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THE FOLDED CUBIC ASSOCIATION SCHEME

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1. The association scheme. Raghavarao and Chandrasekhararao (1964) introduced the cubic association scheme for partially balanced incomplete block designs with three associate classes. There are $v = s^3$ varieties; each variety is represented by a different set of three coordinates (z_1, z_2, z_3), where z_1, z_2, z_3 are integers and $1 \leqq z_1, z_2, z_3 \leqq s$. Two varieties are ith associates if they have $(3-i)$ coordinates the same. Thus $(0, 0, 0)$ has $(0, 1, 0)$ among its first associates and $(0, 1, 1)$ among its second associates.

In this paper we consider an extension of the cubic scheme for the case $s = 4$, $v = 64$. The coordinates z_1, z_2, z_3 do not take the integer values $1, 2, 3, 4$ but, instead, are elements of a Galois field of four elements: $0, 1, x$ and $y(=1+x)$, with addition modulo 2. We now introduce a fourth coordinate defined by $z_1+z_2+z_3+z_4 \equiv 0 \bmod 2$, or equivalently $z_4 \equiv z_1+z_2+z_3$, thus dividing the sixty-four points into four hyperplanes or flats. This leads to the following partially balanced scheme with four associate classes. Two varieties are first associates if two of their four coordinates are identical, e.g. $(1, 0, x, y)$ and $(1, 0, 1, 0)$, and second associates if their representations coincide at only one coordinate, such as $(0, 0, 0, 0)$ and $(0, 1, x, y)$. The three fourth associates of (z_1, z_2, z_3, z_4) are $(z_1+a, z_2+a, z_3+a, z_4+a)$, $a = 1, x, y$. If two varieties are not first, second, or fourth associates, they are third associates. We shall call this new scheme the folded cubic scheme. In the remainder of the paper we shall omit the commas and parentheses in denoting varieties, and write, for example, $01xy$ or 1010.

Since $z_i + z_i = 0$ for all z_i in $GF(2^2)$, it follows that the representations of the varieties fall into three types:

(i) $z_1 = z_2 = z_3 = z_4$: *aaaa*,

(ii) the z_i occur in two pairs: *aabb*, *abab*, *abba*, $a \neq b$,

(iii) the z_i are all different: *abcd*, $a \neq b \neq c \neq d$.

Thus two varieties cannot have more than two coordinates equal, and $n_1 = 18$, $n_2 = 24$, $n_3 = 18$, $n_4 = 3$.

The fourth associates of *aaaa* are *bbbb*, *cccc*, *dddd*; for *aabb* the fourth associates are *bbaa*, *ccdd*, *ddcc*; for *abcd* they are *badc*, *cdab* and *dcba*.

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The P matrices of the scheme are

$$\mathbf{P}_1 = \begin{bmatrix} 6 & 8 & 2 & 1 \\ 8 & 8 & 8 & 0 \\ 2 & 8 & 6 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 6 & 6 & 6 & 0 \\ 6 & 8 & 6 & 3 \\ 6 & 6 & 6 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P}_3 = \begin{bmatrix} 2 & 8 & 6 & 2 \\ 8 & 8 & 8 & 0 \\ 6 & 8 & 2 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{P}_4 = \begin{bmatrix} 6 & 0 & 12 & 0 \\ 0 & 24 & 0 & 0 \\ 12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\mathbf{N}$ be the incidence matrix of a design with this association scheme, and let θ_i, $i = 1, 2, 3, 4$, be the distinct latent roots of $\mathbf{NN}'$ other than the trivial root rk. They are the same as the latent roots of the matrix $\mathbf{\Pi}^*$ (Bose (1963); Bose and Mesner (1959)), which is a square matrix of order four with elements

$$p_{ij}^* = r\delta_{ij} + \textstyle\sum_k p_{ik}^j \lambda_k - n_i \lambda_i,$$

where δ_{ij} is the Kronecker delta.

The latent roots are

$$\begin{aligned} \theta_1 &= r - 6\lambda_1 + 8\lambda_2 - 6\lambda_3 + 3\lambda_4, \\ \theta_2 &= r + 6\lambda_1 - 6\lambda_3 - \lambda_4, \\ \theta_3 &= r - 2\lambda_1 + 2\lambda_3 - \lambda_4, \\ \theta_4 &= r + 2\lambda_1 - 8\lambda_2 + 2\lambda_3 + 3\lambda_4. \end{aligned}$$

These may be written in matrix form as

$$\boldsymbol{\theta} = r\mathbf{1} + \mathbf{Z}^*\mathbf{\Lambda},$$

where $\mathbf{\Lambda}' = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Then (John (1966)) the multiplicities α_i of the latent roots are given by

$$\boldsymbol{\alpha} = -(\mathbf{Z}^{*\prime})^{-1}\mathbf{n},$$

where $\mathbf{n} = (n_1, n_2, n_3, n_4)$.

The multiplicities in this case are $\alpha_1 = 6$, $\alpha_2 = 12$, $\alpha_3 = 36$, $\alpha_4 = 9$.

Viewed geometrically the representation of the varieties in the cubic scheme associates each variety with a point in a finite Euclidean geometry, and the set of points with $z_4 \equiv c$ $(c = 0, 1, x, y)$ lie on parallel hyperplanes. We may obtain a similar association scheme by taking as our definition of z_4 any other set of parallel hyperplanes $z_4 = \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3$ where $\beta_1, \beta_2, \beta_3$ are non-zero elements of GF(2^2). In the general case the first, second and third associates of each variety are defined as before. The fourth associates of (z_1, z_2, z_3, z_4) are $(z_1 + a\beta_1{}^{-1}$, $z_2 + a\beta_2{}^{-1}$, $z_3 + a\beta_3{}^{-1}$, $z_4 + a)$, $a = 1, x, y$. If any of the β_i is zero we no longer have a scheme of this type.

2. The nonexistence of some designs. For any design the latent roots θ must all be nonnegative, since the matrix NN′ is positive semidefinite. Furthermore the parameters must satisfy

$$r(k-1) = \sum \lambda_i n_i = 18\lambda_1 + 24\lambda_2 + 18\lambda_3 + 3\lambda_4, \tag{1}$$

so that either r or $k-1$ must be divisible by 3.

If $k = 4$, we have $r = 6(\lambda_1+\lambda_3)+8\lambda_2+\lambda_4$. Substituting for r, it is seen that all four latent roots are nonnegative for all values of Λ satisfying (1). However, if $k = 3t+1$ and $\lambda_2 = \lambda_4 = 0$, we have $\theta_1 = r(1-t)$, which is negative for $t > 1$. It follows that for $k = 3t+1$, $t > 1$, there are no designs for which λ_2 and λ_4 are simultaneously zero. There are also sets of parameters r, Λ satisfying (1) which can be shown to correspond to nonexistent designs since θ_2, θ_3 or θ_4 would be negative. Examples are

(i) $r = 4,\ k = 16,\ \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2,\ \lambda_4 = 0;\ \theta_2 = -8,$
(ii) $r = 5,\ k = 10,\ \lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 3;\ \theta_3 = -2,$
(iii) $r = 6,\ k = 16,\ \lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 0 = \lambda_4;\ \theta_4 = -16.$

The condition $b \geqq v - \sum' \alpha_i$, where $\sum' \alpha_i$ denotes the sum of the multiplicities of all the zero latent roots may also be used to establish the nonexistence of designs. The set of parameters $r = 5$, $k = 10$, $\Lambda' = (1, 0, 1, 3)$ gives $\theta_1 = \theta_2 = \theta_3 = 2$, $\theta_4 = 18$, and $b = 32$. However, $\sum' \alpha_i = 0$, and there is no such design since $b < v$.

For a symmetric design $|\mathrm{NN}'| = |\mathrm{N}|^2 = r^2\theta_1^{\alpha_1}\theta_2^{\alpha_2}\theta_3^{\alpha_3}\theta_4^{\alpha_4}$. The only odd multiplicity is α_4, so that a necessary condition for a symmetric design to exist with positive latent roots is that θ_4 be a square; (1) also implies that $r \equiv 0$ or $1 \pmod 3$, and that λ_4 is even (or zero). Eliminating λ_4 by the use of (1)

$$\theta_4 = r^2 - 16(\lambda_1+\lambda_3) - 32\lambda_2 = r^2 - 16(\lambda_1 + 2\lambda_2 + \lambda_3).$$

This eliminates for example designs with $r = k = 16$ unless $\lambda_1 + 2\lambda_2 + \lambda_3$ equals 7 or 12 or with $r = k = 9$ unless $\lambda_1 + 2\lambda_2 + \lambda_3$ equals 2 or 5.

For the moment denote varieties by capital letters $U, V, \cdots$ and write $(U, V) = i$ to denote that U and V are ith associates. Consider a block from a design with $\lambda_2 = \lambda_3 = 0$, which contains varieties X, A, S, T, such that $(X, A) = 1$, $(X, S) = (X, T) = 4$. Then $(A, S) = (A, T) = 1$, but this is impossible since $p_{14}^1 = 1$. If $3\lambda_4 > r$ there must be at least one block which contains X and two or more of its fourth associates; if $k > 4$, that block must also contain at least one first associate of X. A similar argument holds if $\lambda_1 = \lambda_2 = 0$ and $(X, A) = 3$. Hence if $k > 4$, $3\lambda_4 > r$, there are no designs with either $\lambda_1 = \lambda_2 = 0$ or with $\lambda_2 = \lambda_3 = 0$. This rules out designs with $b = 64$, $k = 6$, $r = 6$, $\lambda_2 = 0$, $\lambda_4 = 4$ and $\lambda_1 = 1$ or $\lambda_3 = 1$.

3. Some degenerate schemes. If $\lambda_1 = \lambda_3$ we have $\theta_2 = \theta_3$ so that there are only three distinct latent roots. Pooling the first and third associate classes we obtain a new association scheme. Let treatments in the new scheme be first associates if they were first or third associates before, second if they were second associates before and third associates if they were fourth associates before. The new P matrices are

$$P_1 = \begin{bmatrix} 16 & 16 & 3 \\ 16 & 8 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 24 & 12 & 0 \\ 12 & 8 & 3 \\ 0 & 3 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$\theta_1 = r - 12\lambda_1 + 8\lambda_2 + 3\lambda_3$, $\theta_2 = r - \lambda_3$, $\theta_3 = r + 4\lambda_1 - 8\lambda_2 + 3\lambda_3$,

$$\alpha_1 = 6, \qquad \alpha_2 = 48, \qquad \alpha_3 = 9.$$

Except for the numbering of the associate classes this is a hierarchic L_3 scheme, or an $L_3(4,4)$ scheme (Singh and Singh (1964)). The 64 treatments are divided into 16 groups of mutual fourth associates. From each group take the treatment with $z_1 = 0$ as the key member; arrange the key members in a square array with members going into the ith row if $z_2 = i$ and the ith column if $z_3 = i$. The array forms a 4×4 Latin square with z_4 corresponding to letters. Treatments in the same group are third associates. Treatments in different groups are first associates if the key members of their groups have z_2 or z_3 or z_4 equal; otherwise they are second associates.

Pooling the first, second and third associates in the folded cubic gives a G.D. two associate class scheme with sixteen groups of four treatments each.

If $\lambda_1 = \lambda_2 = \lambda_4$, then $\theta_1 = \theta_2$, $\theta_3 = \theta_4$. Thus pooling the first, second and fourth associates converts the folded cubic scheme into a partially balanced scheme with two associate classes. In this new scheme let two varieties be first associates if they are first, second or fourth associates in the folded cubic scheme and second associates otherwise. The two-class scheme has P matrices

$$P_1 = \begin{pmatrix} 32 & 12 \\ 12 & 6 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 30 & 15 \\ 15 & 2 \end{pmatrix};$$

$\theta_1 = r + 5\lambda_1 - 6\lambda_2$, $\alpha_1 = 18$, $\theta_2 = r - \lambda_2$, $\alpha_2 = 45$.

This scheme has the same parameters as the negative Latin square scheme with $i = -5$, $n = -8$ (Mesner (1967)).

If $\lambda_1 = \lambda_4$, $\lambda_2 = \lambda_3$, then $\theta_1 = \theta_3$, $\theta_2 = \theta_4$ and the folded cubic scheme degenerates to a two class scheme. Letting two varieties be first associates in the new scheme if they are first or fourth associates in the original scheme and second associates otherwise we obtain the P matrices

$$P_1 = \begin{pmatrix} 8 & 12 \\ 12 & 30 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 6 & 15 \\ 15 & 26 \end{pmatrix};$$

$\theta_1 = r - 3\lambda_1 + 2\lambda_2$, $\alpha_1 = 42$, $\theta_2 = r + 5\lambda_1 - 6\lambda_2$, $\alpha_2 = 21$.

It will be shown below that this is an $L_3(8)$ association scheme.

4. Some designs with the folded cubic scheme. Associated with any partially balanced scheme are two basic sets of elementary symmetric designs which we denote by ES(i) and E′S(i). In the design ES(i), $i = 1, 2, 3, 4$, the jth block consists of all those varieties which are ith associates of the jth variety. Then $r = k = n_i$ and $\lambda_t = p^t_{ii}$ for $t = 1, 2, 3, 4$. In the design E′S(i) the jth block contains the jth treatment together with all its ith associates; for this design $r = k = n_i + 1$, $\lambda_i = p^i_{ii} + 2$, and $\lambda_t = p^t_{ii}$ $(t \neq i)$.

For the folded cubic scheme it is readily seen that

ES(1) and ES(2) each have $r = k = 18$, $\lambda_1 = \lambda_2 = \lambda_4 = 6$, $\lambda_3 = 2$, and are two associate class designs. E′S(3) is also a two associate class design with $r = k = 19$, $\lambda_1 = \lambda_2 = \lambda_4 = 6$, $\lambda_3 = 4$. ES(2) is a singular GD design with $r = k = 24$, $\lambda_1 = \lambda_2 = \lambda_3 = 8$ and $\lambda_4 = 24$. ES(4) and E′S(4) are disconnected GD designs. E′S(2) has $r = k = 25$, $\lambda_1 = \lambda_3 = 8$, $\lambda_2 = 10$, $\lambda_4 = 24$ and is a three associate class design.

The only symmetric design in these two series with four associate classes is E′S(1). For this design $r = k = 19$, $\lambda_1 = 8$, $\lambda_3 = 2$ and $\lambda_2 = \lambda_4 = 6$. Although $\lambda_2 = \lambda_4$, the latent roots are distinct, $\boldsymbol{\theta}' = (25, 49, 1, 9)$. We may also consider designs of this type using several associate classes, letting ES(i_1, i_2), $i_1 \neq i_2$, denote the design in which the jth block consists of all varieties which are either i_1th or i_2th associates of the jth variety. E′S(i_1, i_2) denotes the corresponding design with the jth variety included in the jth block. The incidence matrix $\mathbf{N}$ of ES(i_1, i_2) is the sum, $\mathbf{B}_{i_1} + \mathbf{B}_{i_2}$, of the corresponding association matrices. The ith association matrix $\mathbf{B}_i$, (Bose and Mesner), is a symmetric $v \times v$ matrix. The element in the sth row and tth column is unity if the sth and tth varieties are ith associates. Otherwise it is zero. If we regard each variety as a 0th associate of itself we have $\mathbf{B}_0 = \mathbf{I}$. The incidence matrix of E′S(i_1, i_2) is $\mathbf{B}_0 + \mathbf{B}_{i_1} + \mathbf{B}_{i_2}$.

There are only two of these designs that have four associate classes, ES(3, 4) and its complement E′S(1, 2). The others all correspond to degenerate schemes. For ES(3, 4) we have $r = k = 21$, $\boldsymbol{\Lambda}' = (10, 6, 4, 8)$, $\boldsymbol{\theta}' = (9, 49, 1, 25)$.

5. Cyclic designs. We consider designs with $b = 64s$ obtained by cycling from a set of s initial blocks of size k. We define the difference between two varieties $v_1 \equiv (a, b, c, d)$, $v_2 \equiv (p, q, r, t)$ as $[v_1 - v_2] = [a-p, b-q, c-r, d-t]$, and note that $[v_1 - v_2] = [v_2 - v_1]$. Each block contains $k(k-1)$ differences $[v_i - v_j]$.

LEMMA. *If the set of $sk(k-1)$ differences from the initial blocks contains*

(i) *each difference of the type $[a, a, b, b]$, $[a, b, a, b]$, $[a, b, b, a]$, where either $a = 0$ or $b = 0$, exactly λ_1 times,*

(ii) *each difference $[a, b, c, d]$ where a, b, c, d are all different, exactly λ_2 times,*

(iii) *each difference $[a, a, b, b]$, $[a, b, a, b]$, $[a, b, b, a]$, where $a \neq 0$, $b \neq 0$, exactly λ_3 times,*

(iv) *each difference $[a, a, a, a]$ where $a \neq 0$, exactly λ_4 times, then the design is a partially balanced design with parameters $b = 64s$, $k = k$, $r = 4s$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.*

The proof of the lemma is essentially the same as the proof of the first fundamental theorem of the method of differences for balanced incomplete blocks, (Bose (1938)) and will not be given.

Design 4.1. Consider the set of six initial blocks

(i) 0000, 0011, 00*xx*, 00*yy*;
(ii) 0000, 0101, 0*x*0*x*, 0*y*0*y*;
(iii) 0000, 0110, 0*xx*0, 0*yy*0;
(iv) 0000, 1001, *x*00*x*, *y*00*y*;
(v) 0000, 1010, *x*0*x*0, *y*0*y*0;
(vi) 0000, 1100, *xx*00, *yy*00.

The complete design has $b = 384$, $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$. However each block appears four times. For example (i) also appears as 0011, 0000, 00*yy*, 00*xx*; 00*xx*, 00*yy*, 0000, 0011 and 00*yy*, 00*xx*, 0011, 0000. Taking each block only once gives design 4.1 with $b = 96$, $k = 4$, $r = 6$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, $E = 0.727$.

Design 4.2. We take a set of eight initial blocks.

(i) 0000, 01*xy*, 0*xy*1, 0*y*1*x*;
(ii) 0000, 0*yx*1, 0*x*1*y*, 01*yx*;
(iii) 0000, 10*xy*, *x*0*y*1, *y*01*x*;
(iv) 0000, *y*0*x*1, *x*01*y*, 10*yx*;
(v) 0000, 1*x*0*y*, *xy*01, *y*10*x*;
(vi) 0000, *yx*01, *x*10*y*, 1*y*0*x*;
(vii) 0000, 1*xy*0, *xy*10, *y*1*x*0;
(viii) 0000, *yx*10, *x*1*y*0, 1*yx*0.

Again each block appears four times. Taking each block only once gives $b = 128$, $k = 4$, $r = 8$, $\lambda_2 = 1$, $\lambda_1 = \lambda_3 = \lambda_4 = 0$, $E = 0.741$.

Design 4.3. There are six initial blocks

(i) 0000, 11*xx*, *xxyy*, *yy*11;
(ii) 0000, 1*x*1*x*, *xyxy*, *y*1*y*1;
(iii) 0000, 1*xx*1, *xyyx*, *y*11*y*;
(iv) 0000, *x*11*x*, *yxxy*, 1*yy*1;
(v) 0000, *x*1*x*1, *yxyx*, 1*y*1*y*;
(vi) 0000, *xx*11, *yyxx*, 11*yy*.

Ignoring repeated blocks we have $b = 96$, $\lambda_3 = 1$, $\lambda_1 = \lambda_2 = \lambda_4 = 0$, $E = 0.737$.

Design 4.4. An unconnected design for $b = 16$, $\lambda_4 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is obtained by taking the sixteen sets of mutual fourth associates, or equivalently by using the initial block 0000, 1111, *xxxx*, *yyyy*.

These four designs combine to give a balanced incomplete design for $v = 64$, $b = 336$, $k = 4$, $r = 21$, $\lambda = 1$, $E = 0.762$.

Design 6.1. The two initial blocks

(i) 0000, 1111, 01*xy*, *xy*01, *x*01*y*, 1*yx*0;
(ii) 0000, 1111, 01*yx*, *xy*10, *x*0*y*1, 1*y*0*x*

give a design with $b = 128$, $k = 6$, $r = 12$, $\lambda_2 = 2$, $\lambda_4 = 4$, $\lambda_1 = \lambda_3 = 0$, $E = 0.812$.

Design 6.2. There are three intial blocks 0000, $zzzz$, 00zz, zz00, z0z0, 0z0z, $z = 1, x, y$. Each block is duplicated. Eliminating duplicates gives $b = 96$, $k = 6$, $r = 9$, $\lambda_1 = 2$, $\lambda_4 = 3$, $\lambda_2 = \lambda_3 = 0$, $E = 0.815$.

Design 8.1. We take three initial blocks

(i) 0000, 1111, 1100, 0011, 1010, 0101, 1001, 0110;
(ii) 0000, $xxxx$, xx00, 00xx, x0x0, 0x0x, x00x, 0xx0;
(iii) 0000, $yyyy$, yy00, 00yy, y0y0, 0y0y, y00y, 0yy0.

Each block is repeated eight times. Taking each block once gives a design with $b = 24$, $k = 8$, $r = 3$, $\lambda_1 = \lambda_4 = 1$, $\lambda_2 = \lambda_3 = 0$, $E = 0.857$, which is isomorphic to a triple lattice $L_3(8)$ design. To obtain the correspondence between the schemes when this is regarded as a two class design, write the elements of blocks (i), (ii) and (iii), in the order given, in the first row, the first column and along the main diagonal of an 8×8 square. Complete the second row by adding $[x, x, x, x]$ to each entry in the first row; complete the third row by adding $[x, x, 0, 0]$ and so on. Assign the letter A to the entries along the main diagonal. Two of the blocks containing $xxxx$ are the second row and the first column; assign to the members of the third block which contains $xxxx$ the letter B; assign to the members of the third block which contains xx00 the letter C, and so on. Two varieties are now first (or fourth) associates if they share a common row, column or letter in the Latin square. Our design is then obtained from the $L_3(8)$ scheme by taking the three replicates "rows," "columns," and "letters." A design with $b = 48$, $\lambda_2 = \lambda_3 = 1$, $\lambda_1 = \lambda_4 = 0$, $E = 0.882$ is obtained by deleting design 8.1 from the orthogonal series balanced design for the 64 varieties.

Design 8.2. There are three initial blocks 0000, $aaaa$, $bbcc$, $ccbb$, $bcbc$, $cbcb$, $bccb$, $cbbc$, where a takes in turn the values $1, x, y$ and a, b, c are different from 0 and each other in any block. Each block is duplicated; taking each block once gives $b = 96$, $k = 8$, $r = 12$, $\lambda_1 = \lambda_3 = 2$, $\lambda_4 = 4$, $\lambda_2 = 0$, $E = 0.877$.

Design 8.3. There are 9 initial blocks with a, b, c different from 0 and each other in any block: 0000, $aaaa$, 00aa, aa00, $bcbc$, $cbcb$, $bccb$, $cbbc$; 0000, $aaaa$, 0a0a, a0a0, $bbcc$, $ccbb$, $bccb$, $cbbc$; 0000, $aaaa$, 0aa0, a00a, $bbcc$, $ccbb$, $bcbc$, $cbcb$. Each block is repeated eight times; omitting repeats gives a design with $b = 72$, $k = 8$, $r = 9$, $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = 3$, $E = 0.875$.

6. The scheme for $v = 27$. For $s = 3$, $v = 27$, the varieties in the cubic scheme may be written as (z_1, z_2, z_3) where $z_i = 0, 1, 2$; we add a fourth coordinate z_4, where $z_1 + z_2 + z_3 + z_4 \equiv 0 \pmod 3$. It will be more convenient to define two varieties to be first, second or third associates if they have zero, two or one coordinates in common respectively. In this case the scheme is a PBIB scheme with three associate classes. (The fourth associate class from the previous case is absorbed in the second class, since adding 1 to z_1, z_2, z_3 or adding 2 to z_1, z_2, z_3 leaves z_4 unchanged with arithmetic mod 3.) Then $n_1 = 6$, $n_2 = 12$, $n_3 = 8$, and the P matrices are identical with those of the cubic scheme with $s = 3$. To see that the schemes are equivalent, associate variety (z_1, z_2, z_3, z_4) in the folded scheme with the variety

(y_1, y_2, y_3) in the cubic scheme, where $y_i = z_i + z_4 \pmod 3$, $i = 1, 2, 3$. It is then easily verified that the jth associates of any treatment are the same in the two schemes. For example, the first associates of 0000 in the folded scheme are $(t, t, 2t, 2t)$, $(t, 2t, t, 2t)$ and $(2t, t, t, 2t)$, $t = 0, 1$. Their images in the cubic scheme are $(0, 0, t)$, $(0, t, 0)$, $(t, 0, 0)$ which are first associates of $(0, 0, 0)$.

7. The case $s = 5$. When $s = 5$ the coordinates z_i take the values 0, 1, 2, 3 and 4 with arithmetic modulo 5. We again add a fourth coordinate z_4 such that $z_1 + z_2 + z_3 + z_4 \equiv 0 \bmod 5$. The previous definition of associate classes no longer leads to a partially balanced scheme. Among the third associates of 1234 are 0000 and 2120; 1234 has only two first associates in common with 0000, namely 1004 and 0230, but it has four first associates in common with 2120, namely 1130, 1220, 2134 and 2224. We can however modify our definitions slightly to obtain a partially balanced scheme with five classes.

As before two varieties are said to be first associates if they have two identical coordinates and second associates if they have a single coordinate in common. The varieties which contain no zero coordinates fall into three types which we shall call (iii), (iv) and (v). They are distinguished as follows:

(iii) the coordinates fall into two pairs *aabb*, *abab*, *abba*, where $a + b \equiv 0 \pmod 5$,
(iv) three coordinates are the same *aaab*, *aaba*, *abaa*, *baaa* with $b \equiv 2a \pmod 5$,
(v) all four coordinates are different.

We define two varieties $(z_{11}, z_{12}, z_{13}, z_{14})$ and $(z_{21}, z_{22}, z_{23}, z_{24})$ to be third, fourth or fifth associates if their difference $[z_{11} - z_{21}, z_{12} - z_{22}, z_{13} - z_{23}, z_{14} - z_{24}]$ is of type (iii), (iv) or (v) respectively. With this definition 1234 is a fifth associate of 0000 and a third associate of 2120. It can be shown that this definition of the associate classes gives a partially balanced scheme with $n_1 = 24$, $n_2 = 48$, $n_3 = 12$, $n_4 = 16$ and $n_5 = 24$. The first two P matrices of the scheme are

$$P_1 = \begin{bmatrix} 7 & 12 & 2 & 0 & 2 \\ 12 & 18 & 2 & 8 & 8 \\ 2 & 2 & 2 & 2 & 4 \\ 0 & 8 & 2 & 2 & 4 \\ 2 & 8 & 4 & 4 & 6 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6 & 9 & 1 & 4 & 4 \\ 9 & 17 & 7 & 4 & 10 \\ 1 & 7 & 0 & 2 & 2 \\ 4 & 4 & 2 & 2 & 4 \\ 4 & 10 & 2 & 4 & 4 \end{bmatrix}.$$

We also have $p_{33}^3 = p_{44}^4 = p_{55}^5 = 3$ and $p_{34}^3 = p_{45}^3 = 0$. There does not appear to be any way of combining the associate classes to obtain a four class scheme, and by analogy from the case $s = 4$ it is to be expected that defining z_4 by $z_4 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$ where $\alpha_1, \alpha_2, \alpha_3$ are nonzero elements of GF(5) will give rise to similar schemes. Designs with this scheme for $s = 5$ have not yet been investigated. Similar extensions of these schemes for $s > 5$ with $m > 4$ associate classes can be conjectured. The large number of varieties required when $s > 5$ suggests, however, that this may not be of practical interest.

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