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A NECESSARY CONDITION FOR GLIVENKO–CANTELLI CONVERGENCE IN E_n

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Let μ be a probability measure on the Borel sets, β, in k dimensional Euclidean space E_k and $X_1, X_2, \cdots$ a sequence of independent random vectors with values in E_k such that $P(X_i \in A) = \mu(A)$ for every A in β, $i = 1, 2, \cdots$. When $k \geqq 2$, a necessary condition is given for

$$\sup_{f \in \mathscr{M}} \left| n^{-1} \sum_{i=1}^{n} f(X_i) - \int f \, d\mu \right| \rightarrow_{\text{a.s.}} 0, \tag{1}$$

where $\mathscr{M}$ is the class of all monotone functions on E_k. This theorem answers a conjecture of Blum [1].

For $k \geqq 2$, the graph of a real-valued function with domain E_{k-1} which is strictly monotone in each variable is said to be a strictly monotone graph in E_k. Let L be the set of points $\mathbf{x}$ in E_k with $\mu(\mathbf{x}) > 0$. Define $\mu^L(A)$ as $\mu(AL^c)$ for all A in β.

THEOREM. *For $k \geqq 2$, if there is a strictly monotone graph, B in E_k with $\mu^L(B) > 0$, then* (1) *does not hold.*

PROOF. Suppose there is a strictly monotone graph B in E_k with $\mu^L(B) > 0$. Without loss of generality it is supposed that B is the graph of a function on E_{k-1} which is decreasing in each variable. Corresponding to $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ in E_k let

$$A_x = \{(y_1, y_2, \cdots, y_k) \colon \; y_i \geqq x_i, \; i = 1, 2, \cdots, k\}.$$

Let $A' = \bigcup_{x \in B} A_x$ and $A'' = A' - B$. Let f' and f'' be the characteristic functions of $A' - LB$ and A''. Given a sample sequence, $\{\mathbf{x}_1, \mathbf{x}_2, \cdots\}$ $\mathbf{x}_i$ in E_k, for each positive integer n define $A(n)$ as $\bigcup_{x_i \in A', 1 \leqq i \leqq n} A_{x_i}$ and f_n as the characteristic function of $A(n) - LB$. For f in $\mathscr{M}$, let $S_n(f) = n^{-1} \sum_{i=1}^{n} f(x_i)$. It follows that $S_n(f') = S_n(f_n)$ and that $\{f_n\}$ is a non-decreasing sequence of monotone functions on E_k. Since $A(n) - LB \subset A'' \cup [\{\mathbf{x}_1, \mathbf{x}_2, \cdots\} \cap BL^c]$ for all n and $\mu[\{\mathbf{x}_1, \mathbf{x}_2 \cdots\} \cap BL^c] = 0$, $\lim f_n \leqq_{\text{a.s.}} f''$ and $\int \lim f_n \, d\mu \leqq \int f'' \, d\mu]$: Hence

$$\int f_n \, d\mu \rightarrow \int \lim f_n \, d\mu \leqq \int f'' \, d\mu.$$

From $S_n(f') \rightarrow \int f' \, d\mu$ it follows that

$$\sup_{f \in \mathscr{M}} \left| S_n(f) - \int f \, d\mu \right| \geqq \left| S_n(f_n) - \int f_n \, d\mu \right| \rightarrow_{\text{a.s.}}$$
$$\int f' \, d\mu - \int \lim f_n \, d\mu \geqq \int f' \, d\mu - \int f'' \, d\mu = \mu^L(B).$$

Hence (1) does not hold.

Define a class $\mathscr{A}$ of sets in β as follows: A is in $\mathscr{A}_1$, if $(x_1, \cdots, x_k)$ in A and $y_i \leqq x_i$, $i = 1, \cdots, k$ imply that $(y_1, \cdots, y_k)$ is in A. Let $\mathscr{A}_i$, $i = 1, 2, \cdots, 2^k$ be

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classes defined as $\mathscr{A}_1$ with the inequalities reversed one at a time. $\mathscr{A} = \bigcup_{i=1}^{2^k} \mathscr{A}_i$. The empiric measure $\mu^n(A)$ is the proportion of the first n random vectors which are in A. In [1], Blum proved that if μ is absolutely continuous with respect to Lebesgue measure then

$$P[\lim_{n\to\infty} \sup_{A\in\mathscr{A}} |\mu^n(A) - \mu(A)| = 0] = 1. \tag{2}$$

He conjectured that (2) holds with no assumption on μ. As a special case of the Theorem, it is necessary for (2) that μ^L be 0 on all strictly monotone graphs in E_k.

REFERENCE

[1] BLUM, J. R. (1955). On the convergence of empiric distribution functions. *Ann. Math. Statist.* **26** 527–529.