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NOTES

HARMONIC FUNCTIONS AND HITTING DISTRIBUTIONS FOR MARKOV PROCESSES[1]

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1. Introduction. In one dimension it is well known that a homeomorphism φ takes a continuous standard Markov process (Y, T_E) into a process having Brownian hitting distributions if and only if φ is a harmonic function for (Y, T_E). It is shown that in the complex plane the homeomorphism φ takes a continuous standard Markov process (Y, T_E) into a process having Brownian hitting distributions if and only if the functions φ^k, $k \geqq 0$, of a complex variable are complex harmonic for (Y, T_E) (i.e., real and imaginary parts are harmonic). Specifically, if (Y, T_E) has Brownian hitting distributions then $(\varphi(Y), T_{\varphi(E)})$ has Brownian hitting distributions if and only if φ or φ conjugate is analytic in E. It seems that these arguments may be extended to higher dimensions and the spherical harmonic functions; they are presented here in this framework because of their simplicity.

2. Notation. We shall call $(Y, T_E) = (Y_t, T_E, F_t, P_x)$ a process if, in the notation of Dynkin [1], it is a continuous standard Markov process with sample paths in a region E of the complex plane, and its harmonic functions are continuous. $(T_{\partial E}(\omega))$ is the infimum of $t > 0$ such that Y_t is in ∂E, if that infimum exists, and $+\infty$ otherwise. The function φ is said to be harmonic for $(Y, T_{\partial E})$ if $\varphi(x) = \mathscr{E}_x \varphi(Y_{T_{\partial U}})$ for x in E, U open, $\bar{U} \subset E$.) It follows that harmonic functions satisfy a maximum modulus theorem (thus the harmonic functions are continuous if the excessive functions are upper semi-continuous); hence, it is well known (c.f. [1]) that the following are equivalent for any real function φ:

(A) If T is a stopping time, $T < T_{\partial E}$ a.s. then $\varphi(x) = \mathscr{E}_x \varphi(Y_T)$.

(B) Given x, there exists a neighborhood $D \subset E$ of x such that $\varphi(x) = \mathscr{E}_x \varphi(Y_{T_{\partial N}})$ for all neighborhoods N of x, $N \subset D$.

Accordingly, we will take either of these to be our definition of harmonic, and we say a function is complex harmonic if its real and imaginary parts are harmonic.

3. Theorems. Let $(X, \tilde{T}_{\partial D})$ and $(Y, T_{\partial D})$ be two processes. (Stopping times for X will be denoted by $\tilde{T}$.)

We say that $(g, B) \in \mathscr{H}_X$ (alternatively $(g, B) \in \mathscr{H}_Y$) when B is a bounded region, $\bar{B} \subset D$, and g is a real harmonic function for the process $(X, \tilde{T}_{\partial B})$ (alternatively $(Y, T_{\partial B})$).

In the following theorems and discussions the process $(X, \tilde{T}_{\partial D})$ will always be

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understood to be standard Brownian motion stopped on leaving D, and D will always be an open region.

THEOREM 1. *If* $\mathscr{H}_X \subset \mathscr{H}_Y$, *then* $(X, \tilde{T}_{\partial D})$ *and* $(Y, T_{\partial D})$ *have the same class of hitting distributions, and* $\mathscr{H}_X = \mathscr{H}_Y$.

PROOF. Let a "smooth" region be a bounded region G, $\bar{G} \subset D$, such that the classical Dirichlet problem has a continuous solution in $\bar{G}$ for any continuous function defined on ∂G.

Let E be a smooth region, and let B_n be an increasing nest of smooth regions such that $\bar{B}_n \subset E$, and $\bigcup_{n=1}^{\infty} B_n = E$. (Smooth regions form a base for the usual topology in R^n.) Let $g(\cdot)$ be a continuous function defined on ∂E; we will show $\mathscr{E}_x g(X(\tilde{T}_{\partial E})) = \mathscr{E}_x g(Y(T_{\partial E}))$, and the result follows below for all bounded regions.

Let $\varphi(x) = \mathscr{E}_x g(X(\tilde{T}_{\partial E}))$. Then $(\varphi, B_n) \in \mathscr{H}_X \subset \mathscr{H}_Y$ for all n, hence $\varphi(x) = \mathscr{E}_x \varphi(Y(T_{\partial B_n}))$ when x is in B_n.

Letting n go to $+\infty$, $Y(T_{\partial B_n})$ converges a.s. to $Y(T_{\partial E})$ and since $\lim_n \varphi(\xi_n) = \varphi(\xi_0)$, where $\xi_n \in E$, $\xi_n \to \xi_0$, $\xi_0 \in \partial E$, it follows that $\mathscr{E}_x f(Y(T_{\partial E})) = \varphi(x) = \mathscr{E}_x(f(X(\tilde{T}_{\partial E})))$, (for details see Itô, McKean [2]), thus completing the proof for smooth regions.

Now let E, $E \subset D$, be any bounded region, and let C be any open subset of D. We will show that $P_x(X(\tilde{T}_{\partial E}) \in \partial E \cap C) = P_x(Y(T_{\partial E}) \in \partial E \cap C)$, hence $(X, \tilde{T}_{\partial D})$ and $(Y, \tilde{T}_{\partial D})$ have the same hitting probabilities for subsets of ∂E that are open in the relativized topology on ∂E, thus proving Theorem 1.

Let A_n be an increasing nest of smooth regions, $x \in A_n$, $A_n \subset E$, $\bigcup_{n=1}^{\infty} A_n = E$, and let

$$Z_n = X(\tilde{T}_{\partial A_n})$$
$$\xi_n = Y(T_{\partial A_n}).$$

Then, from the above and the fact that $(X, \tilde{T}_{\partial D})$, $(Y, T_{\partial D})$ have the strong Markov property, it can be seen that the joint distribution of $(Z_1, \cdots, Z_n)$ is the same as the joint distribution of $(\xi_1, \cdots, \xi_n)$. Z_n and ξ_n have a.s. limits, call them Z_0 and ξ_0 respectively.

We must show $P_x\{Z_0 \in C\} = P_x\{\xi_0 \in C\}$, but

$$\{Z_0 \in C\} = \textstyle\bigcup_{n \geq 1} \bigcap_{m \geq n} \{Z_i \in C, i = n, \cdots, m\} \text{ a.s.}$$

and

$$\{\xi_0 \in C\} = \textstyle\bigcup_{n \geq 1} \bigcap_{m \geq n} \{\xi_i \in C, i = n, \cdots, m\} \text{ a.s.}$$

From the above we see $P_x\{Z_i \in C, i = n, \cdots, m\} = P_x\{\xi_i \in C, i = n, \cdots, m\}$, and the result follows taking monotone limits.

THEOREM 2. *A necessary and sufficient condition that* $(Y, T_{\partial D})$ *have Brownian hitting distributions is that the functions* z^k, $k \geq 0$, *be complex harmonic for* $(Y, T_{\partial D})$.

PROOF. If $(Y, T_{\partial D})$ has Brownian hitting distributions, then any function analytic in a simply connected subregion B of D is a zero of Laplace's equation in B, and therefore is complex harmonic for $(Y, T_{\partial B})$. Hence z^k is complex harmonic in a simply connected neighborhood of every point of D, thus proving the first statements. On the other hand, suppose that z^k are harmonic for $(Y, T_{\partial D})$ and that f is a function harmonic for $(X, \hat{T}_{\partial D})$, we will show that f is harmonic for (Y, T_D), and the result follows by Theorem 1. If f is harmonic for $(X, \hat{T}_{\partial D})$, then given x_0 in D, there exists a simply connected open neighborhood N of x_0 whose compact closure lies in D. In $\bar{N}$, f is a zero of Laplace's equation, hence may be written as the real part of an analytic function φ, and φ may be written as the uniform limit of polynomials in z. Hence φ is the uniform limit of functions that are complex harmonic for $(Y, T_{\partial N})$, and as such is complex harmonic for $(Y, T_{\partial N})$. Thus f is harmonic for $(Y, T_{\partial D})$, as was to be shown.

If φ is a continuous one to one function on a region D, $\varphi(D)=E$, the above theorem settles the question of when the process $(\varphi(Y), \hat{T}_{\partial E})$ will have Brownian hitting distributions; for $(\varphi(Y), \hat{T}_{\partial E})=(X, \hat{T}_{\partial E})$ has Brownian hitting distributions if and only if the functions z^k, $k \geqq 0$, are complex harmonic for $(X, \hat{T}_{\partial E})$; that is, if and only if the functions $\varphi^k(\cdot)$, $k \geqq 0$ are complex harmonic for $(Y, T_{\partial D})$.

COROLLARY. *Let φ be a continuous one to one function on the region D, $\varphi[D]=E$. Then a necessary and sufficient condition that φ take (Y, T_D) into a process $(\varphi(Y), \hat{T}_E)$ with Brownian hitting distribution is that the functions $\varphi^k(\cdot)$ be complex harmonic for (Y, T_D) for all $k \geqq 0$.*

In particular, if (Y, T_D) has Brownian hitting distributions, then it is necessary and sufficient that φ or φ conjugate be analytic in D.

PROOF. In view of the above remarks, all that need be demonstrated is that if φ^k, $k>0$ is a zero of Laplace's equation in D, then either φ or $\bar{\varphi}$ satisfies the Cauchy–Riemann equations. If in some simple connected subregion E of D, $\varphi(z)=u(x+iy)+iv(x+iy)$, u, v, x, y reals, then writing out the value of the Laplacian operating on φ and also φ^2 in E we see that the functions $c(x,y)=(\partial v/\partial y)/(\partial u/\partial x)$ and $d(x,y)=(\partial v/\partial x)/(\partial u/\partial y)$ satisfy $c+d=0$ and $(1-c^2)+(1-d^2)=0$ in D; but the only real solutions to these equations are $c=\pm 1$, $d=\mp 1$, as was to be shown.

If in the above corollary we take E and D equal to the complex plane without infinity, then the only conformal mappings of E onto D are the affine transformations. ($+\infty$ is a singularity of such a mapping. If the mapping is $1-1$ it cannot be an essential singularity, hence the mapping must be a polynomial, but being $1-1$ it has only one zero.)

Thus the only homeomorphisms of the plane preserving Brownian hitting distributions are affine transformations followed by reflections about the real axis. This also seems to be the case in higher dimensions and in general the above results seem to have analogous statements in R^n in terms of expansions involving the spherical harmonic functions.

REFERENCES

[1] DYNKIN, E. B. (1965). *Markov Processes* **1**, **2**. Springer-Verlag, Berlin.

[2] ITÔ, K. and MCKEAN, H. P., Jr. (1965). *Diffusion Processes and their Sample Paths*, Springer-Verlag, Berlin.