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ESTABLISHING THE POSITIVE DEFINITENESS OF THE SAMPLE COVARIANCE MATRIX

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When dealing with a nonsingular multivariate normal distribution, one makes repeated use of the fact that the sample covariance matrix is positive definite with probability one if the sample size N is larger than the dimension of the random vectors. A direct proof of this fact which does not depend upon other results is unknown to me; and I would here like to suggest such a proof.

THEOREM. *Suppose* $\mathbf{X}_1, \cdots, \mathbf{X}_N$ *is a random sample from a p-variate normal distribution whose covariance matrix* $\sum^{p\times p}$ *is of full rank. Then the sample covariance matrix* $A = \sum_{i=1}^N (\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})'$ *is positive definite with probability one if and only if* $N > p$.

PROOF. By a suitable linear transformation, we may represent $A^{p\times p}$ as $A^{p\times p} = BB'$ where $B^{p\times N-1} = (\mathbf{Z}_1, \cdots, \mathbf{Z}_{N-1})$ and the $\mathbf{Z}_i$ are mutually independent $N(\mathbf{0}, \sum^{p\times p})$ variables. By [1] (Theorem 7, page 399) it will suffice to show $B^{p\times N-1}$ has rank p with probability one if and only if $N > p$. It is clear that adding more columns cannot diminish the rank of B. Equally clear is the fact that rank $B < p$ if $N \leqq p$. Thus it will suffice to show that B has rank p with probability one when $N = p+1$.

For any set $\{\mathbf{a}_1, \cdots, \mathbf{a}_{p-1}\}$ of vectors in R^p, let $S\{\mathbf{a}_i; i = 1, \cdots, p-1\}$ be the subspace spanned by $\mathbf{a}_1, \cdots, \mathbf{a}_{p-1}$. Note that $P(\mathbf{Z}_i \in S\{\mathbf{a}_i; i = 1, \cdots, p-1\}) = 0$ for any given set of p-dimensional vectors $\mathbf{a}_1, \cdots, \mathbf{a}_{p-1}$ by the fact that $\sum^{p\times p}$ is nonsingular. Let F denote the joint distribution function of $\mathbf{Z}_2, \cdots, \mathbf{Z}_p$.

Now,

$$\begin{aligned}
&P(\text{rank } B < p)\\
&\quad = P[\mathbf{Z}_1, \cdots, \mathbf{Z}_p \text{ are linearly dependent}]\\
&\quad \leqq \textstyle\sum_{i=1}^p P[\mathbf{Z}_i \in S\{\mathbf{Z}_1, \cdots, \mathbf{Z}_{i-1}, \mathbf{Z}_{i+1}, \cdots, \mathbf{Z}_p\}]\\
&\quad = p \cdot P[\mathbf{Z}_1 \in S\{\mathbf{Z}_2, \cdots, \mathbf{Z}_p\}]\\
&\quad = p \cdot E(P[\mathbf{Z}_1 \in S\{\mathbf{Z}_2, \cdots, \mathbf{Z}_p\} \mid \mathbf{Z}_2, \cdots, \mathbf{Z}_p])\\
&\quad = p \cdot \textstyle\int_{R^{p(p-1)}} P[\mathbf{Z}_1 \in S\{\mathbf{z}_2, \cdots, \mathbf{z}_p\} \mid \mathbf{Z}_2 = \mathbf{z}_2, \cdots, \mathbf{Z}_p = \mathbf{z}_p]\, dF(\mathbf{z}_2, \cdots, \mathbf{z}_p)\\
&\quad = p \cdot \textstyle\int_{R^{p(p-1)}} P(\mathbf{Z}_1 \in S\{\mathbf{z}_2, \cdots, \mathbf{z}_p\})\, dF\\
&\quad = p \cdot \textstyle\int_{R^{p(p-1)}} 0\, dF\\
&\quad = 0.
\end{aligned}$$

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It can be shown that questions of measurability present no problem.

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REFERENCE

[1] Scheffé, Henry (1959). *The Analysis of Variance*. Wiley, New York.