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A PROPERTY OF POISSON PROCESSES AND ITS APPLICATION TO MACROSCOPIC EQUILIBRIUM OF PARTICLE SYSTEMS

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0. Introduction. It was shown by Derman [2] page 545, that if one takes a denumerably infinite state transient Markov chain with stationary measure μ and:

(i) at time 0 inserts $A_i(0)$ particles into state i, $i = 1, 2, \cdots$, where the $A_i(0)$ are independent Poisson variables with parameters $EA_i = \mu_i$;

(ii) lets each particle change states according to the transition matrix of the Markov chain, the particles behaving independently of one another: then for any n, the set of variables $\{A_i^{(n)}\ i = 1, 2, \cdots\}$, where $A_i^{(n)}$ is the number of particles in state i at time n, are independent and Poisson distributed with $EA_i^{(n)} = \mu_i$.

Port [8] studied the above system and derived the independent Poisson property with common parameter, for several collections of random variables describing various aspects of the behavior of the particle system.

One way of viewing this system is to regard the initial spatial process as a non-homogeneous Poisson process over the integers, and each particle (point of the Poisson process) as being independently mapped by a random sequence valued function into the sequence of its future states. The particle system thus generates a counting process on a space whose points are countable sequences of integers. This counting process can be represented by the collection of sequences $\{g_i(T_i)\}$ where the T_i are the initial states of the particles and the $\{g_i\}$ are i.i.d. random mappings. It then follows from a result apparently due to Karlin [4] page 497, and discussed by the author [1], that the counting process generated by $\{g_i(T_i)\}$ is Poisson, and strictly stationary. This strict stationarity generalizes the first order stationary of Derman, and from it the results of Port follow.

In general if we take a Markov process with stationary transition probability, state space $(\mathscr{X}, \mathscr{C})$ and σ-finite stationary measure μ, and at time 0 start with a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process and move each particle independently according to the transition law of the Markov process, then we obtain a Poisson process on the space of particle paths. The measure of this Poisson process is strictly stationary and coincides with the measure on path space generated by μ and the transition law of the Markov process. This strong equilibrium property is derived in Section 3. The main tool is a generalization of Karlin's result (Section 2).

1. Definitions. Let $(\mathscr{X}, \mathscr{C}, \mu)$ be a measure space and let η be a random non-negative integer-valued (including $+\infty$) set function on $(\mathscr{X}, \mathscr{C})$. Define η to be

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Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ distributed if for every m and corresponding sets $C_1, \cdots, C_m \in \mathscr{C}$ and nonnegative (including $+\infty$) integers $r_1, \cdots, r_m$:

$$\Pr(\eta(C_j) = r_j, j = 1, \cdots, m) = \prod_{j=1}^{m} p(\mu(C_j), r_j)$$

$$\text{where} \quad p(\lambda, \alpha) = \frac{\lambda^\alpha e^{-\lambda}}{\alpha!}, \qquad \lambda < \infty \qquad \alpha < \infty,$$

$$= 1, \qquad \lambda = \infty \qquad \alpha = \infty, \qquad \lambda = 0 \qquad \alpha = 0$$

$$= 0 \qquad \text{elsewhere.}$$

We will be concerned with the case where μ is σ-finite. In this case it is easy to prove the existence of a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process (see comment (i), Section 4).

Define the measurable space $(\mathscr{X}, \mathscr{C})$ to be chunkless if $\{x\} \in \mathscr{C}$ for each $x \in \mathscr{X}$. We will be assuming that $(\mathscr{X}, \mathscr{C})$ is chunkless in Theorem 1. This assumption is convenient but not necessary. It is further discussed in Section 4, comment (ii).

Define the random point set $\{T_i, i = 1, 2, \cdots\}$ of a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process with μ σ-finite and $(\mathscr{X}, \mathscr{C})$ chunkless as follows. For each realization of η there are a countable number of points each assigned a positive finite integer as its measure. Order these points in an arbitrary manner obtaining a set $\{S_i, i = 1, 2, \cdots\}$. Define $T_i = S_j$ if $\sum_{k=1}^{j-1} \eta(S_k) < i \leqq \sum_1^j \eta(S_k)$. If the number of S_j points are finite then terminate the T_i sequence at $i = n(\mathscr{X})$. Note that if $\mu(\mathscr{X}) < \infty$ then with positive probability the random point set will be empty.

2. Random transformations of Poisson processes. Theorem 1 which follows is a fairly straightforward generalization of a result of Karlin [4] page 497, which in turn is a generalization of a result of Doob [3] page 404. The method of proof we employ is found in Doob's theorem and is also employed by Karlin. The author independently derived Karlin's result, using a different type of proof, but the author's proof does not easily generalize. See Brown [1] for a discussion of the above mentioned results and some applications.

THEOREM 1. *Let η be a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process, with μ σ-finite and $(\mathscr{X}, \mathscr{C})$ chunkless. Let $(\Omega, \mathscr{B}, P)$ be a probability space, $(\mathscr{Y}, \mathscr{D})$ a measurable space and g a map from $(\Omega, \mathscr{B}) \times (\mathscr{X}, \mathscr{C})$ to $(\mathscr{Y}, \mathscr{D})$ satisfying:*

(i) $g(\cdot, x)$ is a measurable function from $(\Omega, \mathscr{B})$ to $(\mathscr{Y}, \mathscr{D})$ for all $x \in \mathscr{X}$.

(ii) $P(g(\cdot, x) \in D)$ considered as a function from $(\mathscr{X}, \mathscr{C})$ to the real line is Borel measurable for each $D \in \mathscr{D}$. Let $\{T_i, i = 1, \cdots, \eta(\mathscr{X})\}$ be the random point set of the η process, and let $\{g_i, i = 1, \cdots, \eta(\mathscr{X})\}$ be an i.i.d. *sequence distributed as g and independent of $\{T_i, i = 1, 2, \cdots, \eta(\mathscr{X})\}$. Then the collection $\{g_i(T_i), i = 1, \cdots, \eta(\mathscr{X})\}$ generates a Poisson $(\mathscr{Y}, \mathscr{D}, \mu_g)$ process where $\mu_g(D) = \int_{\mathscr{X}} P(g(y) \in D)\, d\mu(y)$.*

PROOF. Since μ is σ-finite we can decompose $\mathscr{X}$ into a countable number of disjoint sets $X_1, X_2, \cdots$ with $\bigcup_i X_i = \mathscr{X}$, $\mu(X_i) < \infty$ for all i. Let $Y_m = \bigcup_1^m X_i$.

Conditional on $\eta(Y_m) = n$, the T_i points are distributed as an i.i.d. sample of size n, with distribution $P(C) = \mu(C \cap Y_m)/\mu(Y_m)$ for $C \in \mathscr{C}$. This can be seen by writing

$P(\eta(C_j) = k_j, j = 1, \cdots, l \mid \eta(Y_m) = n) = P(\eta(C_j) = k_j, j = 1, \cdots, l, \eta(Y_m - \bigcup_i C_j) = n - \sum_1^l k_j)/P(\eta(Y_m) = n)$, and using the Poisson property of η. It thus follows from the assumptions on g that each point falling in Y_m has a probability $p_{j,m}$ of being mapped by g into C_j where

$$P_{j,m} = 1/\mu(Y_m) \int_{Y_m} P(g(y) \in C_j)\, d\mu(y).$$

We thus can compute the conditional probability of the event $\{\eta_{g,m}(C_j) = k_j, j = 1, \cdots, l\}$ given $\{\eta(Y_m) = n\}$. Removing the condition by multiplying by $P\{\eta(Y_m) = n)\}$ and summing over n we obtain that $\{\eta_{g,m}(C_j), j = 1, \cdots, l\}$ are independently Poisson distributed with parameters $E\eta_{g,m}(C_j) = \int_{Y_m} P(g(y) \in C_j)\, d\mu(y)$. The result follows by letting $m \to \infty$.

3. Application to macroscopic equilibrium of particle systems. Consider a Markov process with chunkless state space $(\mathscr{X}, \mathscr{C})$, real index set T containing a first element which we denote by 0, and stationary transition probability $P_t(x, \cdot)$ satisfying the conditions of [7] page 568. Assume the existence of a σ-finite stationary measure, that is a σ-finite measure μ satisfying:

$$\mu(C) = \int_{\mathscr{X}} P_t(x, C)\, d\mu(x)$$

for all $t \in T$, $C \in \mathscr{C}$.

Let $\mathscr{Y} = \times_{t \in T} \mathscr{X}_t$, $\mathscr{D} = \times_{t \in T} \mathscr{C}_t$. Then μ and $P_t(x, \cdot)$ generate a strictly stationary measure γ on $(\mathscr{Y}, \mathscr{D})$. The value of γ on finite dimensional rectangles being given by:

$$\gamma(\mathscr{X}_1^n C_{t_i}) = \int \cdots \int_{x \in \mathscr{X}; y_i \in C_i; i=1, \cdots n-1} \mu(dx) P_{t_1}(x, dy_1) P_{t_2 - t_1}^{(y_1, dy_2)} \cdots P_{t_n - t_{n-1}}^{(y_{n-1}, C_{t_n})}.$$

Suppose at time 0, we start with a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process and let each particle independently move around in $\mathscr{X}$ according to the underlying Markov process. Define:

$\Omega_x = \{$functions $\omega_x: T \to \mathscr{X}$ with $\omega_x(0) = x\}$;

$\mathscr{B}_x = \sigma$-algebra of subsets of Ω_x generated by sets of the form $\{\omega_x(0) = x, \omega_x(t_i) \in C_i, i = 1, \cdots, n$ with $C_i \in \mathscr{C}\}$.

P_x—probability measure on $(\Omega_x, \mathscr{B}_x)$ obtained by starting the Markov process in state x at time 0.

$$\Omega = \times_{x \in \mathscr{X}} \Omega_x, \qquad \mathscr{B} = \times_{x \in \mathscr{X}} \mathscr{B}_x, \qquad P = \times_{x \in \mathscr{X}} P_x.$$

Define the random function $g: (\Omega, \mathscr{B}) \times (\mathscr{X}, \mathscr{C}) \to (\mathscr{Y}, \mathscr{D})$ by $g(\omega, x) = \omega_x$. It follows that g satisfies conditions (i) and (ii) of Theorem 1; condition (i) because $[g(\cdot, x)]^{-1}(D) = D \times [\times_{y \in \mathscr{X}; y \neq x} \Omega_y] \in \mathscr{B}$; condition (ii) because of the conditions imposed on the transition probability.

We thus can apply Theorem 1 and obtain the result that the moving particle system $\{g_i(T_i)\}$ generates a Poisson $(\mathscr{Y}, \mathscr{D}, \gamma)$ process where γ is described by (1). Since γ is a stationary measure on $(\mathscr{Y}, \mathscr{D})$ the moving particle system is in statistical equilibrium.

EXAMPLE 1. Derman [2] and Port [8] and [9] considered a denumerable state transient Markov chain possessing a stationary measure μ. At time 0 $A_i(0)$ particles are assigned to state i, $i = 1, 2, \cdots$ where the $A_i(0)$ are independent Poisson variables with parameters $EA_i(0) = \mu_i$. Each particle goes through a sequence of state transitions governed by the transition matrix of the chain. The particles are assumed to behave independently of one another.

It follows that the moving particle system generates a Poisson process on the space of sequences of states, the Poisson measure coinciding with the stationary measure on this space generated by μ and the transition matrix. Theorem 3 page 407 and Lemma 1 page 408 of Port [8], Lemma 7.1 of Port [9], and Theorem 2 page 545 of Derman [2] are special cases of this result.

EXAMPLE 2. Let X be a vector-valued ($X(t, w) \in E^n$) spatially homogeneous Markov process with stationary transition probability ($P(X_{t+s} \in A \mid X_s = x) = P(X_t \in \{A - x\} \mid X_0 = x)$). Doob [3] page 406 showed that Lebesgue measure is a stationary measure for such a process.

Thus if at time 0 particles are distributed in E^n according to a homogeneous Poisson process, and each particle performs a motion according to the transition law of the Markov process, then as was shown in Section 3, the counting process generated by the collection of particle paths is a strictly stationary Poisson process over the space of vector-valued functions on $[0, \infty)$.

More generally, if we start with a homogeneous Poisson process on E^n and consider the collection of functions $\{\{T_i + X_i(t), t \geqq 0\}, i = 1, 2, \cdots\}$ where $\{X_i\}$ is an i.i.d. sequence of random vector-valued functions independent of $\{T_i\}$ and possessing strictly stationary increments (for any m and corresponding pairs (s_i, t_i) $i = 1, \cdots, m$, the joint distribution of $X_{t+t_i} - X_{t+s_i}$ $i = 1, \cdots, m$, is independent of t), then it follows from Theorem 1 that the counting process generated by the particle paths will be strictly stationary Poisson.

Doob [3] page 406 proved that if $\{T_i\}$ is the random point set of a homogeneous Poisson process on E^n and if $\{\{X_j(t) - T_j, t \geqq 0\} j = 1, 2, \cdots\}$ is an i.i.d. sequence of random vector-valued functions independent of $\{T_i\}$, then at any time t the spatial process will be homogeneous Poisson. This stationarity property is a weaker one than we have considered but it holds for a larger class of motions. Thus Doob's result says that the number of particles whose state is contained in A at time t is Poisson-distributed with parameter independent of t. Our result says that under a more restricted motion described above, for any $m, t_1, \cdots, t_m, A_1, \cdots, A_m$, the distribution of the number of particles whose state is contained in A_i at time $t + t_i$, $i = 1, \cdots, m$, is Poisson with a parameter independent of t.

Doob considered motions of the particles in which the displacements $\{X_j(t) - T_j, t \geqq 0\}$ are assumed independent of each other and of $\{T_i\}$. The result derived in Section 3 allows consideration of certain types of displacements $\{X_j(t) - T_j, t \geqq 0\}$ in which the jth displacement is dependent on the initial state T_j. This is so because the Markov process governing the position of the particles need not be spatially homogeneous. In the non-spatially homogeneous case the stationary measure will differ from Lebesgue measure. For example, let the particles move with velocity one on $(-\infty, 0)$ and with velocity 2 on $(0, \infty)$. Then the stationary measure $\mu(A) = \ell(A \cap (-\infty, 0)) + \frac{1}{2}\ell(A \cap (0, \infty))$ where ℓ is proportional to Lebesgue measure.

Note that Theorem 1 restricts the motions we may consider to those in which for all j the jth displacement process $\{X_j(t) - T_j, t \geqq 0\}$ is conditionally independent of $\{X_i(t) - T_i, t \geqq 0, i \neq j\}$ and of $\{T_i\}$, given T_j.

EXAMPLE 3. We may have a situation in which the real-valued process $\{D(t), t \geqq 0\}$ representing the position of a particle is non-Markovian, but the velocity pattern $\{V(t), t \geqq 0\}$ is Markovian. In this case the process $\{W(t), t \geqq 0\} = \{(V(t), D(t)), t \geqq 0\}$ is Markovian. If μ is a stationary measure of the V process and the V process is spatially homogeneous, then $\mu \times \ell$, where ℓ is Lebesgue measure is a stationary measure for the W process. To see this, let P^* be the transition probability for W and P the transition probability for V. Now:

$$(1) \qquad \int\int P_t^*((x, v) \to (a, b) \times (v_1, v_2))\, dx\, d\mu(v)$$
$$= \int [P_t(A \mid V(0) = v) \cdot \int [P(B_x \mid A, V(0) = v)\, dx]\, d\mu(v)$$

where $A = \{V(t) \in (v_1, v_2)\}$, $B_x = \{\int_0^t V(s)\, ds \in (a - x, b - x)\}$.

Now (1) can be reduced to:

$$(2) \qquad (b - a) \int P_t(A \mid V(0) = v)\, d\mu(v)$$

as follows from Doob [3] page 406. From the stationarity of μ it follows that $(2) = (b - a)\mu(v_1, v_2)$, and thus the result is proved.

This example was motivated by remarks in Spitzer [10]. Spitzer points out that if V is an Ornstein–Uhlenbeck process then a Poisson $\mu \times \ell$ spatial-velocity process, where μ is invariant for V, is preserved under independent Ornstein–Uhlenbeck motions of the particles. The Ornstein–Uhlenbeck process (normal process with mean 0 and covariance kernel $K(s, t) = \alpha \exp(-\beta |t - s|)$, $\alpha > 0$, $\beta > 0$) has a finite stationary measure (a $N(0, \alpha)$ distribution).

Another example is motion under constant velocity. In this case every measure is stationary for the velocity process. Thus if particles are spatially distributed according to a homogeneous Poisson process at time 0, and choose i.i.d. velocities $\{V_i\}$ which they maintain forever, then the collection $\{(T_i + V_i t, V_i), t \geqq 0, i = 1, 2, \cdots\}$ generates a strictly stationary Poisson process. Various cases of this result have appeared in traffic theory literature.

4. Comments and additions.

(i) In Theorem 1 we assume the existence of Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ processes where μ is σ-finite. One can construct a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process as follows: Since μ is σ-finite, $\mathscr{X} = \bigcup_i X_i$ where $\mu(X_i) < \infty$ for all i and the X_i are disjoint and denumerable in number. Construct independent random variables $\{Y_i, i = 1, 2, \cdots\}$ where Y_i is Poisson with parameter $\mu(X_i)$. Then for each i, construct Y_i i.i.d. random variables $Z_{i,1}, \cdots, Z_{i,Y_i}$ with distribution $P(Z_i \in C) = \mu(C \cap X_i)/\mu(X_i)$, $C \in \mathscr{C}$. The sets $\{Z_{i,k}\}$ and $\{Z_{j,k}\}$ are chosen to be independent for $i \neq j$. The collection $\{Z_{i,k} k = 1, \cdots, Y_i, i = 1, 2, \cdots\}$ generates a Poisson $(\mathscr{X}, \mathscr{C}, \mu)$ process. This construction was pointed out to the author by K. Itô in a lecture at Cornell.

(ii) In Theorem 1 we assumed that $\mathscr{X}$ was chunkless. This assumption can be dropped with a slightly modified version of the theorem holding.

Define a set $C \in \mathscr{C}$ to be a chunk of $(\mathscr{X}, \mathscr{C})$ if the null set is the only proper subset of C which belongs to $\mathscr{C}$. It is easy to show that each realization of η can be uniquely expressed as:

$$\eta(C) = \sum_{i:A_i \subset C} \eta(A_i)$$

where the $\{A_i\}$ form a denumerable (perhaps empty) collection of chunks of $(\mathscr{X}, \mathscr{C})$ and each $\eta(A_i) > 0$. The collection $\{A_i\}$, of course, depends on the realization of η. We replace the random point set $\{T_i\}$ in the chunkless case by a random chunk set $\{A_i\}$ in the general case. In the general case we will assume that $\{g_i\}$ is independent of $\{A_i\}$, rather than of $\{T_i\}$ as in Theorem 1.

Recall that by assumption (ii) on g (Theorem 1) $\Pr(g(y) \in D)$ is a measurable function for all $D \in \mathscr{D}$. This implies that for any chunk B of $(\mathscr{X}, \mathscr{C})$ with probability one, all points lying in B will be mapped by g into a common chunk of $(\mathscr{Y}, \mathscr{D})$. Since $\{g_i\}$ is independent of $\{A_i\}$ by assumption and both are countable collections, the above implies that with probability 1 the realization of η and $\{g_i\}$ will be such that for each g_i and A_j, g_i will map all points of A_j into the same chunk of $(\mathscr{Y}, \mathscr{D})$. We can thus unambiguously replace the random point set $\{g_i(T_i)\}$ with a random chunk set $\{g_i(A_i)\}$. It then will follow by the same argument as in Theorem 1 that $g_i(T_i)$ generates a Poisson $(\mathscr{Y}, \mathscr{D}, \mu_g)$ process.

(iii) Theorem 1 can be applied to the branching Poisson model of P. Lewis, [5] and [6]. Under this model primary events $\{T_i\}$ occur according to a (not necessarily homogeneous) Poisson process. Each primary event T_i gives rise to a random set of secondary events $\{S_{i,j} j = 1, \cdots, N_i\}$ of random size N_i. We assume that for each i, $\{S_{i,j}, j = 1, \cdots N_i\}$ is conditionally independent of $\{S_{i',j}, j = 1, \cdots, N_{i'}, i' \neq i\}$ and of $\{T_j\}$ given T_i (Lewis makes the slightly stronger assumption that the collections $\{S_{i,j} - T_i j = 1, \cdots, N_i\}$ are i.i.d.).

It follows from Theorem 1 that the collection of sets $g_i(T_i) = \{T_i, S_{i,j} j = 1, \cdots, N_i\}$ $i = 1, 2, \cdots$ generates a Poisson process on a space $(\mathscr{Y}, \mathscr{D})$ whose points are countable subsets of real numbers. From this it follows that the number of secondary events in any Borel set has a compound Poisson distribution; the number of events (both primary and secondary) in any Borel set is compound Poisson distributed: the number of $g_i(T_i) \in A$ is Poisson distributed for any set $A \in \mathscr{D}$; the number of $g_i(T_i)$ with $T_i \leqq a$, $\sup_j |S_{i,j} - T_i| \leqq b$ is Poisson distributed, etc.

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