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CONVERGENCE OF SUMS OF RANDOM VARIABLES CONDITIONED ON A FUTURE CHANGE OF SIGN[1]

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1. Introduction and notation. Let $\{\xi_k\}$ be a sequence of independent and identically distributed random variables which have a non-lattice distribution and which are in the domain of attraction of a non-degenerate stable law of index $\alpha (0 < \alpha \leqq 2)$. Let S_n denote the partial sum $\xi_1 + \cdots + \xi_n$. It is the purpose of this paper to study the limiting behavior of $\{S_k, k \leqq n\}$ as $n \to \infty$ under the condition that $S_n S_{n+1} < 0$. It turns out that the results depend on whether $\alpha < 1$ or $\alpha \geqq 1$ as is so often the case in the study of stable laws and their domains of attraction.

We will assume for simplicity that no centering constants are needed for the convergence of the normalized partial sums. So, there are constants $b_n > 0$ so that S_n/b_n converges in distribution to an appropriate stable law, which we will call $X(1)$. For each n, define a stochastic process $X_n(t)$ by

$$X_n(t) = S_{[(n+1)t]}/b_n \quad \text{for} \quad 0 \leqq t < 1$$
$$= S_n/b_n \quad \text{for} \quad t = 1.$$

These processes are regarded as random elements of Skorokhod's space $D[0, 1]$ (see Chapter 3 of [1]). It then follows from a theorem of Skorokhod that $X_n(t)$ converges weakly in $D[0, 1]$ to a stable process $X(t)$ whose one-dimensional distributions are the same as those of $t^{1/\alpha}X(1)$ (see Theorem 1 of [5]).

The processes of interest here are $(X_n(t) \mid S_n S_{n+1} < 0)$, which again are regarded as random elements of $D[0, 1]$. In order to insure that the conditioning event $(S_n S_{n+1} < 0)$ has positive probability for each n, we will assume that the stable law $X(1)$ is not one-sided. The main result of this paper is that the processes $(X_n \mid S_n S_{n+1} < 0)$ converge weakly to a limiting process which concentrates at the origin at time one if $1 \leqq \alpha \leqq 2$ and which concentrates on the whole line at time one if $\alpha < 1$.

In two earlier papers, the author studied the limiting behavior of processes of the form $(X_n(t) \mid X_n(1) \in E^n)$ where E^n is a sequence of Borel subsets of the real line ([5] and [6]). Theorem 1 of this paper may be regarded as an application of the results of [6].

2. The case $1 \leqq \alpha \leqq 2$. We will show that in this case $(X_n(t) \mid S_n S_{n+1} < 0)$ converges to the process which is obtained by "tying" $X(t)$ down to the origin at time one. For $\alpha = 2$, this process is the ordinary Brownian Bridge.

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In order to obtain the desired result for $\alpha \geqq 1$, we will need the following two simple lemmas. In the first one, no restriction is made on the value of α.

LEMMA 1.

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P(|\xi_{n+1}| \leqq \delta \mid S_n S_{n+1} < 0) = 0.$$

PROOF. Choose an $\varepsilon > 0$ so that $P(|\xi_{n+1}| > \varepsilon) > 0$. Then by Theorem 1 of [7],

$$\begin{aligned} P(S_n S_{n+1} < 0) &\geqq P(\xi_{n+1} > \varepsilon)P(-\varepsilon < S_n < 0) \\ &\quad + P(\xi_{n+1} < -\varepsilon)P(0 < S_n < \varepsilon) \\ &= \varepsilon b_n^{-1} p(0)[1+o(1)]P(|\xi_{n+1}| > \varepsilon) \end{aligned}$$

where $p(\cdot)$ is the density of $X(1)$ and $o(1) \to 0$ as $n \to \infty$. So,

$$\liminf_{n\to\infty} b_n P(S_n S_{n+1} < 0) > 0. \tag{1}$$

On the other hand,

$$\begin{aligned} P(|\xi_{n+1}| \leqq \delta, S_n S_{n+1} < 0) &\leqq P(0 \leqq \xi_{n+1} \leqq \delta)P(-\delta < S_n < 0) \\ &\quad + P(-\delta \leqq \xi_{n+1} \leqq 0)P(0 < S_n < \delta). \end{aligned}$$

Again using Theorem 1 of [7], we have

$$\limsup_n b_n P(|\xi_{n+1}| \leqq \delta, S_n S_{n+1} < 0) \leqq \delta p(0),$$

thus concluding the proof of the lemma.

LEMMA 2. *Assume* $1 \leqq \alpha \leqq 2$. *Then, for each* $y > 0$,

$$P(|\xi_{n+1}| \geqq y b_n \mid S_n S_{n+1} < 0) \to 0$$

as $n \to \infty$.

PROOF. Consider first the case $1 \leqq \alpha < 2$. From the known tail behavior of the distribution function of ξ_i (see page 176 of [4]), we see that for each $y > 0$,

$$nP(|\xi_{n+1}| \geqq y b_n) \to B y^{-\alpha} \tag{2}$$

where $0 < B < \infty$. On the other hand,

$$\begin{aligned} nP(S_n S_{n+1} < 0) &= \int_{0+}^{\infty} P(X_n(1) \in dx) nP(\xi_{n+1} < -x b_n) \\ &\quad + \int_{-\infty}^{0-} P(X_n(1) \in dx) nP(\xi_{n+1} > -x b_n). \end{aligned}$$

Since $P(X_n(1) \in dx)$ converges weakly to $p(x)\,dx$, and since $\int |x|^{-\alpha} p(x)\,dx = \infty$, it follows from (2) that $nP(S_n S_{n+1} < 0) \to \infty$ as $n \to \infty$. Using (2) again, we see that the lemma holds.

Now suppose $\alpha = 2$. By Theorem 1, page 172 of [4]

$$\frac{P(|\xi_{n+1}| \geqq y b_n) y^2 b_n^{\,2}}{\int_{-y b_n}^{y b_n} x^2 P(\xi_{n+1} \in dx)} \to 0$$

for each $y > 0$. It then follows from a simple estimate that

$$b_n P(|\xi_{n+1}| \geqq y b_n) \to 0$$

for each $y > 0$. This, together with (1) yields the conclusion of the lemma for $\alpha = 2$.

THEOREM 1. *Assume* $1 \leqq \alpha \leqq 2$. *Then the processes* $(X_n(t) \mid S_n S_{n+1} < 0)$ *converge weakly in* $D[0, 1]$ *to the process* $Y(t)$ *which has finite dimensional distributions given by*

$$P(Y(t_1) \in A_1, \cdots, Y(t_k) \in A_k)$$
$$= \int_{A_1} \cdots \int_{A_k} \frac{p\left(-\frac{z_k}{(1-t_k)^{1/\alpha}}\right)}{(1-t_k)^{1/\alpha} p(0)} P(X(t_1) \in dz_1, \cdots, X(t_k) \in dz_k)$$

for $0 < t_1 < \cdots < t_k < 1$; $Y(0) \equiv 0$, $Y(1) \equiv 0$. *Here* $A_1, \cdots, A_k$ *are Borel subsets of* R^1.

PROOF. That there exists a random element Y of $D[0, 1]$ with the given finite dimensional distributions is a consequence of Theorem 4 of [5]. Fix a Borel subset A of $D[0, 1]$ which satisfies $P(Y \in \partial A) = 0$, and define a sequence of measures $k_n(dx)$ on the real line and a sequence of functions $f_n(x)$ by $k_n(dx) = P(\xi_{n+1} \in dx \mid S_n S_{n+1} < 0)$ and $f_n(x) = P(X_n \in A \mid -x^2 < xS_n < 0)$. Note that $f_n(x)$ is not defined for all $x \in R$, since the conditioning event may have zero probability for small x, but it will be defined for all x for which it is used in what follows. Then, we may write

$$\begin{aligned} P(X_n \in A, S_n S_{n+1} < 0) &= \textstyle\int_{-\infty}^{\infty} P(X_n \in A, -x^2 < xS_n < 0) P(\xi_{n+1} \in dx) \\ &= \textstyle\int_{-\infty}^{\infty} f_n(x) P(-x^2 < xS_n < 0) P(\xi_{n+1} \in dx) \\ &= \textstyle\int_{-\infty}^{\infty} f_n(x) P(\xi_{n+1} \in dx, S_n S_{n+1} < 0). \end{aligned}$$

Here the first and third equalities follow from Fubini's Theorem, since X_n and ξ_{n+1} are independent. So, we have

$$P(X_n \in A \mid S_n S_{n+1} < 0) = \textstyle\int_{-\infty}^{\infty} f_n(x) k_n(dx). \tag{3}$$

By Theorem 4 of [6], if $\delta > 0$ and $\delta_n \downarrow 0$, $f_n(x_n) \to P(Y \in A)$ for any sequence $\{x_n\}$ with x_n in $[-\delta_n b_n, -\delta] \cup [\delta, \delta_n b_n]$. By Lemma 2, there is a sequence $\delta_n \downarrow 0$ so that $k_n\{x \mid |x| \geqq \delta_n b_n\} \to 0$. Choose $\varepsilon > 0$. By Lemma 1, there is a $\delta > 0$ so that $\limsup_{n\to\infty} k_n\{x \mid |x| \leqq \delta\} \leqq \varepsilon$. So, from (3) it follows that

$$\begin{aligned} \limsup_{n\to\infty} P(X_n \in A \mid S_n S_{n+1} < 0) &\leqq P(Y \in A) \limsup_{n\to\infty} k_n\{\delta \leqq |x| \leqq \delta_n b_n\} \\ &\quad + \limsup_{n\to\infty} k_n\{|x| \leqq \delta\} \leqq P(Y \in A) + \varepsilon. \end{aligned}$$

Similarly,

$$\liminf_{n\to\infty} P(X_n \in A \mid S_n S_{n+1} < 0) \geqq P(Y \in A)(1 - \varepsilon).$$

Since ε was arbitrary, $P(X_n \in A \mid S_n S_{n+1} < 0) \to P(Y \in A)$. Since this is true for each A for which $P(Y \in \partial A) = 0$, it now follows from Theorem 2.1 of [1] that $(X_n \mid S_n S_{n+1} < 0)$ converges to Y weakly in $D[0, 1]$.

3. The case $\alpha < 1$. One might expect that any weak limit of $(X_n(t) \mid S_n S_{n+1} < 0)$ would concentrate at the origin at time one. This turned out to be true in the case considered in Section 2. When $\alpha < 1$, however, a different phenomenon occurs. The limiting process concentrates instead on the whole line at time one. The first step in studying this case is the following lemma.

LEMMA 3. *Assume* $\alpha < 1$. *Then*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} nP(0 < S_n/b_n < \varepsilon, S_{n+1} < 0) = 0.$$

PROOF. Let F be the distribution function of ξ_i, and let $H(x) = 1 - F(x) + F(-x)$ for $x \geqq 0$. Then

$$\begin{aligned} P(0 < S_n/b_n < \varepsilon, S_{n+1} < 0) &= \textstyle\int_0^\varepsilon P(S_n/b_n \in dx) F(-b_n x) \\ &\leqq \textstyle\sum_{k=0}^{[b_n]} P(0 < S_n/b_n - k\varepsilon/b_n \leqq \varepsilon/b_n) F(-k\varepsilon) \qquad (4) \\ &\leqq K/b_n[\varepsilon + o(1)] \textstyle\sum_{k=0}^{[b_n]} F(-k\varepsilon) \end{aligned}$$

where K is an upper bound for the density $p(\cdot)$ and $o(1) \to 0$ as $n \to \infty$ uniformly for all k and all bounded $\varepsilon > 0$. The second inequality is a consequence of Theorem 1 of [7]. Since F is monotone,

$$\varepsilon \textstyle\sum_{k=1}^{[b_n]} F(-k\varepsilon) \leqq \int_{-\varepsilon b_n}^0 F(x)\, dx \leqq \int_0^{\varepsilon b_n} H(x)\, dx.$$

Since ξ_i belongs to the domain of attraction of the stable law $X(1)$ and $\alpha < 1$, it follows from Theorem 2, page 175, of [4] that $H(x)$ is a regularly varying function with exponent $-\alpha$. Applying Theorem 1, page 273, of [3] (note that while there is a small error in the proof of this theorem, it can be corrected whenever the functions involved are measurable), we see that $(\varepsilon b_n) H(\varepsilon b_n) / \int_0^{\varepsilon b_n} H(x)\, dx \to 1 - \alpha$. From page 176 of [4], it follows that $nH(\varepsilon b_n) \to D\varepsilon^{-\alpha}$ where D is a positive, finite constant. So

$$\begin{aligned} \limsup_{n \to \infty} \frac{n}{b_n} \sum_{k=0}^{[b_n]} F(-k\varepsilon) &\leqq \limsup_{n \to \infty} \frac{\int_0^{\varepsilon b_n} H(x)\, dx}{(\varepsilon b_n) H(\varepsilon b_n)} \cdot nH(\varepsilon b_n) \\ &= \frac{D}{(1-\alpha)\varepsilon^\alpha}. \end{aligned}$$

The statement of the lemma now follows from (4)

THEOREM 2. *Assume* $\alpha < 1$. *Then the processes* $(X_n \mid S_n S_{n+1} < 0)$ *converge weakly in* $D[0, 1]$ *to a process* Z *with distribution given by*

$$P(Z \in A) = \frac{B_1 \int_0^\infty P(X \in A, X(1) \in dz) z^{-\alpha} + B_2 \int_{-\infty}^0 P(X \in A, X(1) \in dz) |z|^{-\alpha}}{B_1 \int_0^\infty p(z) z^{-\alpha}\, dz + B_2 \int_{-\infty}^0 p(z) |z|^{-\alpha}\, dz} \qquad (5)$$

for any measurable subset A *of* $D[0, 1]$, *where* B_1 *and* B_2 *are positive constants.*

PROOF. Note that the expression on the right of (5) is indeed a probability measure on $D[0, 1]$. Let A be a measurable subset of $D[0, 1]$ with $P(Z \in \partial A) = 0$. It then follows from (5) that $P(X \in \partial A) = 0$. Let $G = \{x(\cdot) \in D[0, 1] \,|\, x(1) \in (u, v)\}$ for any choice of u, v with $-\infty < u < v < \infty$. Then

$$P(X \in \partial(A \cap G)) \leqq P(X \in \partial G) + P(X \in \partial A) = 0,$$

so $P(X_n \in A, X_n(1) \in dz)$ converges weakly to $P(X \in A, X(1) \in dz)$ as a sequence of measures on $(-\infty, \infty)$. Now we may write

$$P(X_n \in A \,|\, S_n S_{n+1} < 0)$$
$$= \frac{\int_{0+}^{\infty} P(X_n \in A, X_n(1) \in dz) F(-b_n z) + \int_{-\infty}^{0-} P(X_n \in A, X_n(1) \in dz)[1 - F(-b_n z)]}{\int_{0+}^{\infty} P(X_n(1) \in dz) F(-b_n z) + \int_{-\infty}^{0-} P(X_n(1) \in dz)[1 - F(-b_n z)]}.$$

We see from page 176 of [4] that $nF(-b_n z) \to B_1 z^{-\alpha}$ for $z > 0$, and $n[1 - F(-b_n z)] \to B_2 |z|^{-\alpha}$ for $z < 0$. Since all these functions are monotone in z, the convergence is uniform for z bounded away from zero. So, using Lemma 3, we may conclude that

$$P(X_n \in A \,|\, S_n S_{n+1} < 0) \to P(Z \in A).$$

By Theorem 2.1 of [1], $(X_n \,|\, S_n S_{n+1} < 0)$ converges weakly to Z in $D[0, 1]$.

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