The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 1970–1977

A CHARACTERISTIC PROPERTY OF THE MULTIVARIATE NORMAL DENSITY FUNCTION AND SOME OF ITS APPLICATIONS

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0. Introduction. Price [9] proved by using Dirac δ-functions the following two theorems in a different form.

THEOREM 0.1. *If $X_1, \cdots, X_n$ have an n-variate normal distribution with unit variances and if $g_1, \cdots, g_n$ are functions respectively of $X_1, \cdots, X_n$ admitting Laplace transforms, then*

$$\frac{\partial}{\partial \sigma_{jk}} E\left[\prod_{m=1}^{n} g_m(X_m)\right] = E\left[\frac{\partial^2}{\partial X_j \partial X_k} \prod_{m=1}^{n} g_m(X_m)\right] \tag{0.1}$$

where σ_{jk} is the correlation coefficient of X_j and X_k.

THEOREM 0.2. *If* (0.1) *holds for arbitrary functions $g_1, \cdots, g_n$, then $X_1, \cdots, X_n$ have an n-variate normal distribution.*

For $n = 2$, McMahon [8] showed that

$$\frac{\partial}{\partial \sigma_{12}} E[g(X_1, X_2)] = E\left[\frac{\partial^2}{\partial X_1 \partial X_2} g(X_1, X_2)\right] \tag{0.2}$$

where g is a function of both X_1 and X_2, admitting 2-dimensional Laplace transform. Papoulis [9] generalized (0.2) to the case where $|g(x_1, x_2)| \leqq a \exp(x_1^{\alpha} + x_2^{\alpha})$, $a > 0$, $\alpha < 2$, is required instead of requiring the existence of the Laplace transform. Brown [1] pointed out that the result and method of Papoulis generalize directly to higher dimensions, providing certain generalizations of Price's theorems.

A major motivation in the above-mentioned work lies in obtaining $E[g(X_1, \cdots, X_n)]$ for a multivariate normal distribution by solving a partial differential equation of the kind in (0.2). This method has appeared in books and papers and has sometimes been used where the theoretical justification is lacking. Pawula in [7] gives an alternate approach where the covariance matrix is modified so that each non-diagonal element is multiplied by α, and differentiation is with respect to α.

Section 1 presents a property of the multivariate normal density function. This fundamental theorem generalizes the work of Plackett [8]. Section 2 provides a generalization of Price's theorem with a rigorous proof and presents some of its applications as corollaries. Corollary 2.2 uses Price's theorem as a theoretical tool to solve a problem that arose in an estimation problem [2]. Characterizations of the multivariate normal are studied in Section 3, and in Section 4. The relation of moments to independence, analogous to "decorrelations" discussed in Linnik [3], is studied.

Received December 2, 1968.

1. Fundamental theorem.

LEMMA 1.1. *Let $f \equiv f(x_1, x_2; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ be the bivariate normal density functions with mean vector $\boldsymbol{\mu} = (\mu_j)$ and covariance matrix $\boldsymbol{\Sigma} = (\sigma_{jk})$. Then*

$$\frac{\partial f}{\partial \sigma_{12}} = \frac{\partial^2 f}{\partial x_1 \, \partial x_2} \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{jj}} = \tfrac{1}{2}\frac{\partial^2 f}{\partial x_j^{\,2}}, \qquad j = 1, 2.$$

PROOF. If one writes f in terms of the standardized normal density function, the proof is straightforward.

THEOREM 1.1 (Fundamental theorem). *Let $f \equiv f(x_1, \cdots, x_n; \boldsymbol{\mu}\boldsymbol{\Sigma})$ be the n-variate normal density function with mean vector $\boldsymbol{\mu} = (\mu_j)$ and covariance matrix $\boldsymbol{\Sigma} = (\sigma_{jk})$. Then*

$$\frac{\partial f}{\partial \sigma_{jk}} = \frac{\partial^2 f}{\partial x_j \, \partial x_k}\left(1 - \frac{\delta_{jk}}{2}\right), \qquad j, k = 1, 2, \cdots, n$$

where $\delta_{jk} = 1$ if $j = k$ and $= 0$ otherwise.

PROOF. Without loss of generality assume that $(j, k) = (1, 2)$.

$$\frac{\partial f(x_1, \cdots, x_n)}{\partial \sigma_{12}} = f(x_3, \cdots, x_n) \cdot \frac{\partial f(x_1, x_2 \mid x_3, \cdots, x_n)}{\partial \sigma_{12}}$$

because $f(x_3, \cdots, x_n)$ is functionally independent of σ_{12}. Let $\sigma_{12.3\cdots n}$ be the conditional covariance of X_1 and X_1 given $X_3, \cdots, X_n$. It can be verified from result (8a·2·11) in Rao [14] that among the conditional means and conditional variance-covariances of X_1 and X_2 given $X_3, \cdots, X_n$, $\sigma_{12.3\cdots n}$ is the only conditional parameter of the conditional bivariate normal density function which is a function of σ_{12}, and that its partial derivative with respect to σ_{12} is one. The conclusion follows from Lemma 1.1.

2. A generalization of Price's theorem and related results.

THEOREM 2.1. *Let the random variables $X_1, \cdots, X_n$ have an n-variate normal distribution with density function f, and let g be an n-dimensional function of bounded variation on finite intervals satisfying: $Eg(X_1, \cdots, X_n)$ exists,*

$$\text{(2.1)} \qquad \int x_j f(x_1, \cdots, x_n) g(x_1, \cdots, x_n)\, dx_j \to 0 \quad \text{as} \quad |x_k| \to \infty \quad \textit{and}$$

$$x_k \int f(x_1, \cdots, x_n) g(x_1, \cdots, x_n)\, dx_j \to 0 \quad \text{as} \quad |x_k| \to \infty.$$

Then

$$\frac{\partial E[g(X_1, \cdots, X_n)]}{\partial \sigma_{jk}} = \int \cdots \int f(x_1, \cdots, x_n)\, dG(x_j, x_k)\, \frac{dx_1 \cdots dx_n}{dx_j\, dx_k},$$

where $G(x_j, x_k) = g(x_1, \cdots, x_n)$ with all coordinates except the jth *and the* kth *held fixed.*

PROOF. The conclusion follows from Leibniz's rule, Fubini's theorem, two-dimensional integration by parts (for the formula see Young [11], page 287), and conditions (2.1).

NOTE 2.1. Conditions (2.1) can be replaced by using the Lebesgue dominated convergence theorem with

$$x_j f(x_1, \cdots, x_n) g(x_1, \cdots, x_n) \to 0 \quad \text{as} \quad |x_j| \to \infty \tag{2.2}$$

and there exists a function $h(x_1, \cdots, x_n)$ satisfying

$$\int h(x_1, \cdots, x_n)\, dx_i < \infty,\ i = j, k \quad \text{and} \quad |x_i g(x_1, \cdots, x_n) f(x_1, \cdots, x_n)| \leqq h(x_1, \cdots, x_n).$$

COROLLARY 2.1 (An extension of Price's theorem). *In addition to the assumptions of Theorem* 2.1 *let g possess second order partial derivatives. Then*

$$\frac{\partial E[g(X_1, \cdots, X_n)]}{\partial \sigma_{jk}} = E\left[\frac{\partial^2 g(X_1, \cdots, X_n)}{\partial X_j \partial X_k}\right] \cdot (1 + \delta_{jk})/2,$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$.

PROOF. The result for $j \neq k$ is obvious. When $j = k$ the proof is similar to the proof of Theorem 2.1.

COROLLARY 2.2. *In addition to the assumptions of Corollary* 2.1 *suppose*

$$g(E[X_1], \cdots, E[X_n]) = E[g(X_1, \cdots, X_n)] \tag{2.3}$$

for all values of the parameter space. Then g is a linear function.

PROOF. By alternately using Corollary 2.1 and property (2.3) one finds $E[\partial^2 g(X_1, \cdots, X_n)/\partial X_j \partial X_k] = 0$. Since the normal family of density functions is a complete family, $\partial^2 g(X_1, \cdots, X_n)/\partial X_j \partial X_k = 0$ for all j, k. The conclusion follows by partial integration.

COROLLARY 2.3. *Let $X_1, \cdots, X_n$ have an n-variate normal distribution then for nonnegative integers $r_1, \cdots, r_n$,*

$$\frac{\partial}{\partial \sigma_{jk}} E(X_1^{r_1} \cdots X_n^{r_n}) = r_j r_k E(X_1^{r_1} \cdots X_n^{r_n} / X_j X_k) \quad \text{if} \quad j \neq k,$$

$$= r_j(r_j - 1) E(X_1^{r_1} \cdots X_n^{r_n} / 2X_j^2) \quad \text{if} \quad j = k.$$

PROOF. The conclusion follows by Corollary 2.1.

EXAMPLE 2.1. Let X_1, X_2, X_3, X_4 have a 4-variate normal distribution with zero means. Then $E(X_1 X_2 X_3 X_4) = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}$ (cf. Parzen [6] page 93 for an alternate proof).

PROOF. This is a direct application of Corollary 2.3 followed by partial integration using the boundary condition defined by $\sigma_{jk} = 0$ for all j, k.

COROLLARY 2.4. *Let $X_1, \cdots X_{2m}$ have a 2m-variate normal distribution with zero means. Then $E(X_1 X_2 \cdots X_{2m}) = \sum \sigma_{uv} \cdots \sigma_{rs}$, with the sum taken over all distinct products of m covariances with each index (from 1 to 2m) occurring exactly once.*

PROOF. The proof is an induction argument, where the case $m = 1$ is obvious ($m = 2$ is just Example 2.1) and where the case for general m is reduced to the previous case by applying Corollary 2.3. Then partial integration is used to finish the proof.

NOTE 2.2. When the mean values μ_j's of X_j's are not all zeros, $E[\prod_{j=1}^{2m} x_j]$ can be obtained by expanding $E[\prod X_j] \equiv E[\prod(Y_j+\mu_j)]$ with $Y_j = X_j - \mu_j$ and using Corollary 2.4 on Y_j's.

COROLLARY 2.5. *Let X_1 and X_2 have a bivariate normal distribution with zero means. Then $P[X_1 \geqq 0, X_2 \geqq 0] = (\arcsin \rho)/2\pi + \frac{1}{4}$ where $\rho = \sigma_{12}/\sigma_1 \sigma_2$ is the correlation coefficient of X_1 and X_2.*

PROOF. Since $P[X_1 \geqq 0, X_2 \geqq 0] = E[I(X_1) \cdot I(X_2)]$ where $I(x)$ is 1 if $x \geqq 0$ and 0 otherwise, the conclusion follows from Theorem 2.1 and partial integration with the boundary condition defined by $\sigma_{12} = 0$.

COROLLARY 2.6. *Let $X_1, \cdots, X_n$ have an n-variate normal distribution with characteristic function φ. Then*

$$\frac{\partial \varphi(\mathbf{t})}{\partial \sigma_{jk}} = -t_j t_k \varphi(\mathbf{t}) \qquad j, k = 1, \cdots, n.$$

PROOF. This theorem is an easy consequence of Corollary 2.1.

3. Characterizing the multivariate normal distribution. The characterization theorems presented in this section are essentially converses of results from Section 2. Consider a family of random variables $X_1, \cdots, X_n$ (n fixed), with variance-covariance matrix $\Sigma = (\sigma_{jk})$. This family has all possible values for σ_{jk}, and thus contains the singular cases. This section gives some conditions which imply that this family is the n-variate normal family. The referee of this paper suggested it would be interesting to have similar characterizations for families with a restricted parameter space (e.g. Σ is strictly positive definite). It remains an open problem if such theorems exist. In this section we assume that derivatives exist for $\sigma_{jk} > 0$ and that the functions differentiated are continuous in σ_{jk} at $\sigma_{jk} = 0$.

THEOREM 3.1. *Let $\varphi(t_1, \cdots, t_n)$ be the joint characteristic function of the random variables $X_1, \cdots, X_n$, and let σ_{jk} be the covariance of X_j, X_k for all j, k. If*

$$\frac{\partial \varphi(t_1, \cdots, t_n)}{\partial \sigma_{jk}} = -t_j t_k \varphi(t_1, \cdots, t_n)$$

for all $j < k$, then $X_1, \cdots, X_n$ have an n-variate normal distribution.

PROOF. Using partial integration on the hypotheses we find that

$$\Psi \equiv \ln \varphi = -\sum_{j<k} t_j t_k \sigma_{jk} + c,$$

where c is the appropriate function, constant with respect to the σ_{jk}'s. We write

$$\Psi = i\sum_{j=1}^n \mu_j t_j - \tfrac{1}{2}\sum_{j=1}^n \sigma_j^2 t_j^2 - \sum_{j<k} t_j t_k \sigma_{jk} + h, \tag{3.1}$$

where h is the appropriate function and where $\mu_j = \sum[X_j]$, $\sigma_j^2 = \mathrm{Var}(X_j)$. To obtain the conclusion we need only show that $h \equiv 0$, in which case Ψ is the cumulant generating function for the multivariate normal distribution.

Let $\lambda(t) = \ln E[\exp\{itX_1\}]$, and consider the case where $\sigma_{jk} = \sigma_j \sigma_k$ for all j, k. Then $(X_j - \mu_j)/\sigma_j = (X_1 - \mu_1)/\sigma_1$ a.s., $j = 1, \cdots, n$. Thus

$$\Psi = \ln E[\exp\{i\sum_{j=1}^n \sigma_j t_j X_1/\sigma_1 + i\sum_{j=1}^n t_j(\mu_j - \mu_1 \sigma_j/\sigma_1)\}].$$

Equating this to (3.1) we find that

$$\begin{aligned} &\lambda(\textstyle\sum_{j=1}^n \sigma_j t_j/\sigma_1) + i\sum_{j=2}^n t_j(\mu_j - \mu_1 \sigma_j/\sigma_1) \\ &\qquad = i\textstyle\sum_{j=1}^n \mu_j t_j - (\frac{1}{2})\sum_{j=1}^n \sigma_j^2 t_j^2 - \sum_{j<k} t_j t_k \sigma_j \sigma_k + h. \end{aligned} \tag{3.2}$$

Now consider the case where $\sigma_{j1} = -\sigma_j \sigma_1$ (and therefore $\sigma_{jk} = \sigma_j \sigma_k$ if $1 < j \leqq k$). Then $(X_j - \mu_j)/\sigma_j = -(X_1 - \mu_1)/\sigma_1$ a.s. for $j = 2, \cdots, n$. The same argument as above leads to

$$\begin{aligned} &\lambda(t_1 - \textstyle\sum_{j=2}^n \sigma_j t_j/\sigma_1) + i\sum_{j=2}^n t_j(\mu_j + \mu_1 \sigma_j/\sigma_1) \\ &\qquad = i\textstyle\sum_{j=1}^n \mu_j t_j - (\frac{1}{2})\sum_{j=1}^n \sigma_j^2 t_j^2 + \sum_{j=2}^n t_1 t_k \sigma_1 \sigma_k - \sum_{2\leqq j<k\leqq n} t_j t_k \sigma_j \sigma_k + h. \end{aligned} \tag{3.3}$$

Eliminating h from (3.2) and (3.3) we find that

$$\lambda(t_1 + s) - \lambda(t_1 - s) + 2(t_1 \sigma_1^2 - i\mu_1)s = 0, \quad \text{where} \quad s = \sum_{j=2}^n \sigma_j t_j/\sigma_1. \tag{3.4}$$

Now (3.4) holds for arbitrary t_1. So letting $t_1 = s$,

$$\lambda(2s) - \lambda(0) + 2(s^2 \sigma_1^2 - i\mu_1 s) = 0.$$

However $\lambda(0) = \ln E[\exp\{i \cdot 0 \cdot X_1\}] = 0$. Therefore $\lambda(z) = i\mu_1 z - \sigma_1^2 z^2/2$. Substituting into (3.2) yields $h \equiv 0$, which finishes the proof.

COROLLARY 3.1. *Let $(X_1, \cdots, X_n)$ be a random vector whose moments of all orders exist. If g is an arbitrary function selected from the following three functions (arbitrary but fixed) and if for $j \neq k$,*

$$\frac{\partial E[g(X_1, \cdots, X_n)]}{\partial \sigma_{jk}} = E\left[\frac{\partial^2 g(X_1, \cdots, X_n)}{\partial X_j \partial X_k}\right],$$

then $(X_1, \cdots, X_n)$ have an n-variate normal distribution.

(a) $g(x_1, \cdots, x_n) = \prod_{j=1}^n x_j^{r_j}$ *for arbitrary nonnegative integers $r_1, \cdots, r_n$*
(b) $g(x_1, \cdots, x_n) = \exp\{\sum_{j=1}^n u_j x_j\}$ *for arbitrary imaginary constants $u_1, \cdots, u_n$*
(c) $g(x_1, \cdots, x_n) = \sin(u_0 + \sum_{j=1}^n u_j x_j)$, *for arbitrary constants $u_0, u_1, \cdots, u_n$.*

PROOF. The conclusion follows by writing the characteristic function, φ, of $X_1, \cdots, X_n$ in terms of these three functions and applying Theorem 3.1.

THEOREM 3.2 (The converse of the fundamental theorem). *Let $X_1, \cdots, X_n$ be absolutely continuous random variables with joint density function f and covariance matrix (σ_{jk}). If*

$$\frac{\partial f(x_1, \cdots, x_n)}{\partial \sigma_{jk}} = \frac{\partial^2 f(x_1, \cdots, x_n)}{\partial x_j \partial x_k} \tag{3.5}$$

are continuous for all $j < k$, and integrable in coordinates other than x_j and x_k, then f is an n-variate normal density function.

PROOF. Applying Leibniz's rule to the derivative of the characteristic function $\varphi(t_1, \cdots, t_n)$ with respect to σ_{jk}, followed by the use of (3.5) and Fubini's theorem, gives

$$\frac{\partial \varphi(t_1, \cdots, t_n)}{\partial \sigma_{jk}} = -t_j t_k \varphi(t_1, \cdots, t_n).$$

The conclusion follows from Theorem 3.1.

4. Moments and independence.

THEOREM 4.1. *Let $X_1, \cdots, X_n$ have an n-variate normal distribution with zero means. Then*

$$E[X_1 \cdots X_{2m}] = \sum \pi \sigma_{jk},$$

where the sum is taken over all distinct products where each value of j, k, $(1 \leqq j < k \leqq 2m)$, occurs exactly once.

PROOF. Wang and Uhlenbeck in [10] page 322 proved this result, and Parzen quotes in [6] page 93.

THEOREM 4.2. *Let $X_1, \cdots, X_n$ have an n-variate normal distribution with zero means, and let $r_1, \cdots, r_n$ be a nonnegative integer whose sum $r_1 + \cdots + r_n$ is even. Then*

$$E(X_1^{r_1} \cdots X_n^{r_n}) = \sum (r_1!) \cdots (r_n!) 2^{-s} \prod_{j<k} [(\sigma_{jk})^{m_{jk}}/(m_{jk}!)]$$

where the sum is taken over all nonnegative integers $m_{jk} \equiv m_{kj}$ such that $m_{jj} + s_j = r_j$, $j = 1, \cdots, n$ and where $s = \sum_{j=1}^n m_{jj}$, $s_j = \sum_{k=1}^n m_{kj}$.

PROOF. Some or all of the $\{x_i\}$'s in Theorem 4.1 could be equal. Substituting $Y_j = X_i$ for i such that $r_0 + \cdots + r_{j-1} < i < r_1 + \cdots + r_j$, $j = 1, 2, \cdots, n$, where $r_0 = 0$ gives the conclusion.

COROLLARY 4.1. *Let X have a normal distribution with zero mean and variance σ^2. Then $E[X^{2r}] = [(2r)!/(r!)][\sigma_{11}/2]^r$, $r = 0, 1, \cdots$.*

THEOREM 4.3. *If r_1 and r_2 are fixed positive integers for which r_1+r_2 is even, and if $E[(X_1)^{r_1}(X_2)^{r_2}] = E(X_1)^{r_1} \cdot E(X_2)^{r_2}$ where X_1, X_2 have a bivariate normal distribution with zero means, then X_1 and X_2 are independent.*

PROOF. Let $\rho = \sigma_{12}/(\sigma_{11}\sigma_{22})^{\frac{1}{2}}$. From Theorem 4.2 we see that $E[(X_1)^{r_1}(X_2)^{r_2}]$ equals

$$(4.1) \qquad (r_1!)(\sigma_{11})^{r_1/2}(r_2!)(\sigma_{22})^{r_2/2}\sum \rho^{m_{12}}/2^{m_{11}+m_{22}}(m_{11}!)(m_{12}!)(m_{22}!)$$

with the sum taken over all nonnegative integers m_{11}, m_{12}, m_{22} for which $2m_{11}+m_{12} = r_1$ and $2m_{22}+m_{12} = r_2$. If both r_1 and r_2 are odd positive integers, then m_{12} is an odd nonnegative integer. Then since $E[(X_1)^{r_1}(X_2)^{r_2}] = E(X_1)^{r_1} \cdot E(X_2)^{r_2} = 0$, (all odd moments are 0), we see $\rho = 0$. If, on the other hand, both $r_1 \equiv 2m_1$ and $r_2 \equiv 2m_2$ are even positive integers, then m_{12} is an even nonnegative integer and we write $m_{12} = 2m$. From Theorem 4.2 we have $E[(X_i)^{r_i}] = (r_i!)(\sigma_{ii})^{r_i/2}/2^{m_i}(m_i!)$, $i = 1, 2$. Combining this with (4.1) we get

$$E[(X_1)^{r_1}(X_2)^{r_2}] = E(X_1)^{r_1} \cdot E(X_2)^{r_2}\sum\frac{(m_1!)(m_2!)2^{m_1+m_2}}{(m_{11}!)(m_{12}!)(m_{22}!)2^{m_{11}+m_{22}}}$$

with the sum taken over all nonnegative integers m_{11}, m_{12}, m_{22} for which $m_{ii}+m = m_i$, $i = 1, 2$. Substituting $m_{11} = m_1-m$, $m_{22} = m_2-m$, and $m_{12} = 2m$ one finds $\sum[(X_1)^{r_1}(X_2)^{r_2}]$ equals

$$E(X_1)^{r_1} \cdot E(X_2)^{r_2}\textstyle\sum_{m=0}^{m_0} C(m_1, m_2, m)(2\rho)^{2m},$$

where $m_0 = \min\{m_1, m_2\}$ and $C(m_1, m_2, m) = \binom{m_1}{m}\binom{m_2}{m}/\binom{2m}{m}$. Since $C(m_1, m_2, 0)(2\rho)^0 = 1$, we have $\sum_{m=1}^{m_0} C(m_1, m_2, m)(2\rho)^{2m} = 0$. Now $C(m_1, m_2, m) > 0$. Therefore $\rho = 0$. The conclusion follows.

NOTE 4.1. It is clear that if two of the exponents in $E[(X_1)^{r_1}(X_2)^{r_2}(X_3)^{r_3}]$ are odd integers, then the factorization into the product $E(X_1)^{r_1} \cdot E(X_2)^{r_2} \cdot E(X_3)^{r_3}$, which is zero, does not imply the independence of X_1, X_2, and X_3. One might hope that if the factorization holds for fixed even positive integers r_1, r_2, and r_3, then X_1, X_2, and X_3 are independent. For a counterexample take X_1, X_2, X_3 to have a trivariate normal distribution with zero means and correlation coefficients $\rho_{12} = 3/4$, $\rho_{13} = -3/4$, and $\rho_{23} = 3/4$. The following theorem gives a partial result.

THEOREM 4.4. *Let $X_1, \cdots, X_n$ have a non-degenerate n-variate normal distribution with zero means and nonnegative covariances. If*

$$E[(X_1)^{2r_1}\cdots(X_n)^{2r_n}] = E(X_1)^{2r_1}\cdots E(X_n)^{2r_n}$$

for arbitrary but fixed positive integers $r_1, \cdots, r_n$, then $X_1, \cdots, X_n$ are independent.

PROOF. Expanding $E[(X_1)^{r_1}\cdots(X_n)^{r_n}]-E(X_1)^{r_1}\cdots E(X_n)^{r_n}$ by using Theorem 4.2 one obtains a sum of nonnegative terms. This sum must be zero, and therefore each term must be zero. One of these terms is a multiple of

$$(\sigma_{12})^2(\sigma_{11})^{2r_1-2}(\sigma_{22})^{2r_2-2}(\sigma_{33})^{2r_3}\cdots(\sigma_{nn})^{2r_n}.$$

Therefore $\sigma_{12} = 0$, and similarly $\sigma_{jk} = 0$ for $j \neq k$. The conclusion follows.

LEMMA 4.1. *Let* $X_1, \cdots, X_n$ *have an n-variate normal distribution with zero means. Let* $S_j = 1$ *if* $X_j \geqq 0$ *and* $S_j = 0$ *otherwise*, $j = 1, 2, \cdots, n$. *Let* ρ_{jk} *and* R_{jk} *be the correlation coefficients of* X_j, X_k *and of* S_j, S_k *respectively. Then* $R_{jk} = (2/\pi) \arcsin \rho_{jk}$ (N.B. R_{jk} *and* ρ_{jk} *have the same sign and assume the values* 0 *and* ± 1 *together*).

PROOF. Now $\mathrm{Var}(S_j) = \frac{1}{4}$, and by Corollary 2.5, $\mathrm{Cov}(S_j, S_k) \equiv P[X_j \geqq 0, X_k \geqq 0] - \frac{1}{4} = (2\pi)^{-1} \arcsin \rho_{jk}$. The conclusion follows.

THEOREM 4.5. *Using the notation of Lemma* 4.1 *the following are equivalent*:

(a) $X_1, \cdots, X_n$ *are independent*,
(b) $X_1, \cdots, X_n$ *are pairwise uncorrelated*,
(c) $S_1, \cdots, S_n$ *are independent*,
(d) $S_1, \cdots, S_n$ *are pairwise uncorrelated.*

PROOF. Obvious.

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