The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 2166–2168

ON THE SUPREMUM OF S_n/n

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Let $X_1, X_2, \cdots$ be independent and identically distributed. We give a simple proof based on stopping times of the known result that $\sup(|X_1 + \cdots + X_n|/n)$ has a finite expected value if and only if $E|X|\log|X|$ is finite. Whenever $E|X|\log|X| = \infty$, a simple nonanticipating stopping rule τ, not depending on X, yields $E(|X_1 + \cdots + X_\tau|/\tau) = \infty$.

Let $X_1, X_2, \cdots$ be an i.i.d. sequence and set $S_n = X_1 + \cdots + X_n$. Marcinkiewicz and Zygmund [2] proved that if

$$E|X_1|\log|X_1| < \infty \tag{1}$$

then

$$E\sup_n |S_n/n| < \infty. \tag{2}$$

D. L. Burkholder [1] proved the converse and showed further that (1), (2), and (3) are equivalent, where

$$E\sup_n |X_n/n| < \infty. \tag{3}$$

We give a simple proof of Burkholder's results by using stopping times. We extend his results by proving that (1)–(5) are equivalent, where

$$\sup_{\text{rule } N} E|S_N/N| < \infty \tag{4}$$

$$\sup_{\text{rule } N} E|X_N/N| < \infty, \tag{5}$$

the sup in (4) and (5) being taken over all nonanticipative stopping rules (times) N. As an immediate corollary we see that whenever anticipative stopping with reward $\sup|S_n/n|$ gives infinite expected reward, then there is a (nonanticipative) stopping rule which also has infinite expected reward. In fact, this rule is very simple: stop the first time that $|X_n| > cn$; the rule thus does not depend on the distribution of X (except for the constant c).

Finally, we give a one-sided version of the equivalence of (1)–(5) in terms of S_n/n rather than $|S_n/n|$.

We learned from D. L. Burkholder after writing this paper that Burgess Davis [4] and Richard F. Gundy [5] have also obtained stopping time proofs of Burkholder's theorem [1], among other results. We have decided to publish our proof because of its simplicity.

A universal stopping time. We begin the proof of the equivalence of (1)–(5) with the implication (5) $\Rightarrow$ (1). It is clearly no loss of generality to suppose that $p_1 = P(|X_1| < 1) > 0$ and we do this for convenience. Taking $N \equiv 1$ in (5) shows that

Received December 5, 1969.

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$E|X_1| < \infty$. To prove that $E|X_1|\log|X_1| < \infty$ as well, take $N =$ the first $n \geqq 1$ for which $|X_n| > n$; $N = \infty$ if there is no such n. We then have from (5)

$$(6) \qquad \infty > E_{[N<\infty]}|X_N/N| = \sum_{n=1}^\infty P(N=n)n^{-1}E[|X_n| \mid N=n] = \sum_{n=1}^\infty P(N=n)n^{-1}E[|X_n| \mid |X_n| \geqq n]$$

since the X's are independent. If F denotes the common distribution function of the X's (6) yields

$$(7) \quad \infty > \sum_{n=1}^\infty P(N \geqq n)n^{-1}\int_{|x|\geqq n}|x|\,dF(x) \geqq P(N=\infty)\sum_{n=1}^\infty n^{-1}\int_{|x|\geqq n}|x|\,dF(x).$$

But $P(N=\infty) > 0$ because $P(N=\infty) = \prod_{n=1}^\infty P(|X_n| \leqq n)$ and $\sum_{n=1}^\infty [1 - P(|X_n| \leqq n)] = \sum_{n=1}^\infty \int_{|x|>n} dF(x) = \int_{-\infty}^\infty (\sum_{n<|x|} 1)\,dF(x) \leqq \int_{-\infty}^\infty |x|\,dF(x) = E|X_1| < \infty$. Thus from (7),

$$\sum_{n=1}^\infty n^{-1}\int_{|x|\geqq n}|x|\,dF(x) = \int_{-\infty}^\infty |x|(\sum_{n\leqq|x|} n^{-1})\,dF(x) < \infty$$

and so $\int_{-\infty}^\infty |x|\log|x|\,dF(x) < \infty$ and (1) is proved.

We next prove (4) $\Rightarrow$ (1). We define N exactly as before and again we have that $E|X_1| < \infty$. Observing that (4) gives

$$(8) \quad \infty > E_{[N<\infty]}|S_N/N| \geqq E_{[N<\infty]}|X_N/N| - E_{[N<\infty]}[(|X_1| + \cdots + |X_{N-1}|)/N]$$

we see that (1) follows exactly as before if we can prove that the last term on the right of (8) is finite. This term is

$$(9) \qquad \sum_{n=1}^\infty P(N=n)n^{-1}\sum_{k<n}E[|X_k| \mid N=n] = \sum_{n=1}^\infty P(N=n)n^{-1}\sum_{k<n}E[|X_k| \mid |X_k| < k]$$

by independence of the X's again. We have

$$E[|X_k| \mid |X_k| < k] = \frac{\int_{|x|<k}|x|\,dF(x)}{\int_{|x|<k}dF(x)} \leqq \frac{E|X_1|}{p_1} < \infty$$

and so (9) and the last term on the right of (8) are finite. Thus (4) $\Rightarrow$ (1).

The implication (2) $\Rightarrow$ (4) is trivial because for every rule N, $|S_N/N| \leqq \sup|S_n/n|$. Similarly, (3) $\Rightarrow$ (5). Since $\cdots, (|X_1|+|X_2|)/2, |X_1|/1$ is a martingale, ([3] page 317) shows immediately that (1) $\Rightarrow$ (2) and consequently (1) $\Rightarrow$ (3). Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (1) and the equivalence is proved.

Burkholder also gives a one-sided version of the equivalence of (1)–(3) replacing (1)–(3) by

$$(1') \qquad EX_1\log X_1^+ < \infty$$

$$(2') \qquad E\sup_n(S_n/n) < \infty$$

$$(3') \qquad E\sup_n(X_n/n) < \infty.$$

He proves that (1′)–(3′) are equivalent (and each is equivalent to an additional statement about X_1 in terms of conditional expectation operators) *provided that* $E|X_1| < \infty$. We can easily modify our proof to obtain the one-sided version and to prove that (1′)–(5′) are equivalent, where

$$\sup_{\text{rule } N} E(S_N/N) < \infty \tag{4′}$$

$$\sup_{\text{rule } N} E(X_N/N) < \infty, \tag{5′}$$

provided that $E|X_1| < \infty$. Indeed the only change required in the proof of (5) ⇒ (1) to prove (5′) ⇒ (1′) is to let N' = the first $n \geqq 1$ for which $X_n > n$; $N' = \infty$ otherwise. The proof (5′) ⇒ (1′) then proceeds exactly as before. The proof of (4′) ⇒ (1′) is also exactly as before. Again (2′) ⇒ (4′), (3′) ⇒ (5′) trivially and the final implications (1′) ⇒ (2′) and (1′) ⇒ (3′) are proved by observing that $\cdots, (X_1^+ + X_2^+)/2, X_1^+/1$ is a martingale and again applying the martingale theorem ([3] page 317).

We should point out that of course the one-sided version becomes false if $EX_1^- = \infty$ since the negative side of X_1 could then overwhelm the positive side and hence the implication (4′) ⇒ (1′) would break down.

We remark that in case (1) fails, there is a stopping rule N^* which makes the left side of (4) or (5) infinite and satisfies $P(N^* < \infty) = 1$ as well. Indeed, if N is the stopping rule we used above (with $P(N = \infty) > 0$) we can easily find an integer valued random variable T independent of N and having a sufficiently long tail for which $N^* = \min(N, T)$ does the trick. By shifting the sequence $X_1, X_2, \cdots$ and defining T as a function of X_1, we can even take N^* to be defined on the original sample space.

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