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LINEAR RANK STATISTICS UNDER ALTERNATIVES INDEXED BY A VECTOR PARAMETER[1]

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1. Introduction. Since the publication of Hájek's papers [3] and [4] on the asymptotic distributions of linear rank statistics under the null hypothesis of randomness and under contiguous location-shift alternatives, several other fruitful applications of the same basic techniques have been made. Some of these later applications may be found in the papers by Matthes and Truax [7] and by Adichie [1], as well as in the book [5] by Hájek and Šidák. With the exception of [1], the contiguous alternatives examined in these references have depended upon a one-dimensional parameter. Adichie considered a two-dimensional parameter indexing simple regression alternatives, but by making the parameter components linearly dependent in the asymptotics, essentially reduced the problem.

This paper presents a general treatment of the asymptotic distributions of linear rank statistics under contiguous alternatives indexed by a q-dimensional parameter. The approach adopted follows that of Hájek and Šidák [5]. It is assumed that under the null hypothesis, the observations are independent and identically distributed random variables. The linear rank statistics considered are of the forms

$$S_c = \sum_{i=1}^N \mathbf{c}_i'\mathbf{a}_N(R_{Ni}), \qquad \text{and} \tag{1.1}$$

$$S_c^+ = \sum_{i=1}^N \mathbf{c}_i'\mathbf{a}_N(R_{Ni}^+)\operatorname{sign}(X_i), \tag{1.2}$$

where the p-dimensional column vectors $\mathbf{c}_1, \cdots, \mathbf{c}_N$ and $\mathbf{a}_N(1), \cdots, \mathbf{a}_N(N)$ are, respectively, constants and values of a score function $\mathbf{a}_N(\cdot)$. Such statistics arise naturally in the present context and also in the study of locally most powerful rank tests. The results of this paper may be used to derive asymptotic distribution theory under vector parameter alternatives for the Kolmogorov–Smirnov, Cramér–von Mises and Rényi statistics as well as for various simple quadratic rank statistics. Section 4 illustrates.

2. Limiting distributions under the null hypothesis. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with common density f, defined on R^1. Let R_{Ni} denote the rank of X_i among $X_1, \cdots, X_N$. The first situation envisaged in this section is that of sequences of sample sizes $\{N_\nu\}$, of p-dimensional vector constants $\{(\mathbf{c}_{\nu 1}, \cdots, \mathbf{c}_{\nu N_\nu})\}$, of linear rank statistics $\{S_{c\nu}\}$, defined as in (1.1), and of null hypotheses $\{H_\nu\}$. Under H_ν, the joint density of $(X_1, \cdots, X_{N_\nu})$ is assumed to be

$$p_\nu(\mathbf{x}) = \prod_{i=1}^{N_\nu} f(x_i). \tag{2.1}$$

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Our aim is to establish general conditions for the asymptotic normality of the statistics S_{cv} under H_v as $v \to \infty$.

For simplicity of notation, the subscript v will usually be dropped. Unless otherwise indicated, all vectors appearing in this section are p-dimensional column vectors. The norm $||\cdot||$ is the usual Euclidean norm. A basic result to be used repeatedly is the following central limit theorem, under whose assumptions the Lindberg condition holds.

THEOREM 2.1. *Let* $\mathbf{Y}_1, \mathbf{Y}_2, \cdots$ *be a sequence of independent identically distributed random variables with mean* $\mathbf{m}$ *and covariance matrix* V. *If*

$$\max_{1 \leqq i \leqq N} ||\mathbf{a}_i|| \to 0, \qquad \sum_{i=1}^N ||\mathbf{a}_i||^2 \leqq d^2 < \infty, \tag{2.2}$$

then the statistics

$$S_a = \sum_{i=1}^N \mathbf{a}_i{}'\mathbf{Y}_i \tag{2.3}$$

are asymptotically normal $(\mu_a, \sigma_a{}^2)$ *with*

$$\mu_a = \mathbf{m}' \sum_{i=1}^N \mathbf{a}_i, \qquad \sigma_a{}^2 = \sum_{i=1}^N \mathbf{a}_i{}' V \mathbf{a}_i. \tag{2.4}$$

Let F denote the distribution function with density f. If $U_i = F(X_i)$, the random variables $U_1, U_2, \cdots$ are independent, uniformly distributed on (0, 1). The rank of U_i among $U_1, \cdots U_N$ is still R_{Ni}. Let $\varphi(u)$, $u \in (0, 1)$ be a p-dimensional vector valued function, square integrable in the sense that its Euclidean norm $||\varphi(u)||$ belongs to $L_2(0, 1)$. Define

$$\mathbf{a}_N(i, \varphi) = E[\varphi(U_1) \mid R_{N1} = i]. \tag{2.5}$$

The following two lemmas follow immediately from Theorem V.1.4a and Theorem V.1.4b of [5].

LEMMA 2.1. *If* $\varphi(u)$ *is square integrable and* $\mathbf{a}_N(i, \varphi)$ *is defined by* (2.5), *then*

$$\lim_{N \to \infty} E ||\mathbf{a}_N(R_{N1}, \varphi) - \varphi(U_1)||^2 = 0. \tag{2.6}$$

LEMMA 2.2. *If* $\varphi(u)$ *is square integrable and* $\mathbf{a}_N(i, \varphi)$ *is defined by* (2.5), *then*

$$\lim_{N \to \infty} \int_0^1 ||\mathbf{a}_N(1 + [uN], \varphi) - \varphi(u)||^2 \, du = 0. \tag{2.7}$$

Let $\bar{\varphi} = \int_0^1 \varphi(u)\, du$ and let $D = \int_0^1 [\varphi(u) - \bar{\varphi}][\varphi(u) - \bar{\varphi}]'\, du$. The next two theorems, the main results of this section, are analogues of Theorem V.1.5a and Theorem V.1.6a of [5],

THEOREM 2.2. *Let the scores* $\mathbf{a}_N(i, \varphi)$ *be associated with a square integrable function* $\varphi(u)$ *as in* (2.5). *If*

$$\max_{1 \leqq i \leqq N} ||\mathbf{c}_i - \bar{\mathbf{c}}|| \to 0, \qquad \sum_{i=1}^N ||\mathbf{c}_i - \bar{\mathbf{c}}||^2 \leqq d^2 < \infty, \tag{2.8}$$

then, under H_v, *the statistics*

$$S_c{}^\varphi = \sum_{i=1}^N \mathbf{c}_i{}'\mathbf{a}_N(R_{Ni}, \varphi) \tag{2.9}$$

are asymptotically normal (μ_c, σ_c^2) *with*

$$\mu_c = \bar{\mathbf{c}}' \textstyle\sum_{i=1}^N \mathbf{a}_N(i, \varphi), \qquad \sigma_c^2 = \sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' D (\mathbf{c}_i - \bar{\mathbf{c}}), \tag{2.10}$$

or with $\sigma_c^2 = \operatorname{Var} S_c^{\varphi}$.

PROOF. As in Theorem 3.2, it is sufficient to prove the result under the additional assumption

$$\textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' D (\mathbf{c}_i - \bar{\mathbf{c}}) \to c^2 < \infty. \tag{2.11}$$

If $c^2 = 0$, the theorem is trivial. If $c^2 > 0$, we may write

$$S_c^{\varphi} = \textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' \mathbf{a}_N(R_{Ni}, \varphi) + \bar{\mathbf{c}}' \sum_{i=1}^N \mathbf{a}_N(i, \varphi). \tag{2.12}$$

Define

$$T_c = \textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' \varphi(U_i) + \bar{\mathbf{c}}' \sum_{i=1}^N \mathbf{a}_N(i, \varphi). \tag{2.13}$$

An argument analogous to the one in [5] yields

$$\begin{aligned} &E[(T_c - S_c^{\varphi})^2 \mid \mathbf{U}^{(\cdot)} = \mathbf{u}^{(\cdot)}] \\ &\quad \leqq N(N-1)^{-1} \textstyle\sum_{i=1}^N \|\mathbf{c}_i - \bar{\mathbf{c}}\|^2 E[\|\mathbf{a}_N(R_{N1}, \varphi) - \varphi(U_1)\|^2 \mid \mathbf{U}^{(\cdot)} = \mathbf{u}^{(\cdot)}]. \end{aligned} \tag{2.14}$$

Thus, if $\tau_c = \sum_{i=1}^N \|\mathbf{c}_i - \bar{\mathbf{c}}\|^2 / \sigma_c^2$

$$E[(T_c - S_c^{\varphi})\sigma_c^{-1}]^2 \leqq N\tau_c (N-1)^{-1} E\|\mathbf{a}_N(R_{N1}, \varphi) - \varphi(U_1)\|^2. \tag{2.15}$$

Assumption (2.8) implies that $N \to \infty$ and, together with (2.11), that τ_c is bounded asymptotically. Therefore, Lemma 2.1 and (2.15) show that $(T_c - S_c^{\varphi})\sigma_c^{-1} \to_P 0$. Moreover, T_c is asymptotically normal (μ_c, σ_c) by Theorem 2.1. The theorem follows.

The next theorem extends the asymptotic normality of Theorem 2.2 to a larger class of score functions $\mathbf{a}_N(\cdot)$. (cf. Theorem V.1.6a of [5]).

THEOREM 2.3. *Let the scores* $\mathbf{a}_N(i)$ *be associated with a square integrable function* $\varphi(u)$ *in the sense that*

$$\lim_{N \to \infty} \textstyle\int_0^1 \|\mathbf{a}_N(1 + [uN]) - \varphi(u)\|^2 \, du = 0. \tag{2.16}$$

If (2.8) *is satisfied, then, under* H_v, *the statistics*

$$S_c = \textstyle\sum_{i=1}^N \mathbf{c}_i' \mathbf{a}_N(R_{Ni}) \tag{2.17}$$

are asymptotically normal (μ_c, σ_c^2) *with*

$$\mu_c = ES_c, \qquad \sigma_c^2 = \textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' D (\mathbf{c}_i - \bar{\mathbf{c}}), \tag{2.18}$$

or with $\sigma_c^2 = \operatorname{Var} S_c$.

Because of Lemma 2.2, the scores $\mathbf{a}_N(i, \varphi)$ satisfy condition (2.16). Other simple score functions $\mathbf{a}_N(\cdot)$ satisfying (2.16) are provided by the following two lemmas, which are direct consequences of Lemma V.1.6a and Lemma V.1.6b of [5].

LEMMA 2.3. *If each component of* $\varphi(u)$ *is expressible as a finite sum of square integrable and monotone functions, then*

$$\mathbf{a}_N(i) = \varphi(i/N+1), \qquad 1 \leqq i \leqq N, \tag{2.19}$$

has the property (2.16).

LEMMA 2.4. *If the function* $\varphi(u)$ *is square integrable, then*

$$\mathbf{a}_N(i) = N\int_{(i-1)/N}^{i/N} \varphi(u)\,du, \qquad 1 \leqq i \leqq N, \tag{2.20}$$

has the property (2.16).

The preceding arguments extend to rank statistics $\{S_{c_\nu}^+\}$, defined as in (1.2), under symmetric hypotheses $\{H_\nu^+\}$. Under H_ν^+, the joint density of $(X_1, \cdots, X_{N_\nu})$ is of the form (2.1); however the density f is assumed to be symmetric about the origin.

Let R_{Ni}^+ denote the rank of $|X_i|$ among $(|X_1|, \cdots, |X_N|)$, let sign (X_i) equal 1, 0, or -1 according to whether X_i is positive, zero, or negative, and let $D^+ = \int_0^1 \varphi(u)\varphi(u)'\,du$. Then we have

THEOREM 2.4. *Let the scores* $\mathbf{a}_N(i)$ *be associated with a square integrable function* $\varphi(u)$ *in the sense of* (2.16). *If*

$$\max_{1\leqq i\leqq N} ||\mathbf{c}_i|| \to 0, \qquad \sum_{i=1}^N ||\mathbf{c}_i||^2 \leqq d^2 < \infty, \tag{2.21}$$

then, under H_ν^+, *the statistics*

$$S_c^+ = \sum_{i=1}^N \mathbf{c}_i'\mathbf{a}_N(R_{Ni}^+)\,\text{sign}\,(X_i) \tag{2.22}$$

are asymptotically normal $(0, \gamma_c^2)$ *with*

$$\gamma_c^2 = \sum_{i=1}^N \mathbf{c}_i' D^+ \mathbf{c}_i, \tag{2.23}$$

or with $\gamma_c^2 = \operatorname{Var} S_c^+$.

3. Limiting distributions under contiguous alternatives. The first situation envisaged in this section is that of sequences of sample sizes $\{N_\nu\}$, of p-dimensional vector constants $\{(\mathbf{c}_{\nu 1}, \cdots, \mathbf{c}_{\nu N_\nu})\}$, of linear rank statistics $\{S_{c\nu}\}$, defined as in (1.1), of null hypotheses $\{H_\nu\}$, and of alternatives $\{K_\nu\}$. Under H_ν, the joint density of $(X_1, \cdots, X_{N_\nu})$ is assumed to be

$$p_\nu(\mathbf{x}) = \prod_{i=1}^{N_\nu} f(x_i, \boldsymbol{\theta}_0), \qquad \boldsymbol{\theta}_0 \in R^q, \tag{3.1}$$

where $\boldsymbol{\theta}_0$ is known. Under K_ν, the joint density is assumed to be

$$q_\nu(\mathbf{x}) = \prod_{i=1}^{N_\nu} f(x_i, \boldsymbol{\theta}_{\nu i}), \qquad \boldsymbol{\theta}_{\nu i} \in R^q, 1 \leqq i \leqq N_\nu. \tag{3.2}$$

Our aim is to derive the asymptotic distributions of the statistics $S_{c\nu}$ as $\nu \to \infty$ for sequences of densities $\{q_\nu\}$ contiguous to $\{p_\nu\}$.

As in the one-dimensional parameter case dealt with by Hajék and Šidák [5], the principal theoretical tools used are some results due to LeCam [6]. For notational convenience, the subscript ν will again be dropped and $\boldsymbol{\theta}_0$ will be taken to be the null vector $\mathbf{0}$. The following regularity assumption on the densities will be made.

DEFINITION 3.1. *Let $\Theta \subset R^q$ be an open subset containing* $\mathbf{0}$. *A family of densities $\{f(x, \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\}$ is said to satisfy condition* A *if*

(i) *For almost all x, $f(x, \boldsymbol{\theta})$ is continuously differentiable with respect to $\boldsymbol{\theta}$ whenever $\boldsymbol{\theta} \in \Theta$.*

(ii) $\lim_{\|\boldsymbol{\theta}\| \to 0} I_{jj}(\boldsymbol{\theta}) = I_{jj}(\mathbf{0}) < \infty, j = 1, 2, \cdots, q.$

By $I_{jj}(\boldsymbol{\theta})$ is meant the (j, j)th element of the information matrix

$$I(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}[\dot{\mathbf{f}}(X, \boldsymbol{\theta})\dot{\mathbf{f}}(X, \boldsymbol{\theta})'/f^2(X, \boldsymbol{\theta})], \tag{3.3}$$

where $\dot{\mathbf{f}}(x, \theta)$ denotes the q-dimensional column vector of first partial derivatives with respect to $\boldsymbol{\theta}$. The continuous differentiability of $f(x, \boldsymbol{\theta})$ is equivalent to the existence and continuity of $\dot{\mathbf{f}}(x, \boldsymbol{\theta})$ (cf. [8] page 146). For use in the following lemmas, let

$$\begin{aligned} w(x, \boldsymbol{\theta}) &= 2[f(x, \boldsymbol{\theta})]^{\frac{1}{2}}, \\ \mathbf{S}(x, \boldsymbol{\theta}) &= \dot{\mathbf{f}}(x, \boldsymbol{\theta})[f(x, \boldsymbol{\theta})]^{-1}. \end{aligned} \tag{3.4}$$

In these terms,

$$\begin{aligned} I(\boldsymbol{\theta}) &= \int_{-\infty}^{\infty} \dot{\mathbf{w}}(x, \boldsymbol{\theta})\dot{\mathbf{w}}(x, \boldsymbol{\theta})'\, dx \\ &= E_{\boldsymbol{\theta}}[\mathbf{S}(X, \boldsymbol{\theta})\mathbf{S}(X, \boldsymbol{\theta})']. \end{aligned} \tag{3.5}$$

LEMMA 3.1. *If $f(x, \boldsymbol{\theta})$ satisfies condition* A, *then*

$$\lim_{\|\boldsymbol{\theta}\| \to 0} \int_{-\infty}^{\infty} \dot{w}_j(x, \boldsymbol{\theta})\dot{w}_j(x, \mathbf{0})\, dx = I_{jj}(\mathbf{0}), \qquad j = 1, 2, \cdots, q. \tag{3.6}$$

PROOF. The Cauchy–Schwarz inequality and part (ii) of condition A imply that

$$\limsup_{\|\boldsymbol{\theta}\| \to 0} \int_{-\infty}^{\infty} |\dot{w}_j(x, \boldsymbol{\theta})\dot{w}_j(x, \mathbf{0})|\, dx \leqq I_{jj}(\mathbf{0}). \tag{3.7}$$

On the other hand, part (i) of condition A gives

$$\lim_{\|\boldsymbol{\theta}\| \to 0} \dot{w}_j(x, \boldsymbol{\theta})\dot{w}_j(x, \mathbf{0}) = \dot{w}_j^2(x, \mathbf{0}), \tag{3.8}$$

for almost all x. The result follows by Theorem II.4.2 of [5].

LEMMA 3.2. *If $f(x, \boldsymbol{\theta})$ satisfies condition* A *and if*

$$\max_{1 \leqq i \leqq N} \|\boldsymbol{\theta}_i\| \to 0, \tag{3.9}$$

then, for arbitrary $\varepsilon > 0$,

$$\max_{1 \leqq i \leqq N} P(|f(X_i, \boldsymbol{\theta}_i)/f(X_i, \mathbf{0}) - 1| > \varepsilon) \to 0 \tag{3.10}$$

under H_v.

PROOF. Let $\mathbf{e}_i = \boldsymbol{\theta}_i/||\theta_i||$. Then, under H_ν,

$$E|f(X_i,\boldsymbol{\theta}_i)/f(X_i,\mathbf{0})-1| = \int_{-\infty}^{\infty}|\int_0^{||\boldsymbol{\theta}_i||}\mathbf{e}_i'\dot{\mathbf{f}}(x,t\mathbf{e}_i)\,dt|\,dx$$
$$(3.11)\qquad \leqq \int_0^{||\boldsymbol{\theta}_i||}\int_{-\infty}^{\infty}||\dot{\mathbf{f}}(x,t\mathbf{e}_i)||\,dx\,dt$$
$$\leqq \int_0^{||\boldsymbol{\theta}_i||}[\operatorname{trace} I(t\mathbf{e}_i)]^{\frac{1}{2}}\,dt,$$

the last step using the Cauchy–Schwarz inequality. Because of (3.9) and condition A, the right-hand side of (3.11) tends to 0.

LEMMA 3.3. *If $f(x, \boldsymbol{\theta})$ satisfies condition* A, *then*

$$(3.12)\qquad \int_{-\infty}^{\infty}\dot{\mathbf{f}}(x,\mathbf{0})\,dx = 0.$$

PROOF. It is equivalent to show that for an arbitrary unit vector $\mathbf{e}\in R^q$,

$$(3.13)\qquad \int_{-\infty}^{\infty}\mathbf{e}'\dot{\mathbf{f}}(x,\mathbf{0})\,dx = 0.$$

Now, for any $\tau > 0$ and any such $\mathbf{e}$,

$$(3.14)\qquad |\int_{-\infty}^{\infty}\mathbf{e}'\dot{\mathbf{f}}(x,\mathbf{0})\,dx| = |\int_{-\infty}^{\infty}[\tau^{-1}(f(x,\tau\mathbf{e})-f(x,\mathbf{0}))-\mathbf{e}'\dot{\mathbf{f}}(x,\mathbf{0})]\,dx|$$
$$\leqq \tau^{-1}\int_0^\tau\int_{-\infty}^{\infty}||\dot{\mathbf{f}}(x,t\mathbf{e})-\dot{\mathbf{f}}(x,\mathbf{0})||\,dx\,dt.$$

Furthermore, by the Minkowski and Cauchy–Schwarz inequalities,

$$\int_{-\infty}^{\infty}||\dot{\mathbf{f}}(x,t\mathbf{e})-\dot{\mathbf{f}}(x,\mathbf{0})||\,dx$$
$$\leqq \int_{-\infty}^{\infty}\left\|\frac{\dot{\mathbf{f}}(x,t\mathbf{e})}{f^{\frac{1}{2}}(x,t\mathbf{e})}-\frac{\dot{\mathbf{f}}(x,\mathbf{0})}{f^{\frac{1}{2}}(x,\mathbf{0})}\right\|\mathbf{f}^{\frac{1}{2}}(x,t\mathbf{e})\,dx$$
$$(3.15)\qquad +\int_{-\infty}^{\infty}\left\|\frac{\dot{\mathbf{f}}(x,\mathbf{0})}{f^{\frac{1}{2}}(x,\mathbf{0})}\right\||f^{\frac{1}{2}}(x,t\mathbf{e})-f^{\frac{1}{2}}(x,\mathbf{0})|\,dx$$
$$\leqq [\int_{-\infty}^{\infty}||\dot{\mathbf{w}}(x,t\mathbf{e})-\dot{\mathbf{w}}(x,\mathbf{0})||^2\,dx]^{\frac{1}{2}}$$
$$+2^{-1}[\operatorname{trace} I(\mathbf{0})]^{\frac{1}{2}}[\int_{-\infty}^{\infty}|w(x,t\mathbf{e})-w(x,\mathbf{0})|^2\,dx]^{\frac{1}{2}}.$$

Considering separately each term on the right-hand side of (3.15), we find

$$(3.16)\qquad \int_{-\infty}^{\infty}||\dot{\mathbf{w}}(x,t\mathbf{e})-\dot{\mathbf{w}}(x,\mathbf{0})||^2\,dx$$
$$= \operatorname{trace} I(t\mathbf{e})+\operatorname{trace} I(\mathbf{0})-2\operatorname{trace}[\int_{-\infty}^{\infty}\dot{\mathbf{w}}(x,t\mathbf{e})\dot{\mathbf{w}}(x,\mathbf{0})'\,dx]\to 0$$

as $t\to 0$ because of Lemma 3.1 and condition A. By analogous argument,

$$(3.17)\qquad \int_{-\infty}^{\infty}|w(x,t\mathbf{e})-w(x,\mathbf{0})|^2\,dx = 8-2\int_{-\infty}^{\infty}w(x,t\mathbf{e})w(x,\mathbf{0})\,dx\to 0$$

as $t\to 0$. Thus, the lemma is established by letting $\tau\to 0$ in (3.14) and noting (3.15), (3.16) and (3.17).

The next two lemmas relate the asymptotic behavior of the statistics

$$(3.18)\qquad W_{\boldsymbol{\theta}} = 2\sum_{i=1}^N\{[f(X_i,\boldsymbol{\theta}_i)/f(X_i,\mathbf{0})]^{\frac{1}{2}}-1\}$$

and

$$(3.19)\qquad T_{\boldsymbol{\theta}} = \sum_{i=1}^N\boldsymbol{\theta}_i'\mathbf{S}(X_i,\mathbf{0}),$$

thereby leading up to the application of LeCam's results on contiguity. Their proofs are based upon (3.16) and are similar to the arguments for Lemma VI.2.1a and Lemma VI.2.1b in [5].

LEMMA 3.4. *If* $f(x, \boldsymbol{\theta})$ *satisfies condition* A *and*

$$\max_{1 \leqq i \leqq N} ||\boldsymbol{\theta}_i|| \to 0, \qquad \textstyle\sum_{i=1}^N ||\boldsymbol{\theta}_i||^2 \leqq d^2 < \infty, \tag{3.20}$$

then, under H_v,

$$\mathrm{Var}(W_\theta - T_\theta) \to 0. \tag{3.21}$$

LEMMA 3.5. *If* $f(x, \boldsymbol{\theta})$ *satisfies condition* A, (3.20) *is satisfied, and*

$$\textstyle\sum_{i=1}^N \boldsymbol{\theta}_i' I(\mathbf{0})\boldsymbol{\theta}_i \to b^2 < \infty, \tag{3.22}$$

then, under H_v,

$$EW_\theta \to -\tfrac{1}{4}b^2. \tag{3.23}$$

Define the likelihood ratio

$$L_\theta = q_v/p_v = \textstyle\prod_{i=1}^N f(X_i, \boldsymbol{\theta}_i)/\prod_{i=1}^N f(X_i, \mathbf{0}). \tag{3.24}$$

The following theorem establishes general conditions under which the densities $\{q_v\}$ are contiguous to the densities $\{p_v\}$ (cf. Theorem VI.2.1 of [5]).

THEOREM 3.1. *If* $f(x, \boldsymbol{\theta})$ *satisfies condition* A, (3.20) *and* (3.22) *are satisfied, then, under* H_v,

$$\log L_\theta - T_\theta + b^2/2 \to_p 0. \tag{3.25}$$

Moreover, $\log L_\theta$ *is asymptotically normal* $(-b^2/2, b^2)$ *and the densities* $\{q_v\}$ *are contiguous to the densities* $\{p_v\}$.

PROOF. Lemma 3.3, Lemma 3.4 and Lemma 3.5 show that under H_v,

$$E(W_\theta - T_\theta + b^2/4)^2 = \mathrm{Var}(W_\theta - T_\theta) + (EW_\theta - ET_\theta + b^2/4)^2 \to 0. \tag{3.26}$$

Consequently

$$W_\theta - T_\theta + b^2/4 \to_p 0. \tag{3.27}$$

Since $ET_\theta = 0$, $\mathrm{Var}\, T_\theta \to b^2$, it may be concluded from Theorem 2.1 that the statistics T_θ are asymptotically normal $(0, b^2)$. Then, in view of (3.27), the statistics W_θ are asymptotically normal $(-b^2/4, b^2)$ under H_v. This fact, Lemma 3.2, and LeCam's second lemma (Lemma VI.1.3 of [5], extended to the degenerate case as on page 219) imply the contiguity of the densities q_v to the densities p_v and also (3.25).

The next theorem, the main result of the section and an analogue of Theorem VI.2.4 in [5], proves the asymptotic normality of the statistics S_c under alternatives K_v contiguous to the null hypotheses H_v. For use in the proof, we introduce the uniform scores

$$\varphi(u, f) = \mathbf{S}(F^{-1}(u), \mathbf{0}). \tag{3.28}$$

where F is the distribution function with density $f(x, \mathbf{0})$. If $U_i = F(X_i)$, then $U_1, U_2, \cdots$ are independent, uniformly distributed on (0, 1) under H_ν. The statistics T_θ, defined in (3.19), may be rewritten as

$$T_\theta = \textstyle\sum_{i=1}^N \boldsymbol{\theta}_i' \boldsymbol{\varphi}(U_i, f). \tag{3.29}$$

Let the scores $\mathbf{a}_N(i)$ be associated with a square integrable function $\boldsymbol{\varphi}(u)$, $0 < u < 1$ as in (2.16). Define the $p \times p$ matrix D as in Theorem 2.2, and let the $p \times q$ matrix $B = \int_0^1 \boldsymbol{\varphi}(u)\boldsymbol{\varphi}(u, f)'\, du$.

THEOREM 3.2. *If* $f(x, \boldsymbol{\theta})$ *satisfies condition* A, *if*

$$\max_{1 \leqq i \leqq N} ||\mathbf{c}_i - \bar{\mathbf{c}}|| \to 0, \qquad \textstyle\sum_{i=1}^N ||\mathbf{c}_i - \bar{\mathbf{c}}||^2 \leqq d^2 < \infty, \tag{3.30}$$

$$\max_{1 \leqq i \leqq N} ||\boldsymbol{\theta}_i|| \to 0, \qquad \textstyle\sum_{i=1}^N ||\boldsymbol{\theta}_i||^2 \leqq e^2 < \infty, \tag{3.31}$$

and if (2.16) *holds, then, under* K_ν, *the statistics* $S_c = \sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' \mathbf{a}_N(R_{Ni})$ *are asymptotically normal* $(\mu_{c\theta}, \sigma_c^2)$ *with*

$$\mu_{c\theta} = \textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' B \boldsymbol{\theta}_i, \qquad \sigma_c^2 = \sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' D (\mathbf{c}_i - \bar{\mathbf{c}}). \tag{3.32}$$

PROOF. It is sufficient to prove the theorem under the additional assumptions

$$\textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' D (\mathbf{c}_i - \bar{\mathbf{c}}) \to b_1^2 < \infty, \tag{3.33}$$

$$\textstyle\sum_{i=1}^N \boldsymbol{\theta}_i' I(\mathbf{0}) \boldsymbol{\theta}_i \to b_2^2 < \infty, \tag{3.34}$$

$$\textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' B \boldsymbol{\theta}_i \to b_{12} < \infty. \tag{3.35}$$

Indeed, were the theorem false, there would exist sequences $\{(\mathbf{c}_{\nu 1}, \cdots, \mathbf{c}_{\nu N_\nu})\}$ and $\{(\boldsymbol{\theta}_{\nu 1}, \cdots, \boldsymbol{\theta}_{\nu N_\nu})\}$ with the property that for all subsequences of these sequences, (3.30) and (3.31) would hold but the conclusion of the theorem would not. Therefore, the theorem would fail for those particular subsequences also satisfying (3.33), (3.34) and (3.35); the existence of these latter subsequences follows from the Bolzano–Weierstrass theorem.

The proofs for Theorem 2.2 and Theorem 2.3 demonstrate that under H_ν, $S_c - T_c \to_p 0$, where

$$T_c = \textstyle\sum_{i=1}^N (\mathbf{c}_i - \bar{\mathbf{c}})' \boldsymbol{\varphi}(U_i). \tag{3.36}$$

Theorem 3.1 shows that $\log L_\theta - T_\theta + b_2^2/2 \to 0$ under H_ν. Thus, under H_ν, $(S_c, \log L_\theta)$ has the same joint asymptotic distribution as $(T_c, T_\theta - b_2^2/2)$.

From Lemma 3.3 and (3.33), (3.34), (3.35),

$$ET_c = ET_\theta = 0 \tag{3.37}$$

$$\operatorname{Var} T_c \to b_1^2, \qquad \operatorname{Var} T_\theta \to b_2^2, \qquad \operatorname{Cov}(T_c, T_\theta) \to b_{12}.$$

An arbitrary linear combination $\alpha_1 T_c + \alpha_2 T_\theta$ may be expressed in the form

$$\alpha_1 T_c + \alpha_2 T_\theta = \textstyle\sum_{i=1}^N \mathbf{d}_i' \boldsymbol{\psi}(U_i), \tag{3.38}$$

where

$$\mathbf{d}_i = \begin{pmatrix} \alpha_1(\mathbf{c}_i - \bar{\mathbf{c}}) \\ \alpha_2 \boldsymbol{\theta}_i \end{pmatrix}, \qquad \psi(u) = \begin{pmatrix} \varphi(u) \\ \varphi(u,f) \end{pmatrix}. \tag{3.39}$$

It is immediate from assumptions (3.30) and (3.31) that $\max_{1 \leqq i \leqq N} ||\mathbf{d}_i|| \to 0$ and $\sum_{i=1}^N ||\mathbf{d}_i||^2 \leqq c^2 < \infty$. From this, Theorem 2.1, and the previous paragraph, we find that under H_ν, $(S_c, \log L_\theta)$ is asymptotically normal (μ, Σ), with

$$\mu = \begin{pmatrix} 0 \\ -b_2^2/2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} b_1^2 & b_{12} \\ b_{12} & b_2^2 \end{pmatrix}. \tag{3.40}$$

By contiguity, established in Theorem 3.1, and LeCam's third lemma (Lemma VI.1.4 of [5]), we conclude that S_c is asymptotically normal (b_{12}, b_1^2) under K_ν. The theorem is proved.

For the asymptotic distribution of statistics $\{S_{c_\nu}^+\}$ under alternatives $\{K_\nu^+\}$, there is an analogue of Theorem 3.2. Under K_ν^+, the joint density of $(X_1, \cdots, X_{N_\nu})$ is of the form (3.2); however $f(x, \boldsymbol{\theta}_0)$ is assumed to be symmetric about the origin. For convenience $\boldsymbol{\theta}_0$ is again taken to be the null vector $\mathbf{0}$.

Let

$$\varphi^+(u,f) = \varphi(\tfrac{1}{2} + \tfrac{1}{2}u, f), \tag{3.41}$$

where $\varphi(u,f)$ is defined by (3.28). Define the $p \times p$ matrix D^+ as in Theorem 2.4 and let the $p \times q$ matrix $B^+ = \int_0^1 \varphi(u)\varphi^+(u,f)'\,du$.

THEOREM 3.3. *If $f(x, \mathbf{0})$ is symmetric about the origin, if $f(x, \boldsymbol{\theta})$ satisfies condition* A, *if*

$$\max_{1 \leqq i \leqq N} ||\mathbf{c}_i|| \to 0, \qquad \sum_{i=1}^N ||\mathbf{c}_i||^2 \leqq d^2 < \infty, \tag{3.42}$$

$$\max_{1 \leqq i \leqq N} ||\boldsymbol{\theta}_i|| \to 0, \qquad \sum_{i=1}^N ||\boldsymbol{\theta}_i||^2 \leqq e^2 < \infty, \tag{3.43}$$

and if (2.16) *holds, then, under K_ν^+, the statistics S_c^+ are asymptotically normal $(\nu_{c\theta}, \gamma_c^2)$ with*

$$\nu_{c\theta} = \sum_{i=1}^N \mathbf{c}_i' B^+ \boldsymbol{\theta}_i, \qquad \gamma_c^2 = \sum_{i=1}^N \mathbf{c}_i' D^+ \mathbf{c}_i. \tag{3.44}$$

4. Applications. One application of the foregoing results is to the two parameter regression problem considered by Adichie [1]. It is readily seen that Theorem 3.1, Theorem 4.1 and Theorem 4.2 of [1] follow (with a small modification) from Theorem 2.4 and Theorem 3.3 of this paper. The modification consists in replacing condition A by the conditions on f used in [1], a possibility due to the particular form of the regression alternatives.

Another application is to the asymptotic distribution theory under alternatives of the generalized Kolmogorov–Smirnov, Cramér–von Mises, and Rényi statistics analyzed in [5] (cf. Section V.3.6). These statistics are continuous functionals in $C[0, 1]$ of the stochastic process

$$T_c(t) = [\sum_{i=1}^N (c_i - \bar{c})^2]^{-\frac{1}{2}} \sum_{i=1}^N (c_i - \bar{c}) a_N(R_i, t). \tag{4.1}$$

The scores $a_N(i, t)$ are given by

$$a_N(i,t) = 0 \qquad i \leqq tN$$
$$(4.2) \qquad = i - tN \qquad tN \leqq i < tN+1$$
$$= 1 \qquad tN+1 \leqq i.$$

Define $\varphi(u, t)$ by

$$(4.3) \qquad \varphi(u,t) = 0 \quad \text{if} \quad 0 \leqq u \leqq t \leqq 1$$
$$= 1 \quad \text{if} \quad 0 \leqq t < u \leqq 1.$$

Clearly

$$(4.4) \qquad \lim_{N\to\infty} \int_0^1 [a_N(1+[uN], t) - \varphi(u,t)]^2 \, du = 0, \qquad 0 \leqq t \leqq 1.$$

Let F denote the distribution function of the density $f(x, \mathbf{0})$ and define $\psi(t)$ through

$$(4.5) \qquad \psi(t) = \int_{F^{-1}(t)}^{\infty} \dot{\mathbf{f}}(x, \mathbf{0}) \, dx.$$

Theorem 3.1 and Theorem 3.2 imply the following generalization of Theorem VI.3.2 of [5]. The proof is similar.

THEOREM 4.1. *If $f(x, \boldsymbol{\theta})$ satisfies condition* A, *if*

$$(4.6) \qquad \max_{1 \leqq i \leqq N} |c_i - \bar{c}| \to 0, \qquad \sum_{i=1}^N (c_i - \bar{c})^2 \leqq d^2 < \infty,$$

$$(4.7) \qquad \max_{1 \leqq i \leqq N} \|\boldsymbol{\theta}_i\| \to 0, \qquad \sum_{i=1}^N \|\boldsymbol{\theta}_i\|^2 \leqq e^2 < \infty,$$

and if

$$(4.8) \qquad \mu_{c\theta}(t) = [\sum_{i=1}^N (c_i - \bar{c})^2]^{-\frac{1}{2}} [\sum_{i=1}^N (c_i - \bar{c}) \boldsymbol{\theta}_i' \boldsymbol{\psi}(t)],$$

then, under K_v, the process $T_c(t) - \mu_{c\theta}(t)$ converges in distribution in $C[0, 1]$ to the Brownian bridge $Z(t)$.

With the aid of this result, the asymptotic power of a test based upon a continuous functional of $T_c(t)$ may be reduced to the probability of an event involving $Z(t)$. However, numerical evaluation of these latter probabilities is awkward (cf. Anděl [2]).

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