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ABSTRACT OF PAPERS

(*Abstracts of Papers contributed by title*)

70T-88. On pairing random observations from a bivariate distribution. MILTON C. CHEW, JR., Rensselaer Polytechnic Institute.

Suppose a random sample of size n is taken from a known bivariate distribution, whose density $f(x, y)$ possesses a "monotone likelihood ratio," i.e. for all $x_1 < x_2$ and $y_1 < y_2$, one of the following inequalities always holds: $f(x_1, y_1)f(x_2, y_2) \lesseqgtr f(x_1, y_2)f(x_2, y_1)$. Included in such a class are the normal and trinomial distributions. When the rank order of the x and y data is known, but their correct pairwise ordering is not, it is desirable to optimally "reconstruct" the original bivariate sample. Such a situation occurs when one wishes to "decode" the received y-data into the x-data transmitted over a memoryless channel. It is easily seen that the Maximum Likelihood Pairing (MLP) maximizes the probability of a perfect match, or "reconstruction." A more difficult objective is to maximize the expected number of correct (x, y) pairs. In this case it is unlikely (though unproved) that the optimal pairing depends only on the rank order of the data, and not their values. It is shown, however, that the MLP is better than an arbitrary pairing, whose expected number of correct pairs is unity for all n. The problem is treated as one of combinatorics, and use is made of the rencontres numbers used in the random matching problem. (Received July 23, 1970.)

70T-89. Nonexistence of a single-sample selection procedure whose $P(CS)$ is independent of the variances. EDWARD J. DUDEWICZ, The University of Rochester.

Suppose an experimenter has k populations $\pi_1, \cdots, \pi_k$, where observations from π_i are normally distributed $N(\mu_i, \sigma_i^2)$ $(1 \leqq i \leqq k)$ with $\mu_1, \cdots, \mu_k$, $\sigma_1^2, \cdots, \sigma_k^2$ all unknown. Further, suppose the goal is to select that population which has the largest mean, say $\mu_{[k]}$ (where $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ordered means). Bechhofer [*Ann. Math. Statist.* **25** (1954) 16–39] discussed this problem when $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ with σ^2 known, Bechhofer, Dunnett and Sobel [*Biometrika* **41** (1954) 170–176] gave a two-sample solution for the case $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$ with σ^2 unknown, and various authors considered sequential solutions for these two cases. No satisfactory solution is available for the problem with $\sigma_1^2, \cdots, \sigma_k^2$ unknown. One reason is that certain types of single-sample procedures do not exist to provide such a solution. This result, stated but not proved by Bechhofer, Dunnett and Sobel [*loc. cit.*], is given a rigorous treatment, using results of Dantzig [*Ann. Math. Statist.* **11** (1940) 186–192], Stein [*Ann. Math. Statist.* **16** (1945), 243–258], and Hall [*Ann. Math. Statist.* **30** (1959), 964–969]. (Received July 24, 1970.)

70T-90. Confidence intervals for power, with special reference to medical trials. EDWARD J. DUDEWICZ, The University of Rochester.

Suppose that one has observed $X_1, \cdots, X_n$, independent random variables, each normally distributed $N(\delta, \sigma^2)$ where δ $(-\infty < \delta < +\infty)$ and σ^2 $(0 < \sigma^2)$ are both unknown. For the problem of testing H: $\delta = 0$ it is customary to use a t-test (at level $\alpha = 0.05$, e.g.); one then calculates $t = \bar{X}n^{\frac{1}{2}}/\hat{\sigma}$ (where $\hat{\sigma}^2 = \sum(X_i - \bar{X})^2/(n-1)$) and compares it with a critical value $t_\alpha(n)$. This yields a Type I error of α, and a power which is a function of $d = \delta n^{\frac{1}{2}}/\sigma$. Since one does not know σ, one cannot answer the question "What is the probability that I would have detected a difference $|\delta| \geqq 0.2$ (e.g.)?" A common recommendation, usually implicit [*Biometrika Tables for Statisticians* **1**, 3rd ed. (1966) 25] but sometimes explicit [*Experimental Design: Procedures for the Behavioral Sciences*. R. E. Kirk (1968) 108] is essentially that one act as if $d = \delta n^{\frac{1}{2}}/\hat{\sigma}$ in such power evaluations. We note that if one takes a confidence interval on σ^2 with confidence coefficient γ (e.g. $0.90 \leqq \gamma < 1$), one can obtain a confidence interval for the power at δ. This is compared with the usual power estimate, and the latter is found to be misleadingly large. An application illustrating the technique (on data from a study of the effect of digoxin in acute myocardial infarction) is given. Such power analysis applies to F-tests, etc. (Received July 24, 1970.)

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70T-91. The invariance and distribution of the quadratic form $X'\Sigma^{-}X$. V. P. BHAPKAR, University of Kentucky.

If X is a random vector normally distributed with mean $\mathbf{0}$ and covariance matrix Σ (possibly singular) and A is a real symmetric matrix, it is known (see Good, *Biometrika* **56** (1969) 215–216) that $X'AX$ has the chi-squared distribution if, and only if, ΣA is idempotent. If Σ^{-} is any g-inverse of Σ, i.e. if Σ^{-} satisfies the relation $\Sigma\Sigma^{-}\Sigma = \Sigma$, the following results are proved: (i) $X'\Sigma^{-}X$ is distributed as a chi-squared variable with df equal to $R(\Sigma)$. (ii) For any distribution of X with mean $\mathbf{0}$ and covariance matrix Σ, $X'\Sigma^{-}X$ is invariant, with probability one, under any choice of Σ^{-}. (iii) If X is $N(\mu, \Sigma)$ and Σ^{-} is symmetric, $X'\Sigma^{-}X$ has a chi-squared (possibly non-central) distribution if, and only if, $\mu'(\Sigma^{-} - \Sigma^{-}\Sigma\Sigma^{-})\mu = 0$; in that case, the df are equal to $R(\Sigma)$ and the non-centrality parameter is $\mu'\Sigma^{-}\Sigma\Sigma^{-}\mu$. For (iii) we have the corollaries: (a) $X'\Sigma^{-}X$ has a central chi-squared distribution if, and only if, $\Sigma\Sigma^{-}\mu = \mathbf{0}$, $\mu'\Sigma^{-}\mu = 0$. (b) If Σ is a g-inverse of Σ^{-}, the necessary and sufficient condition in (a) simplifies to $\Sigma^{-}\mu = \mathbf{0}$. (Received July 28, 1970.)

70T-92. Estimation of the mode with an application to cardiovascular physiology. EDWARD J. WEGMAN AND D. WOODROW BENSON, JR., University of North Carolina.

This paper reviews several proposed methods for estimating the mode of a probability density function. The method of Chernoff, (1964, *Ann. Inst. Statist. Math.* **16** 31–41), is examined in greater detail and is shown to be a strongly consistent estimate of a somewhat generalized mode under relaxed conditions on the density. The restriction of continuity is removed among others. Finally, an application to determining blood flow characteristics in the ascending aorta is given. (Received August 10, 1970.)

70T-93. Robust procedures for estimating polynomial regression. CORWIN L. ATWOOD, University of Minnesota.

For estimating a regression function $g(x)$ which is a polynomial of degree s on a compact interval, let U_s be the least squares estimator and ξ_s the optimal design. Suppose now that the regression function is a polynomial of degree s plus a small polynomial of higher degree k. We use an estimator of the form $(1-\alpha)U_s + \alpha U_k$, and a design of the form $(1-\beta)\xi_s + \beta\xi_k$, with $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$. Our criterion for an optimal procedure is to minimize the maximum (over x) mean square error. For $k \leqq s+2$, $k \leqq 10$ we find values of (α, β) which, compared to the standard procedures $\alpha = \beta = 0$ and $\alpha = \beta = 1$, show desirable robustness when the true regression function deviates from a polynomial of degree s by only a moderate amount. (Received August 11, 1970.)

70T-94. A characterization of a multivariate t distribution. PI-ERH LI, Florida State University.

We consider the p-variate t distribution defined in Raiffa and Schlaifer (*Applied Statistical Decision Theory* (1961). Division of Research, Harvard Business School). Let $X \mid y \sim N(\mu, y^{-1}\Sigma)$, where $Y \sim \nu^{-1}\chi_\nu^2$. Then X is said to have a p-variate t distribution with mean μ, covariance matrix $\nu(\nu-2)^{-1}\Sigma$, $\nu > 2$ and degrees of freedom ν, denoted by $T_\nu(\mu, \Sigma, p)$. We show that $X \sim T_\nu(\mu, \Sigma, p)$ if and only if for any $\mathbf{a} \neq \mathbf{0}$, $(\mathbf{a}'\Sigma\mathbf{a})^{-\frac{1}{2}}\mathbf{a}'(X-\mu)$ has the Student's t distribution with ν degrees of freedom. Several sampling distributions of statistics derived from the p-variate t distribution are obtained. (Received August 12, 1970.)

70T-95. An invariance principle for martingales. RICHARD DROGIN, University of California, Berkeley.

Many discrete martingales with increments in L_2 can be normalized so that the resulting trajectory is distributed approximately like Brownian motion. This paper will find all such martingales, subject to a natural side condition. Two techniques of normalization are possible: The usual one involving the partial sums of conditional variances of the increments given the past, and the analogous method using the partial sums of squares of the increments. This result is applied to obtain a central limit theorem and an arc sin law for dependent random variables. (Received August 31, 1970.)

70T-96. Some characterizations of Poisson laws. R. C. Srivastava, The Ohio State University.

Consider a binomial distribution with parameters n and π and suppose that n is a nonnegative integer-valued random variable. Let X and Y denote the number of successes and failures respectively. It is well known, that X and Y are independent, if n is distributed according to a Poisson distribution. The purpose of this paper is to prove that the independence of X and Y is a characterizing property of the Poisson distribution and to extend this result to the bivariate case and to obtain a characterization of the Poisson process. More specifically the following theorems are proved. Theorem 1. *Let Z be a discrete random variable taking values* $0, 1, 2, \cdots$. *After observing $Z = n$, n Bernoulli trials with probability π of success are conducted. Let X and Y denote the resulting number of successes and failures. Then Z has a Poisson distribution if and only if, X and Y are independent.* Theorem 2. *Let (Z_1, Z_2) be a discrete bivariate random variable, Z_i $(i = 1, 2)$ taking values* $0, 1, \cdots$. *After observing $(Z_1, Z_2) = (n_1, n_2)$, n_i Bernoulli trials with probabilities π_i of success are conducted. Let $(X_i, Y_i)(i = 1, 2)$ denote the resulting number of successes and failures respectively. Then (Z_1, Z_2) is distributed as a multiple Poisson distribution, if and only if, X_1, Y_1, X_2 and Y_2 are independent.* Theorem 3. *Let $Z(t)$ be a time-homogeneous point process wiih density λ. Classify this process into two processes as follows. Each event of $Z(t)$ belongs to $X(t)$ with probability π and $Y(t)$ with probability $1-\pi$, independently of what happens to other events. Then $Z(t)$ is a Poisson process, if and only if, $X(t)$ and $Y(t)$ are independent.* (Received August 31, 1970.)

70T-97. Two sequential ranking procedures. S. K. Perng, Kansas State University.

Let X_{ij} $(i = 1, 2, \cdots, k; j = 1, 2, \cdots)$ be independent random samples from k populations with unknown finite variances σ_i^2's and unknown means μ_i's such that $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_{k-1} < \mu_k$ and $\mu_k - \mu_{k-1} = \delta^* > 0$. A class of sequential procedures for selecting the population with μ_k is proposed as following: (i) Sample one observation at a time from the ith population and stop sampling according to the stopping rule N_i defined by $N_i =$ smallest $n > 1$ such that $s_{in}^2 \leqq d^2 n/a_n$ for $i = 1, 2, \cdots, k$; where $s_{in}^2 = n^{-1} \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2 + 1/n$, d is a positive constant such that $2d \leqq \delta^*$, and $\{a_n\}$ is a proper sequence of real numbers. (ii) When sampling is stopped at $N_i = n_i$, select the population with the largest sample mean. It is shown that these sequential procedures have certain asymptotical optimal properties. Under the assumption of normality, the performance of the sequential procedure in case of small sample sizes has been considered. A similar sequential ranking process based on maximum likelihood estimates has also been proposed. Under cert ain regularity conditions, some asymptotical optimal properties have been proved. (Received September 8, 1970.)

70T-98. Asymptotic efficiency of one multifactor experiment relative to several one-factor experiments for selecting the normal population with the largest mean. Vijay S. Bawa, Bell Telephone Laboratories, Inc.

In this paper, we consider two single-stage procedures for ranking means of normal populations with a common known variance, for problems involving an r-way $(r \geqq 2)$ classification without interaction. For the problem of selecting the best level of each of the r factors, the asymptotic $(P^* \to 1)$ efficiency of one r-factor experiment relative to r one-factor experiments is determined. Some numerical results are provided for $r = 2$. Applications to some related problems are noted. (Received September 8, 1970.)

70T-99. On sensitive rank tests for comparing the effects of two treatments on a single group. HANS URY, California State Department of Public Health.

The rank test R_n of Cronholm and Revusky [*Psychometrika* **30** (1965) 459–467] in effect doubles the number of comparisons carried out in the Mann–Whitney test between "Treated" and "Control" subjects chosen from a single group of n and triples the asymptotic efficiency against normal alternatives by using a series of $n-1$ subexperiments. However, the full series can be too costly or time-consuming. It is shown that j appropriately chosen subexperiments will permit one to make at least $j/(j+1)$ of the number of comparisons possible under R_n, with large sample efficiency of $j/(j+2)$ relative to R_n against normal alternatives and small sample efficiency greater than that, $j = 2, 3, \cdots, n-2$. The resulting test criterion is a sum of j independent Mann–Whitney test statistics. Small sample efficiency comparisons are carried out for $n = 10$. (Received September 9, 1970.)

70T-100. On the exact distribution of the smallest root of the Wishart matrix using zonal polynomials. P. R. KRISHNAIAH AND T. C. CHANG, Aerospace Research Laboratories.

Let $S:p \times p$ be the central Wishart matrix with n degrees of freedom. Also, let h be the smallest latent root of S and $m = (n-p-1)/2$. In addition, let $E(S) = nI_p$ where I_p is the identity matrix. When m is an integer, the authors proved that the density function of h is given by $C(p,n)\,h^{pm} \times \exp(-ph/2)\,{}_2F_0(-m;(p+2)/2;-2I_{p-1}/h)$ where $C(p,n)$ is a constant, and ${}_2F_0(a;b;A)$ is the hypergeometric function with matrix argument as defined in Constantine (*Ann. Math. Statist.* **34** 1270–1285). The authors have also obtained an expression for the density function of h when m is an integer and $E(S) = n\Sigma$. (Received September 10, 1970.)

70T-101. Fixed alternatives and Wald's formulation of the noncentral asymptotic behavior of the likelihood ratio statistic. T. W. F. STROUD, Queen's University at Kingston.

Wald's result on the limiting noncentral chi-square distribution of the likelihood ratio test statistic under a sequence of local alternatives was stated by Wald in a more general setting, namely, in terms of a sequence of differences of cdf's converging to zero, uniformly in both the parameter and the argument of the cdf's. This general formulation is too strong. For the case of testing the mean of a normal distribution with unknown mean and variance, it is shown in this paper that, if the parameter is held fixed, Wald's expression does not converge to zero uniformly in the argument of the cdf. It is shown generally that convergence of Wald's expression uniformly in the argument of the cdf, when the parameter is fixed, is nearly equivalent to the convergence of a related random variable to a certain normal law. (Received September 14, 1970.)