probability that the true value θ is included in the confidence set selected.

For convenience we shall refer to every $\phi \in \mathscr{D}$ as a procedure, though as stated before not every ϕ corresponds to a confidence procedure. Equivalence of procedure is defined by

DEFINITION 2.1. Procedures ϕ_1 and ϕ_2 are equivalent if $\phi_1(x, y, \theta) = \phi_2(x, y, \theta)$ for almost all (x, y, θ).

Using (5), it is seen that two non-randomized procedures are equivalent if for almost all (x, y), their confidence sets differ at most by null subsets of Ω.

It has been shown in the previous paper (1969) that in the absence of any restrictions on the geometrical form of the confidence sets, no uniquely minimax or admissible procedure can exist as given any procedure another one uniformly superior to it can always be constructed. All the procedures so constructed are however equivalent according to Definition 2.1 and uniqueness can therefore pertain to the equivalence class which contains a given procedure. The uniqueness of ϕ_0 proved in the following therefore means the uniqueness of the equivalence class which contains ϕ_0. For a more detailed discussion of the notion of equivalence, randomized procedures and the alternative restrictions on the geometrical form of the confidence sets we refer to the previous paper (1969).

A notion of strong admissibility for confidence procedures has been defined by the author (1969) as follows.

DEFINITION 2.2. A procedure ϕ_1 is strongly admissible if there exists no other procedure ϕ_2 such that for all $\theta \in \Omega$

(i) $E_\theta v\phi_2(x, y, \cdot) \leqq E_\theta v\phi_1(x, y, \cdot)$, and
(ii) $E_\theta[\phi_2(\cdot, \cdot, \theta)] \geqq E_\theta[\phi_1(\cdot, \cdot, \theta)]$,

and the strict inequality holds either in (i) or (ii) for at least one $\theta \in \Omega$. The definition of weak admissibility is obtained by replacing (i) above by

$$(1^*) \qquad v\phi_2(x, y, \cdot) \leqq v\phi_1(x, y, \cdot) \qquad \text{for almost all } (x, y)$$

and requiring the strict inequality to hold in (ii) only: Strong admissibility of a procedure ϕ implies its weak admissibility.

It is easily seen that uniqueness up to the equivalence of Definition 2.1 of ϕ_0 in having the minimax property implies its strong admissibility up to the equivalence.

3. Main theorem. On making the transformations in (3), the conditional probability density in (1) is seen to be a function of $(x-\theta)$ and y. We accordingly put

$$(8) \qquad g(x-\theta, y) = g(\theta \mid \mathbf{x}).$$

Then by (5), the procedure ϕ_0 corresponding to the confidence sets in (2) is given by

$$(9) \qquad \phi_0(x, y, \theta) = 1, \quad \text{if} \quad g(x-\theta, y) \geqq b;$$
$$= 0, \quad \text{otherwise}.$$

It is easily verified that $v\phi_0(x, y, \cdot)$ is independent of x. We therefore write

$$(10) \qquad v\phi_0(x, y, \cdot) = v_0(y).$$

We further write,

$$(11) \qquad p(x-\theta, y) = \textstyle\prod_{r=1}^n f(x_r - \theta), \qquad \text{and}$$
$$u(y) = \textstyle\int_{-\infty}^{\infty} p(x-\theta, y)\, d\theta,$$

so that the conditional probability density in (8) is given by

$$(12) \qquad p(x-\theta, y) = u(y) \cdot g(x-\theta, y).$$

Let μ denote the measure defined on subsets of the space R_{n-1} of y, by the density $u(y)$, i.e. for every measurable $S \subset R_{n-1}$

$$(13) \qquad \mu(S) = \textstyle\int_S u(y)\, dy$$

where dy is written for $dy_1, dy_2, \cdots, dy_{n-1}$ and will be so written hereafter.

We now assume that the density $f(x)$ satisfies the following conditions.

CONDITION 1. The density $f(x)$ has finite first absolute moment, i.e.

$$(14) \qquad \textstyle\int_{-\infty}^{\infty} |x| f(x)\, dx < \infty.$$

CONDITION 2. $g(x-\theta, y)$ being the density in (8), for almost all $y(\mu)$, $\{\theta: g(x-\theta, y) = b\}$ is a null subset of Ω, i.e.,

$$\int u(y)\,dy \int_{[g=b]} d\theta = 0 \tag{15}$$

where $[g = b]$ is written for short for the θ-set, $\{\theta: g(x-\theta, y) = b\}$. By substituting z for $(x-\theta)$ the left-hand side of (15) is seen to be independent of x.

In (15), and also everywhere hereafter, unless otherwise specified the integral with respect to y is taken over the whole space R_{n-1} of y.

Condition 1 is a sufficient condition for our theorem, while Condition 2 can be shown to be necessary also.

We now prove the following.

THEOREM 3.1. *If the density $f(x)$ satisfies Condition 1 and Condition 2, ϕ_0 is the procedure defined by (9), and ϕ_1 is any other procedure such that for all $\theta \in \Omega$,*

$$\begin{aligned} bE_\theta v\phi_1(x, y) - E_\theta \phi_1(\cdot,\cdot,\theta) &\leqq bE_\theta v\phi_0(x, y) - E_\theta \phi_0(\cdot,\cdot,\theta) \\ &= bv_0 - (1-\alpha), \end{aligned} \tag{16}$$

then $\phi_1(x, y, \theta) = \phi_0(x, y, \theta)$ for almost all (x, y, θ).

PROOF. For a procedure ϕ, we define, following Blyth (1951) a loss function, $L_\phi(x, y, \theta)$ by

$$L_\phi(x, y, \theta) = bv\phi(x, y, \cdot) - \phi(x, y, \theta). \tag{17}$$

For brevity we put $v_1(x, y) = v\phi_1(x, y, \cdot)$, and

$$\begin{aligned} q(x, y, \theta) &= L_{\phi_0}(x, y, \theta) - L_{\phi_1}(x, y, \theta) \\ &= [bv_0(y) - \phi_0(x, y, \theta)] - [bv_1(x, y) - \phi_1(x, y, \theta)]. \end{aligned} \tag{18}$$

It now follows from (16) that

$$\int dy \int_{-\infty}^{\infty} q(x, y, \theta) p(x-\theta, y)\,dx \geqq 0.$$

Hence for any $L > 0$,

$$\int_{-L}^{L} d\theta \int dy \int_{-\infty}^{\infty} q(x, y, \theta) p(x-\theta, y)\,dx \geqq 0. \tag{19}$$

Using (18) and the bounds on ϕ in (4), the integrand in (19) is seen to be bounded in absolute magnitude by $f(x, y) = [2 + bv_0(y) + bv_1(x, y)] \cdot p(x-\theta, y)$, and since by (16) $bE_\theta v_1(x, y) \leqq bv_0 + \alpha$, $f(x, y)$ is integrable on the domain of integration in (19). The order of integration will therefore be changed wherever necessary, without further justification.

NOTE 3.1. The following proof is a close adaptation of the proof of a theorem (Theorem 2.1.1) of Brown (1966), relating to invariant estimators.

Interchanging the order of integration in (19) with respect to θ and (x, y) and then transforming θ by putting $z = x - \theta$, we obtain

$$\int dy \int_{-\infty}^{\infty} dx \int_{x-L}^{x+L} q(x, y, \theta) p(z, y)\,dz \geqq 0. \tag{20}$$

In the integrand $q(x, y, \theta)$ becomes $q(x, y, x-z)$, but for convenience we shall continue to write it as $q(x, y, \theta)$, it being understood that $\theta = x-z$.

Partitioning the domain of integration in (20) we obtain

$$\text{(21)} \quad \text{left-hand side of (20)}$$
$$= \int dy\,\{\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dz + \int_{-3L/2}^{-L/2} dx \int_{-L/2}^{L+x} dz + \int_{L/2}^{3L/2} dx \int_{x-L}^{L/2} dz$$
$$+ \int_{L/2}^{\infty} dz \int_{z-L}^{z+L} dx + \int_{-\infty}^{-L/2} dz \int_{z-L}^{z+L} dx\}\,.\,q(x, y, \theta)p(z, y)\,dz$$
$$= T_1 + T_2 + T_3 + T_4 + T_5 \quad \text{say.}$$

(Note the change in the order of integration in T_4 and T_5).
Now $1 = \int_{-\infty}^{\infty} g(x-\theta, y)\,d\theta \geqq \int_{[g \geqq b]} g(x-\theta, y)\,d\theta \geqq b \int_{[g \geqq b]} d\theta = bv_0(y)$.

Here we have written $[g \geqq b]$ for short for the θ-set $\{\theta\colon g(x-\theta, y) \geqq b\}$.
Thus $bv_0(y) \leqq 1$ for all y and hence by (18)

$$\text{(22)} \qquad q(x, y, \theta) \leqq 1 + bv_0(y) \leqq 2.$$

Using (22), we have in (21),

$$T_4 + T_5 \leqq 2 \int dy\,\{\int_{-\infty}^{-L/2} dz \int_{z-L}^{z+L} dx + \int_{L/2}^{\infty} dz \int_{z-L}^{z+L} dx\}\,p(z, y)\,dz$$
$$\leqq 4L \int dy \int_{|z| > L/2} p(z, y)\,dz$$
$$\text{(23)} \qquad \leqq 8 \int dy \int_{|z| > L/2} |z|\,p(z, y)\,dz$$
$$= 8 \int_{|x_1| > L/2} |x_1| f(x_1)\,dx_1 \prod_{r=2}^{n-1} \int_{-\infty}^{\infty} f(x_r)\,dx_r \qquad \text{by (11),}$$
$$= 8 \int_{|x_1| > L/2} |x_1| f(x_1)\,dx_1 \to 0 \qquad \text{as } L \to \infty \text{ by (14).}$$

Next, in the expression for T_2 in (21) changing x into $x+L$, we get

$$T_2 = \int dy\,\{\int_{-L/2}^{0} dx \int_{-L/2}^{x} dz + \int_0^{L/2} dx \int_{-L/2}^{-x} dz + \int_0^{L/2} dx \int_{-x}^{x} dz\}$$
$$\cdot\, q(x-L, y, \theta)p(z, y)$$
$$\text{(24)} \qquad \leqq 4 \int dy \int_{-L/2}^{0} |z|\,p(z, y)\,dz$$
$$+ \int_0^{L/2} dx \int dy \int_{-x}^{x} q(x-L, y, \theta)p(z, y)\,dz \qquad \text{by (22)}$$
$$\leqq 4 \int dy \int_{-\infty}^{\infty} |z|\,p(z, y)\,dz$$
$$+ \int_0^{L/2} dx\,\{\sup_{\phi_1 \in \mathscr{D}} [\int dy \int_{-x}^{x} q(x-L, y, \theta)p(z, y)\,dz]\}$$
$$= t_1 + t_2 \quad \text{say.}$$

In (24)

$$\text{(25)} \qquad t_1 < \infty \qquad \text{by (14) and (11).}$$

In the expression for t_2 since for each (x, y) by (18), the expression in square brackets vanishes if we put

$$\phi_1(x-L, y, \theta) = \phi_0(x-L, y, \theta),$$

the supremum in curly brackets $\geqq 0$. Hence

(26) $$t_2 \leqq \int_0^\infty dx\,\{\sup_{\phi_1 \in \mathscr{D}}[\int dy \int_{-x}^{x} q(x-L, y, \theta)p(z, y)\,dz]\}.$$

We shall show later that

(27) the right-hand side of (26) $\leqq$ a constant independent of L.

Assuming (27) for the present (to avoid an interruption in the argument) and combining (27) and (25) we obtain

(28) $$T_2 \leqq c_1 \; (c_1 \text{ is some constant independent of } L).$$

By a similar argument we obtain,

(29) $$T_3 \leqq c_1'.$$

Combining (23), (28) and (29) with (21), we obtain that there exists a constant c_2 ($c_2 \geqq c_1 + c_1'$) such that

(30) $$T_1 = \int dy \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} q(x, y, \theta)p(z, y)\,dz \geqq -c_2.$$

Since the integral in (26) is finite, there exists a sequence $\lambda_i \to \infty$, such that the quantity in braces ({ }), in that integral $\to 0$ like $0(\lambda_i^{-1})$ as $i \to \infty$. Hence if k is any fixed integer, putting in (26), $L = 0$,

(31)
$$\begin{aligned}
&\liminf_{i\to\infty} \{\int_{-\lambda_i}^{\lambda_i} dx \int dy \int_{-\lambda_i}^{\lambda_i} q(x, y, \theta)p(z, y)\,dz\} \\
&\quad \leqq \liminf_{i\to\infty} \{\int_{-\lambda_k}^{\lambda_k} dx \int dy \int_{-\lambda_i}^{\lambda_i} q(x, y, \theta)p(z, y)\,dz \\
&\qquad + \int_{-\lambda_i}^{\lambda_i} dx\,\{\sup_{\phi_1 \in \mathscr{D}}[\int dy \int_{-\lambda_i}^{\lambda_i} q(x, y, \theta)p(z, y)\,dz]\} \\
&\quad = \liminf_{i\to\infty} \int_{-\lambda_k}^{\lambda_k} dx \int dy \int_{-\lambda_i}^{\lambda_i} q(x, y, \theta)p(z, y)\,dz.
\end{aligned}$$

Since the choice of k in (31) is arbitrary,

(32)
$$\begin{aligned}
&\liminf_{\lambda\to\infty} \int dy \int_{-\lambda}^{\lambda} dx \int_{-\lambda}^{\lambda} q(x, y, \theta)p(z, y)\,dz \\
&\qquad \leqq \int_{-\infty}^{\infty} dx\,\{\int dy \int_{-\infty}^{\infty} q(x, y, \theta)p(z, y)\,dz\}
\end{aligned}$$

where in the right-hand side, the braces are added to emphasize that the order of integration in it cannot be changed. We next define a function w on $R = R_n$ and some new sets as follows.

(33) $$w(x, y) = \int_{-\infty}^{\infty} |\phi_0(x, y, \theta) - \phi_1(x, y, \theta)| \cdot |g(x-\theta, y) - b|\,d\theta$$

where $g(x-\theta, y)$ is the density in (12).

Let $A > 0$ and $\delta > 0$ be any given numbers which may respectively be arbitrarily large and arbitrarily small.

For every $L \geqq 0$, $S(L)$ is the subset of the sample space R, defined by

(34) $$S(L) = \{(x, y) : w(x, y) > \delta,\ -L-A < x < -L+A\}.$$

$T(L)$ and T, are subsets of $R_1 = (-\infty, \infty)$ defined by

(35) $$T(L) = \left\{x : \mu\{y : (x, y) \in S(L)\} > \frac{\beta_1}{4A}\right\}$$

where $\beta_1 > 0$ is any given arbitrarily small number and μ is the measure on the space R_{n-1} of y given by (13); and $T = \bigcup_{L>0} T(L)$.

From the definitions of T, $T(L)$ and $S(L)$ it follows that

(36) $$\text{for any point } x \in T, \qquad \mu\{y : w(x, y) > \delta\} > \frac{\beta_1}{4A}.$$

Now

(37) $$\begin{aligned} &\int_{-\infty}^{\infty} \{bv_0(y) - \phi_0(x, y, \theta)\}\, p(x-\theta, y)\, d\theta \\ &= bv_0(y) \cdot u(y) - \int_{-\infty}^{\infty} \phi_0(x, y, \theta) p(x-\theta, y)\, d\theta \qquad \text{by (11),} \\ &= u(y) \int_{-\infty}^{\infty} [b - g(x-\theta, y)] \phi_0(x, y, \theta)\, d\theta \qquad \text{by (6) and (12).} \end{aligned}$$

Writing down the corresponding expression for

$$\int_{-\infty}^{\infty} \{bv_1(x, y) - \phi_1(x, y, \theta)\} \cdot p(x-\theta, y)\, d\theta$$

and combining with (37), we have from (18),

(38) $$\begin{aligned} &\int_{-\infty}^{\infty} q(x, y, \theta) p(x-\theta, y)\, d\theta \\ &= u(y) \int_{-\infty}^{\infty} [b - g(x-\theta, y)] \cdot [\phi_0(x, y, \theta) - \phi_1(x, y, \theta)]\, d\theta. \end{aligned}$$

Now from (4) and (9),

$$\phi_1(x, y, \theta) \leqq \phi_0(x, y, \theta) = 1 \quad \text{if} \quad g(x-\theta, y) \geqq b, \qquad \text{and}$$

$$\phi_1(x, y, \theta) \geqq \phi_0(x, y, \theta) = 0 \quad \text{if} \quad g(x-\theta, y) < b.$$

The integrand in the right-hand side of (38) is therefore always non-positive. As its absolute magnitude is equal to the integrand in (33), (38) and (33) combined give

(39) $$\int_{-\infty}^{\infty} q(x, y, \theta) p(x-\theta, y)\, d\theta = -u(y) \cdot w(x, y) \leqq 0.$$

We now integrate both sides of (39) with respect to x and y. Since the right-hand side of (39) is always $\leqq 0$, we have,

(40) $$\begin{aligned} &\int_{-\infty}^{\infty} dx \int dy \int_{-\infty}^{\infty} q(x, y, \theta) p(x-\theta, y)\, d\theta \\ &= \int_{-\infty}^{\infty} dx\, \{-u(y) w(x, y)\}\, dy \\ &\leqq \int_T dx \int_{\{y:\, w(x,y) > \delta\}} \{-u(y) w(x, y)\}\, dy \\ &\leqq -\delta \int_T dx \cdot \mu\{y : w(x, y) > \delta\} \qquad \text{by (13)} \\ &\leqq -\beta_2 \int_T dx \qquad \text{by (36),} \end{aligned}$$

where $\beta_2 = \delta\beta_1/(4A)$.

Hence combining (30), (32) and (40), we obtain

(41) $$-c_2 \leqq \int_{-\infty}^{\infty} dx \{\int dy \int_{-\infty}^{\infty} q(x,y,\theta)p(z,y)\,dz\} \leqq -\beta_2 \int_T dx.$$

Hence

(42) $$\int_T dx \leqq c_2/\beta_2.$$

Since $T = \bigcup_{L>0} T(L)$, (42) implies that

(43) $$\int_{T(L)} dx \to 0 \qquad \text{as} \quad L \to \infty.$$

Let

(44) $$T_1(L) = \{x: -L-A < x < -L+A, \text{ and } x \notin T(L)\}.$$

Then

(45) $$\int_{S(L)} dx \cdot u(y)\,dy = \int_{T(L)} dx\, \mu\{y:(x,y) \in S(L)\} + \int_{T_1(L)} dx\, \mu\{y:(x,y) \in S(L)\}$$
$$\leqq \int_{T(L)} dx + \frac{\beta_1}{4A} 2A \qquad \text{by definition of } T(L) \text{ in (35).}$$

Since β_1 can be taken arbitrarily small, it follows from (45) and (43) that

(46) $$\int_{S(L)} u(y)\,dx\,dy \to 0 \qquad \text{as} \quad L \to \infty.$$

Now reverting to (21), we have

$$T_2 = \int dy \int_{-3L/2}^{-L/2} dx \int_{-L/2}^{L+x} q(x,y,\theta)p(z,y)\,dz$$
$$= \int dy \{\int_{-L-A}^{-L+A} dx \int_{-L/2}^{L+x} dz + \int_{-3L/2}^{-L-A} dx \int_{-L/2}^{L+x} dz$$
$$+ \int_{-L+A}^{-L/2} dx \int_{-L-x}^{L+x} dz + \int_{-L+A}^{-L/2} dx \int_{-L/2}^{-L-x} dz\}\, q(x,y,\theta)p(z,y)$$

(47) $$\leqq \int_{-A}^{A} dx \int dy \int_{-L/2}^{x} q(x-L,y,\theta)p(z,y)\,dz$$
$$+ \int dy \{\int_{-L/2}^{-A} dz \int_{z-L}^{-A-L} dx + \int_{-L+A}^{-L/2} dx \int_{-L-x}^{L+x} dz$$
$$+ \int_{-L/2}^{-A} dz \int_{A-L}^{-z-L} dx\}\, q(x,y,\theta)p(z,y)$$
$$\leqq \int_{-A}^{A} dx \int dy \{\sup_{\lambda_1,\lambda_2} \int_{\lambda_2}^{\lambda_1} q(x-L,y,\theta)p(z,y)\,dz\}$$
$$+ 4\int dy \int_{-\infty}^{-A} [|z|+A] \cdot p(z,y)\,dz$$
$$+ \int_A^{\infty} dx \{\sup_{\phi_1 \in \mathscr{D}} \{\int dy \int_{-x}^{x} q(x,y,\theta)p(z,y)\,dz\}\}$$
$$= I_1 + I_2 + I_3 \quad \text{say.}$$

Here we have used (22) in the last but one step for obtaining the term I_2, and that in the term I_3 the expression in braces is nonnegative.

In (47) A can be made arbitrarily large. It follows from Condition 1, i.e. from (14) that I_2 can be made arbitrarily small by making A sufficiently large, i.e.

(48) $$I_2 \to 0 \qquad \text{as} \quad A \to \infty.$$

Similarly using the result in (27), for $L = 0$, we obtain,

(49) $$I_3 \to 0. \qquad \text{as} \quad A \to \infty.$$

It remains to consider the term I_1. $S(L)$ being the set in (34), let $S_1(L)$ denote the subset of $R = R_n$, defined by

(50) $$S_1(L) = \{(x, y): -A-L < x < -L+A, \text{ and } (x, y) \notin S(L)\}.$$

Then using (22), we have

(51) $$\begin{aligned} I_1 &\leqq 2\int_{S(L)} dx\, dy \int_{-\infty}^{\infty} p(z, y)\, dz \\ &\quad + \int_{S_1(L)} dx\, dy \int_{-\infty}^{\infty} q(x, y, \theta) p(z, y)\, dz \\ &= I_{1,1} + I_{1,2} \quad \text{say.} \end{aligned}$$

Then

(52) $$I_{1,1} = 2\int_{S(L)} u(y)\, dy\, dx \to 0 \qquad \text{by (46).}$$

Next in the integral for $I_{1,2}$,

(53) $$\begin{aligned} |v_0(y) - v_1(x, y)| &= \left|\int_{-\infty}^{\infty} [\phi_0(x, y, \theta) - \phi_1(x, y, \theta)]\, d\theta\right| \qquad \text{by (6).} \\ &\leqq \int_{-\infty}^{\infty} |\phi_0(x, y, \theta) - \phi_1(x, y, \theta)|\, d\theta. \end{aligned}$$

Hence by (18),

(54) $$|q(x, y, \theta)| \leqq b\, |v_0(y) - v_1(x, y)| + |\phi_0(x, y, \theta) - \phi_1(x, y, \theta)|.$$

Using (54) and (53) in the expression for $I_{1,2}$, in (51) and substituting for $p(z, y)$ by (12), we obtain,

(55) $$I_{1,2} \leqq (1+b)\int_{-L-A}^{-L+A} dx \int_K u(y)\, dy \int_{-\infty}^{\infty} |\phi_0(x, y, \theta) - \phi_1(x, y, \theta)| \cdot g(x-\theta, y)\, d\theta$$

where $K = \{y: (x, y) \in S_1(L)\}$.

We partition the inner integral in the right-hand side of (55) by

(56) $$\int_{-\infty}^{\infty} d\theta = \int_{[|g-b| < \delta^{1/2}]} d\theta + \int_{[|g-b| \geqq \delta^{1/2}]} d\theta$$

where $[|g-b| < \delta^{\frac{1}{2}}] = \{\theta: |g(x-\theta, y) - b| < \delta^{\frac{1}{2}}\}$ and similarly for $[|g-b| \geqq \delta^{\frac{1}{2}}]$.

Now in (55),

(57) $$\begin{aligned} &\int_{[|g-b| < \delta^{1/2}]} |\phi_0(x, y, \theta) - \phi_1(x, y, \theta)|\, g(x-\theta, y)\, d\theta \\ &\qquad \leqq 2(b + \delta^{\frac{1}{2}}) \int_{[|g-b| < \delta^{1/2}]} d\theta \qquad \text{by (4).} \end{aligned}$$

We now show that K being any subset of R_{n-1} (independent of δ)

(58) $$\lim_{\delta \to 0} \int_K u(y)\, dy \int_{[|g-b| < \delta^{1/2}]} d\theta = \int_K u(y)\, dy \int_{[g=b]} d\theta = 0$$

by the assumed condition 2.

The integral of $u(y)$ with respect to (θ, y) defines a measure on the Lebesgue measurable sets of the product space $R_{n-1} \times \Omega$. Also in (58) as $\delta \to 0$,

(59) $$\{(\theta, y): y \in K; |g(x-\theta, y)-b| < \delta^{\frac{1}{2}}\} \downarrow \{(\theta, y): y \in K; g(x-\theta, y) = b\},$$

and

(60) $$\begin{aligned}\int_K u(y)\,dy \int_{[|g-b|<\delta^{1/2}]} d\theta &\leqq (b+\delta^{\frac{1}{2}})^{-1} \int_K u(y)\,dy \int_{[|g-b|<\delta^{-1/2}]} g(x-\theta, y)\,d\theta \\ &< (b)^{-1} \int_{R_{n-1}} u(y)\,dy \int_{-\infty}^{\infty} g(x-\theta, y)\,d\theta \\ &= (b)^{-1} < \infty.\end{aligned}$$

(59) and (60), together imply (58) by the property of a measure.

Hence by (57) and (58), we have in (55)

(61) $$\begin{aligned}\int_{\{y:(x,y)\in S_1(L)\}} u(y)\,dy \int_{[|g-b|<\delta^{1/2}]} |\phi_0(x, y, \theta)-\phi_1(x, y, \theta)|\, g(x-\theta, y)\,d\theta \\ = \beta(\delta),\end{aligned}$$

where $\beta(\delta) \to 0$ as $\delta \to 0$.

Next, the expression $g/|g-b|$ being monotonically decreasing as g increases above $(b+\delta^{\frac{1}{2}})$ or decreases below $(b-\delta^{\frac{1}{2}})$, we have

(62) $$g(x-\theta, y)\,[|g(x-\theta, y)-b|]^{-1} \leqq (b+\delta^{\frac{1}{2}})\delta^{-\frac{1}{2}}$$

for $\theta \in [|g-b| \geqq \delta^{\frac{1}{2}}]$.

Using (62), we have in (55)

(63) $$\begin{aligned}&\int_{[|g-b|\geqq\delta^{1/2}]} |\phi_0(x, y, \theta)-\phi_1(x, y, \theta)|\, g(x-\theta, y)\,d\theta \\ &\leqq (b+\delta^{\frac{1}{2}})\delta^{-\frac{1}{2}} \int_{[|g-b|\geqq\delta^{1/2}]} |\phi_0(x, y, \theta)-\phi_1(x, y, 0)|\, |g(x-\theta, y)-b|\,d\theta \\ &\leqq (b+\delta^{\frac{1}{2}})\delta^{-\frac{1}{2}} w(x, y) \qquad \text{by (33)} \\ &\leqq (b+\delta^{\frac{1}{2}})\delta^{\frac{1}{2}} \qquad \text{if } (x, y) \in S_1(L), \text{ by (50) and (34).}\end{aligned}$$

Hence in (55),

(64) $$\begin{aligned}&\int_{\{y:(x,y)\in S_1(L)\}} u(y)\,dy \int_{[|g-b|\geqq\delta^{1/2}]} |\phi_0(x, y, \theta)-\phi_1(x, y, \theta)| \cdot g(x-\theta, y)\,d\theta \\ &\leqq (b+\delta^{\frac{1}{2}})\,\delta^{\frac{1}{2}} \int_{R_{n-1}} u(y)\,dy = (b+\delta^{\frac{1}{2}})\,\delta^{\frac{1}{2}}.\end{aligned}$$

Combining (61) and (64) with (55), we get

(65) $$I_{1,2} \leqq 2A(1+b)\,\{\beta(\delta)+b\delta^{\frac{1}{2}}+\delta\}$$

which $\to 0$, as $\delta \to 0$, by (61).

For given A, the right-hand side of (65) can be made arbitrarily small by making δ sufficiently small. Thus combining (65) and (52) with (51) and then with (48) and (49), we obtain that the right-hand side of (47) can be made less than an arbitrarily small positive number, by making A sufficiently large, then δ sufficiently small for given A and then letting $L \to \infty$.

Hence, in (21),

$$\limsup_{L\to\infty} T_2 \leqq 0. \tag{66}$$

By a similar argument

$$\limsup_{L\to\infty} T_3 \leqq 0. \tag{67}$$

Combining (66), (67) and (23), we obtain from (21),

$$\liminf_{L\to\infty} \int dy \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} q(x,y,\theta)p(z,y)\,dz \geqq 0. \tag{68}$$

The required result is now proved by combining (68) and (32) and noting that by (39), the expression in braces, in the right-hand side of (32) is $\leqq 0$. Hence from (32)

$$\int_{-\infty}^{\infty} q(x,y,\theta)p(x-\theta,y)\,d\theta = 0 \qquad \text{for almost all } (x,y)$$

so that

$$\phi_1(x,y,\theta) = \phi_0(x,y,\theta) \qquad \text{for almost all } (x,y,\theta) \tag{69}$$

which was the result to be proved.

To prove our theorem, it still remains to prove the result which was assumed in (27).

Now in the right-hand side of (26)

$$\begin{aligned}\int_{-x}^{x} & q(x-L,y,\theta)p(z,y)\,dz \\ &= \int_{-x}^{x} [bv_0(y)-\phi_0(x-L,y,\theta)]p(z,y)\,dz \\ &\quad -\int_{-x}^{x} [bv_1(x-L,y)-\phi_1(x-L,y,\theta)]p(z,y)\,dz \\ &= s_1 - s_2 \quad \text{say.}\end{aligned} \tag{70}$$

We remind again that in (70), θ is to be put $= x-L-z$. For obtaining the supremum in (26), ϕ_0 remains fixed, and hence we have to choose ϕ_1 so as to minimize s_2.

Now let

$$s_2' = \int_{-x}^{x} [bv_1(x-L,y)-\phi_1(x-L,y,\theta)]\,g(z,y)\,dz \tag{71}$$

so that in (70), $s_2 = u(y)\cdot s_2'$ by (12).

Now put

$$G(x,y) = \int_{-x}^{x} g(z,y)\,dz. \tag{72}$$

Then in (71),

$$\begin{aligned} s_2' &= bv_1(x-L,y)\cdot G(x,y) - \int_{-x}^{x} \phi_1(x-L,y,\theta)g(z,y)\,dz \\ &= bG(x,y)\int_{-\infty}^{\infty} \phi_1(x-L,y,\theta)\,d\theta - \int_{-x}^{x} \phi_1(x-L,y,\theta)g(z,y)\,dz, \qquad \text{by (6)} \\ &= bG(x,y)\int_{-\infty}^{\infty} \phi_1(x-L,y,\theta)\,dz - \int_{-x}^{x} \phi_1(x-L,y,\theta)g(z,y)\,dz \qquad (73) \\ &\qquad \text{by putting } z = x-L-\theta \text{ in the first integral} \\ &= \int_{|z|>x} bG(x,y)\phi_1(x-L,y,\theta)\,dz \\ &\quad + \int_{|z|<x} [bG(x,y)-g(z,y)]\phi_1(x-L,y,\theta)\,dz. \end{aligned}$$

Clearly the right-hand side of (73) is minimized by replacing ϕ_1 by ϕ_2 defined by

$$\begin{aligned}(74)\qquad &\phi_2(x-L, y, \theta) = 0 \quad \text{if (i) } |z| \geqq x \text{ or } g(z, y) \leqq bG(x, y);\\ &\phi_2(x-L, y, \theta) = 1 \quad \text{if (ii) } |z| < x \text{ and } g(z, y) > bG(x, y).\end{aligned}$$

For brevity, let $v_2(x-L, y) = v\phi_2(x-L, y, \cdot)$.

We next show that $v_2(x, y)$ is bounded above for all (x, y).

For given x, and y, let $B_{x,y}$ denote the set of values of z on which both the inequalities in (74)-(ii) hold.

Then

$$\begin{aligned} v_2(x-L, y) &= \textstyle\int_{B_{x,y}} dz \\ (75)\qquad &\leqq [bG(x, y)]^{-1} \textstyle\int_{B_{x,y}} g(z, y)\, dz && \text{by (74)–(ii)} \\ &\leqq [bG(x, y)]^{-1} \textstyle\int_{-x}^{x} g(z, y)\, dz && \text{as } B_{x,y} \subset (-x, x) \\ &= 1/b && \text{by (72).}\end{aligned}$$

In the above it was assumed that $G(x, y) > 0$. But if for some x, y, $G(x, y) = 0$, then for such x, y, $g(z, y) = 0$ for almost all z, $|z| < x$, so that $B_{x,y}$ is a null set. Hence for such (x, y) $v_2(x-L, y) = 0$, so that (75) continues to hold.

Since substitution of ϕ_2 for ϕ_1 minimizes s_2' and hence s_2 in the right-hand side of (70), we have

$$\begin{aligned}(76)\quad &\text{right-hand side of (70)}\\ &\leqq \textstyle\int_{-x}^{x} \{[bv_0(y) - \phi_0(x-L, y, \theta)] - [bv_2(x-L, y) - \phi_2(x-L, y, \theta)]\}\\ &\quad \cdot p(z, y)\, dz.\end{aligned}$$

Next by an argument similar to that in (39), we have

$$\begin{aligned}(77)\quad &\textstyle\int_{-\infty}^{\infty} \{[bv_0(y) - \phi_0(x-L, y, \theta)] - [bv_2(x-L, y) - \phi_2(x-L, y, \theta)]\}\\ &\quad \cdot p(z, y)\, dz \leqq 0.\end{aligned}$$

Combining (77) and (76) with (70), we obtain

$$\begin{aligned} &\textstyle\int_{-x}^{x} q(x-L, y, \theta) p(z, y)\, dz \\ (78)\qquad &\leqq -\textstyle\int_{|z|>x} \{[bv_0(y) - \phi_0(x-L, y, \theta)] - [bv_2(x-L, y) - \phi_2(x-L, y, \theta)]\}\\ &\quad \cdot p(z, y)\, dz \\ &\leqq \textstyle\int_{|z|>x} [\phi_0(x-L, y, \theta) + bv_2(x-L, y)] p(z, y)\, dz \\ &\leqq 2\textstyle\int_{|z|>x} p(z, y)\, dz && \text{by (75) and (4).}\end{aligned}$$

Hence,

right-hand side of (26)

$$\begin{aligned}(79)\qquad &\leqq 2\textstyle\int_0^\infty dx \int dy \int_{|z|>x} p(z,y)\,dz\\ &= 2\textstyle\int dy \int_{-\infty}^{\infty} dz\, p(z,y) \int_0^{|z|} dx\\ &= 2\textstyle\int dy \int_{-\infty}^{\infty} |z|\, dz\, p(z,y)\\ &= 2\textstyle\int_{-\infty}^{\infty} |x_1| f(x_1)\, dx_1 && \text{by (11)}\\ &= k \quad \text{(independent of } L) && \text{by (14).}\end{aligned}$$

This proves the result assumed in (27) and thus completes the proof of Theorem 3.1.

4. An application. An interesting application of Theorem 3.1. is the following. Let the probability of the rv $\cdot X$ be $p(x-\theta)$ where $p(t)$ strictly decreases as t increases for $t \geqq 0$ and as t decreases for $t \leqq 0$. We assume that only one observation x of X is taken. Then the usual shortest confidence intervals with confidence level $(1-\alpha)$ are given by

$$(80)\qquad \{x-h_2 \leqq \theta \leqq x+h_1\} \qquad \text{where } h_1 > 0, h_2 > 0,$$

are uniquely fixed by

$$(81)\qquad \textstyle\int_{-h_1}^{h_2} p(t)\,dt = 1-\alpha, \qquad \text{and}$$

$$p(-h_1-0) \leqq p(h_2) \leqq p(-h_1+0).$$

It is easily verified that Condition 2 of Theorem 3.1 is satisfied by virtue of the strict monotonicity of $p(t)$. If in addition

$$(82)\qquad \textstyle\int_{-\infty}^{\infty} |t|\, p(t) < \infty$$

holds, then Condition 1 of Theorem 3.1 is also satisfied and hence the confidence intervals in (80) are uniquely minimax up to the equivalence class.

Acknowledgment. I am deeply grateful to the referees for their valuable guidance and in particular for suggesting the application to this problem of the method used by Blackwell (1951) and Brown (1966). In the original version the problem was attempted by another method based on that of Blyth (1951) and the result obtained was of very limited scope.

I am also thankful to Mr. P. M. Joshi for doing the necessary typing work with great neatness and patience.

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