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NOTE ON MOMENTS OF A LOGISTIC ORDER STATISTICS[1]

BY B. K. SHAH

Rockland State Hospital

1. Introduction and summary. For a logistic distribution which is defined by $x = \ln\{F/(1-F)\}$, where F is the probability of a value less than x, Plackett [4], Birnbaum and Dudman [1], and Gupta and Shah [2], obtained the expressions for the moments of the ordered statistics in terms of digamma functions. Once the digamma functions are tabulated and their values are stored in the computer memory, it is easy to calculate the moments (first and second). This procedure was adopted by Birnbaum and Dudman and the author (jointly with others) in the past to calculate the moments of these order statistics for which some tables are available in the literature.

In this note a recurrence relation for the moments of all order statistics for a logistic distribution is derived. This result not only generates higher moments of order statistics but also generates the moments of all order statistics for a logistic distribution, and this eliminates problems such as errors involved in computing and machine time etc. Thus using a computer language such as Fortran or APL, this simple relation is easily programmable for robustness studies, [3] for a logistic distribution.

2. Recurrence relation. Let $x_{1,n} \leqq x_{2,n} \leqq \cdots \leqq x_{n,n}$ be the order statistics in a sample of size n from any continuous distribution. Let the cumulative distribution function (cdf) be denoted by $F(x)$. It is well known that the distribution of the ith order statistics has the probability differential element

$$a_{i,n}(x)\,dx = cF^{i-1}(x)[1-F(x)]^{n-i}\,dF(x), \qquad i = 1, 2, \cdots, n, \tag{2.1}$$

where $c = n!/(i-1)!(n-i)!$, and let

$$\mu_{i,n}^{(k)} = \int_{-\infty}^{\infty} x^k a_{i,n}(x)\,dx, \qquad 1 \leqq i \leqq n, k = 1, 2, \cdots \tag{2.2}$$

with

$$\mu_{i,n}^{(1)} = \mu_{i,n}.$$

Now let us consider a logistic density function $f(x) = \exp(-x)/(1+\exp(-x))^2$, $-\infty < x < \infty$, whose mean is zero and variance is $\pi^2/3$. We also note that $F(x)[1-F(x)] = f(x)$. We will abbreviate $F(x)$ and $f(x)$ by F and f respectively.

LEMMA 1. *For a logistic distribution defined above, one has*

$$\mu_{i+1,n+1}^{(k)} = \mu_{i,n}^{(k)} + (k/i)\mu_{i,n}^{(k-1)}, \qquad k = 1, 2, \cdots, 1 \leqq i \leqq n, \tag{2.3}$$

with $\mu_{i,n}^{(0)} = 1, \quad 1 \leqq i \leqq n.$

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PROOF. Consider

$$\mu_{i,n}^{(k-1)} = E(x_{i,n}^{i-1})$$
$$= c\int_{-\infty}^{\infty} x^{k-1}F^{i-1}(1-F)^{n-i}f\,dx.$$

Integrating by parts by treating x^{k-1} for integration and $F^{i-1}(1-F)^{n-i}f$ for differentiation and noting that $f' = df/dx = f(1-2F)$, and $d\{F^{i-1}(1-F)^{n-i}f\}/dx = iF^{i-1}(1-F)^{n-i}f-(n+1)F^{i}(1-F)^{n-i}f$, we have,

$$\mu_{i,n}^{(k-1)} = c\left[\left\{\frac{x^k}{k}F^{i-1}(1-F)^{n-i}f\right\}_{-\infty}^{\infty} - \frac{1}{k}\int_{-\infty}^{\infty} x^k\frac{d}{dx}\{F^{i-1}(1-F)^{n-i}f\}\,dx\right]$$
$$= -\frac{c}{k}i\int_{-\infty}^{\infty} x^kF^{i-1}(1-F)^{n-i}\,dF + \frac{c(n+1)}{k}\int_{-\infty}^{\infty} x^kF^{i}(1-F)^{n-i}\,dF$$
$$= -\frac{i}{k}\mu_{i,n}^{(k)} + \frac{i}{k}\mu_{i+1,n+1}^{(k)}.$$

Hence the lemma.

Note that for a standardized logistic variate (i.e. whose variance is one), the above lemma is

$$\mu_{i+1,n+1}^{(k)} = \mu_{i,n}^{(k)} + (k/ig)\mu_{i,n}^{(k-1)}, \qquad k = 1,2,\cdots,1 \leqq i \leqq n \tag{2.4}$$

with $\mu_{i,n}^{(0)} = 1, 1 \leqq i \leqq n$ and where $g = \pi/3^{\frac{1}{2}}$.

ILLUSTRATION. (i) Consider $k = 1$ and we know $\mu_{1,1} = 0$, since the mean of logistic density function is zero, we can obtain $\mu_{1,2}$ by putting $n = 1$ and $i = 1$ in (2.4). Since $\mu_{1,2} = -\mu_{2,2}$ thus we have

$$\mu_{2,2} = \mu_{1,1} + (1/g)1 = 1/g = 3^{\frac{1}{2}}/\pi = 0.5513.2889.$$

Hence $\mu_{1,2} = -0.5513$ which agrees with the published tables [1], [2]. Note that Birnbaum and Dudman has given Table 1 for the sample $x_1 \geqq x_2 \cdots \geqq x_n$ instead of the usual increasing order of magnitude.

(ii) For calculating $\mu_{1,3}$, we can calculate $\mu_{3,3}$ by putting $i = 2$, $n = 2$ in (2.4). Thus we have

$$\mu_{3,3} = \mu_{2,2} + 1/(2\cdot g).$$
$$= 1/g + 1/2g$$
$$= 0.8269.9334.$$

Thus in order to find any moments of order statistics in a sample of size n drawn from a logistic distribution one does not have to calculate any integral or any digamma or polygamma functions.

For obtaining the best linear unbiased estimators for scale and location parameters of a logistic distribution, it is necessary to obtain the first and second moments of all order statistics and the covariance formula for any two logistic order statistics. Tarter and Clark [6] gave an expression for obtaining the covariance formula for any two order statistics. Shah [5] also gave a convenient recursive relation to obtain covariances of any two logistic order statistics. With the help of recursive relation 4.21 [5] and the recursive relation (2.4) mentioned above; it will be more easy to calculate the best linear unbiased estimators for any sample sizes.

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REFERENCES

[1] BIRNBAUM, A. and DUDMAN, J. (1963). Logistic order statistics. *Ann. Math. Statist.* **34** 658–663.

[2] GUPTA, S. S. and SHAH, BHUPENDRA, K. (1965). Exact moments and percentage points of the order statistics and the distribution of the range from the logistic distribution. *Ann. Math. Statist.* **36** 907–920.

[3] GASTWIRTH, J. L. and COHEN, A. (1968). The small sample behavior of some robust linear estimators of location. Tech. Report No. 91, Dept. of Statistics, Johns Hopkins Univ.

[4] PLACKETT, R. L. (1958). Linear estimation from censored data. *Ann. Math. Statist.* **29** 131–142.

[5] SHAH, B. K. (1966). On the bivariate moments of order statistics from a logistic distribution. *Ann. Math. Statist.* **37** 1002–1010.

[6] TARTER, M. E. and CLARK, V. A. (1965). Properties of the median and other order statistics of logistic variates. *Ann. Math. Statist.* **36** 1779–1786.