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A COMPLETE CLASS THEOREM FOR MULTIDIMENSIONAL ONE-SIDED ALTERNATIVES

BY MORRIS L. EATON[1]

University of Chicago

1. Introduction. For testing a simple hypothesis in an exponential family, Birnbaum (1955) showed that the class of tests with closed convex acceptance regions, say D_0, forms an essentially complete class of decision functions. The original proof of this result was incomplete and a complete proof appeared in Matthes and Truax (1967) (hereafter referred to as M–T). In discussing the minimal completeness of the class D_0, Birnbaum considered testing $(\psi_1, \psi_2) = (0, 0)$ against $\{(\psi_1, \psi_2) \mid \psi_1 \geqq 0, \ \psi_2 \geqq 0, \ \psi_1 + \psi_2 > 0\}$ where $(\psi_1, \psi_2)'$ is the two-dimensional parameter vector of an exponential family. For this problem, Birnbaum showed that D_0 is not essentially minimal complete and he characterized a subset of D_0 which is essentially minimal complete under certain conditions. In this paper we present a generalization of this two-dimensional one-sided result by Birnbaum.

In recent years, a number of authors have considered the problem of testing that the mean of a multivariate normal distribution is 0 against the alternative that the mean is in a closed convex cone (in particular, the positive orthant). For example, Nuesch (1966) and Perlman (1969) were concerned with the derivation of, and distribution theory for, the likelihood ratio test for such a problem. A related problem is that of testing the equality of components of a mean vector versus an ordered alternative. Bartholomew (1959a,b, 1961a, 1961b) and Kudô (1963) have discussed this problem in detail. Also, Oosterhoff and Van Zwet (1967) considered Birnbaum's original one-sided problem in two dimensions in their paper on the combination of independent test statistics.

In Section 2 of this paper, we present some results concerning convex sets and convex cones. These results are used in Section 3 to establish a complete class theorem for testing a simple hypothesis against certain one-sided alternatives when the underlying distribution is an exponential family. In Section 4, this result is extended to include the case of nuisance parameters.

It is assumed that the reader is familiar with the results and methods in M–T. Certain proofs in this paper are rather abbreviated as the arguments parallel those in M–T.

2. Preliminary results. Let Φ be the class of all non-empty closed convex sets in R^k-k-dimensional Euclidean space. If V is a closed convex cone, $V \neq R^k$ and

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$V \neq \{0\}$, the *normal cone* of V at $0 \in R^k$ is defined by $V^- = \{w \mid (w, x) \leqq 0$ for all $x \in V\}$ where $(\cdot, \cdot)$ is the standard inner product on R^k (see Valentine (1964) page 135). The convex cone V^- is also referred to as the *dual cone* by many authors. Obviously, $V^{--} = V$. For $C \in \Phi$, a vector $a_0 \in R^k(\|a_0\| = 1)$ is *normal* to C if there is a point $c_0 \in \partial C$ such that $C - c_0 \subseteq \{x \mid (x, a_0) \leqq 0\}$. Now let $\Phi(V)$ be those sets $C \in \Phi$ such that if a_0 is normal to C, then $a_0 \in V$.

PROPOSITION 2.1. *$C \in \Phi(V)$ if and only if for each $c_0 \in \partial C$, $V^- \subseteq C - c_0$.*

PROOF. First note that $V^- \in \Phi(V)$. Suppose $V^- \subseteq C - c_0$ for $c_0 \in \partial C$. Let $H = \{x \mid (a, x) = 0\}$, $\|a\| = 1$ be a supporting hyperplane to $C - c_0$ at 0, such that $C - c_0 \subseteq \{x \mid (a, x) \leqq 0\}$. Then H is a supporting hyperplane to V^- at 0 and $V^- \subseteq \{x \mid (a, x) \leqq 0\}$. Thus $a \in V$. Since c_0 was arbitrary, all normal vectors to C are in V so that $C \in \Phi(V)$.

Conversely, suppose $C \in \Phi(V)$ and suppose there exists $c_0 \in \partial C$ such that $V^- \nsubseteq C - c_0$. Then there is a $v_0 \in V^-$ such that $v_0 \notin C - c_0$. Let d_0 be the closest point in $C - c_0$ to v_0. Then $a_0 = (v_0 - d_0)/(\|v_0 - d_0\|)$ is a normal vector to $C - c_0 \in \Phi(V)$, so that $a_0 \in V$. Since $0 \in C - c_0$, it follows that $\|d_0\| < \|v_0\|$. Hence $(a_0, v_0) > 0$ which contradicts the definition of V^-. This completes the proof.

With the aid of Proposition 2.1, it is now possible to show that $\Phi(V)$ is a closed subset of Φ with respect to a natural metric topology on Φ. Let Φ_0 be all closed convex sets, C, such that $0 \in C$ and define $\Phi_{0,r}$ by

$$\Phi_{0,r} = \{C \mid C \in \Phi_0, C \subseteq S_r\} \tag{2.1}$$

for $r = 1, 2, \cdots$ where $S_r = \{x \mid \|x\| \leqq r\}$. Denoting the Hausdorff metric on $\Phi_{0,r}$ by d_r, define the metric d on Φ_0 by

$$d(C_1, C_2) = \sum_{r=1}^{\infty} \frac{d_r(C_1 \cap S_r, C_2 \cap S_r)2^{-r}}{1 + d_r(C_1 \cap S_r, C_2 \cap S_r)}. \tag{2.2}$$

It follows immediately from the argument in M–T (see proof of Theorem 2.1 in M–T) that (Φ_0, d) is a compact metric space.

For $C \in \Phi$, let $g(C)$ be the closest point in C to $0 \in R^k$. g is a well-defined function and $g(C) = 0$ for $C \in \Phi_0$. Note that $C \in \Phi$ corresponds to a unique point in $\Phi_0 \times R^k$, namely $(C - g(C), g(C))$. Using this, define a metric τ on Φ by

$$\tau(C_1, C_2) = d(C_1 - g(C_1), C_2 - g(C_2)) + \|g(C_1) - g(C_2)\|. \tag{2.3}$$

If $B \subseteq \Phi$ is a closed subset of (Φ, τ), then B is compact if and only if $\sup_{C \in B} \|g(C)\| < +\infty$. Hence (Φ, τ) is a locally compact, σ-compact metric space. It is clear that convergence in (Φ, τ) is precisely the convergence described by M–T, although they did not show the convergence corresponded to a topology.

PROPOSITION 2.2. *$\Phi(V)$ is a closed subset of (Φ, τ).*

PROOF. Suppose $C_n \to_\tau C$ with $C_n \in \Phi(V)$. Let $c_0 \in \partial C$ so $C_n - c_0 \to_\tau C - c_0$. Since $g(C - c_0) = 0$, $g(C_n - c_0) \to 0$. Thus $C_n - c_0 - g(C_n - c_0) \to_d C - c_0$. But, $V^- \subseteq$

$C_n - c_0 - g(C_n - c_0)$ for all n. Hence $V^- \subseteq C - c_0$, and from Proposition 2.1, the conclusion follows.

3. An extension of Birnbaum's theorem. Let Y be a random vector in R^k with a density $p_\omega(y) = c(\omega)\exp[\omega' y]$ and a dominating probability measure μ. Here, $\omega' y = \sum_{i=1}^k \omega_i y_i$. The natural parameter space is denoted by Ω. The testing problem considered is $\omega = 0 \in \Omega$ against the alternative $\omega \in \Omega_1 \subseteq \Omega$ and assume that Ω_1 is contained in some half-space—that is, there is a vector $a \neq 0$ such that $\Omega_1 \subseteq \{x \mid (a, x) \leqq 0\}$.

Recall that a test function φ has a convex acceptance region if there exists $C \in \Phi$ such that

$$\begin{aligned} \varphi(y) &= 0 & & y \in \text{ interior } (C) \\ &= \gamma(y) & & y \in \partial C \\ &= 1 & & y \notin C, \end{aligned}$$

where $0 \leqq \gamma(y) \leqq 1$ is a measurable function. Let V be the smallest closed convex cone which contains Ω_1 and consider the set, $D(V)$, of test functions which have closed convex acceptance regions C such that $C \in \Phi(V)$. Assume that $\varphi \equiv 1$ is in $D(V)$. The main result of this section shows that $D(V)$ is an essentially complete class of test functions.

Let Π be the set of all probability measures on Ω_1 which are concentrated on a finite set of points.

PROPOSITION 3.1. *Let $\pi \in \Pi$ be a prior distribution on Ω_1. Then a Bayes solution to the above testing problem is equivalent* $[\mu]$ *to a test in $D(V)$.*

PROOF. Every Bayes solution is equivalent $[\mu]$ to a test function of the form

$$\begin{aligned} \varphi(y) &= 0 \quad \text{if } \textstyle\sum_{i=1}^n \xi_i c(\omega_i)\exp[\omega_1' y] < M, \\ &= 1 \quad \text{if } \textstyle\sum_{i=1}^n \xi_i c(\omega_i)\exp[\omega_i' y] > M; \end{aligned}$$

with possible randomization on the boundary of

$$C = \{y \mid \textstyle\sum_{i=1}^n \xi_i c(\omega_i)\exp - [\omega_i' y] \leqq M\}.$$

Here, π puts mass ξ_i on the point $\omega_i \in \Omega_1$, $i = 1, \cdots, n$. Clearly, $C \in \Phi$. To show $C \in \Phi(V)$, let $f(y) = \sum_{i=1}^n \xi_i c(\omega_i)\exp[\omega_i' y]$. Since V is a convex cone and $\omega_i \in V$, $i = 1, \cdots, n$, it follows immediately that the gradient of f, $\nabla f(y)$, is in V for each $y \in R^k$. However, if a_0 is normal to C, then $a_0 = \nabla f(y)/\|\nabla f(y)\|$ for some y. This completes the proof.

THEOREM 3.1. *The set $D(V)$ is an essentially complete class.*

PROOF. Since every Bayes rule for $\pi \in \Pi$ is in $D(V)$, it is sufficient to show that $D(V)$ is closed in the weak* topology. Suppose $\varphi_n \to_{\omega^*} \varphi_0$ and $C_n \in \Phi(V)$ is associated with $\varphi_n \in D(V)$. If $\underline{\lim}\, \|g(C_n)\| = +\infty$, it is easy to show that $\varphi_0 = 1[\mu]$ so $\varphi_0 \in D(V)$. If $\underline{\lim}\, \|g(C_n)\| < +\infty$, then there exists a subsequence n_i and a set

C_0 such that $C_{n_i} \to_\tau C_0$. By Proposition 2.2, $C_0 \in \Phi(V)$. Using the argument M–T employed to prove Birnbaum's Theorem, it follows that $\varphi_0 = 1[\mu]$ on the complement of C_0 and $\varphi_0 = 0[\mu]$ on the interior of C_0. This completes the proof.

If $\varphi \in D(V)$ is such that $\varphi = 0$ for $y \in C$ and $\varphi = 1$ for $y \notin C$ where $C \in \Phi(V)$, the question of admissibility for φ can sometimes be answered by a result due to Stein (1956). For example, if $\Omega_1 = V - \{0\}$, Stein's result shows that φ is admissible.

4. One-sided alternatives with nuisance parameters. Consider a random vector $(X, Y) \in R^m \times R^k$ which has an exponential density

$$p(x, y; \theta, \omega) = c(\theta, \omega) \exp[\theta' x + \omega' y]$$

with respect to a probability measure μ on $R^m \times R^k$. Let Θ denote the natural parameter space and assume (0, 0) is an interior point of Θ.

The problem considered in this section is that of testing $\omega = 0$ against the alternative that $\omega \in \Omega_1 \subseteq R^k$ where Ω_1 is contained in some half-space. It is assumed that for each $\omega \in \Omega_1$, there exists a $\theta \in R^m$ such that $(\theta, \omega) \in \Theta$.

Let $V \subseteq R^k$ be the smallest closed convex cone containing Ω_1 and let $\Phi(V)$ be as in Section 2. Consider the set, $D^*(V)$, of test functions with the following property: if $\phi \in D^*(V)$, there exists a measurable set $A \subseteq R^m \times R^k$ such that each x section, $A(x) \subseteq R^k$, is in $\Phi(V)$, and

$$\begin{aligned} \phi(x, y) &= 0 \qquad && y \in \text{ interior } (A(x)) \\ &= 1 && y \notin A(x) \end{aligned}$$

with possible randomization on the boundary of $A(x)$.

THEOREM 4.1. *For the testing problem above, $D^*(V)$ is an essentially complete class. Further, given any test function Ψ, there exists $\phi \in D^*(V)$ such that for each $\omega \in \Omega_1$,*

$$E_\omega(\phi(X, Y) \mid X = x) \geqq E_\omega(\Psi(X, Y) \mid X = x)[\nu] \tag{4.1}$$

with equality when $\omega = 0$. Here, ν is the marginal distribution of X when $(\theta, \omega) =$ (0, 0).

PROOF. With the aid of Proposition 2.2 and Theorem 3.1, the proof of this result parallels the proof of M–T's Theorem 3.1.

Note that (4.1) shows for each test function Ψ, there exists $\varphi \in D^*(V)$ such that the conditional power of ϕ (given $X = x$) is no smaller than the conditional power of Ψ. This statement is of course stronger than the assertion that $D^*(V)$ is an essentially complete class.

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