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BEHAVIOR OF MOMENTS OF ROW SUMS OF ELEMENTARY SYSTEMS

BY B. M. BROWN AND G. K. EAGLESON

La Trobe University and University of Sydney

1. Introduction. Bawly [1] studied the convergence of a sequence of distribution functions of row sums of an elementary system, i.e. (Gnedenko [3] page 316) row sums of uniformly small random variables which converge in distribution to an infinitely divisible law with bounded variance. He introduced their so-called *accompanying laws* ([4] page 98, see Section 2), showed them to be asymptotic to the row sum distributions, and thus obtained necessary and sufficient conditions for the convergence in law of the row sums.

In this paper, a study is made of the behavior of the moments of the sequence of row sums, using the accompanying laws, and *their* moments. First, the cumulants of the sequence of row sums are shown to be closely related to those of the accompanying laws. This leads to necessary and sufficient conditions for the convergence of the moments of row sums to those of the limit distribution. These conditions include, as a special case, the Lindeberg conditions of even integer order which are necessary and sufficient for the convergence of moments in the central limit theorem (see [2]). The results of Section 1 of [2] are therefore placed in a natural and more general setting.

Section 4 contains a brief survey of the main results of the paper.

2. Preliminaries. Let $X_{n1}, X_{n2}, \cdots, X_{nj_n}$ be independent random variables for each $n = 1, 2, \cdots$, with $EX_{nj} = 0$ and $DX_{nj} < \infty$ for $j = 1, 2, \cdots, j_n$, $n = 1, 2, \cdots$; where DX denotes the variance of the random variable X. Assume that the $\{X_{nj}\}$ form an *elementary system* (see Gnedenko, page 316), i.e. that

$$\max_{j \leqq j_n} DX_{nj} \to 0 \qquad \text{as} \quad n \to \infty \tag{1}$$

and

$$\sum_j DX_{nj} \leqq \text{some } C < \infty \tag{2}$$

for all $n = 1, 2, \cdots$.

Let $F_{nj}(\cdot)$ denote the distribution function and $f_{nj}(\cdot)$ the characteristic function of X_{nj}. Also, for each $n = 1, 2, \cdots$, let $Y_{n1}, \cdots, Y_{nj_n}$ be independent random variables whose characteristic functions are given by $\phi_{nj}(t) = E\exp(itY_{nj}) = \exp(f_{nj}(t) - 1)$, and write

$$S_n = \sum_j X_{nj}, \qquad \text{and}$$

$$T_n = \sum_j Y_{nj}.$$

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The distribution of the random variables $\{T_n\}$ are called the *accompanying laws* of the random variables $\{S_n\}$, and have characteristic functions

$$\begin{aligned} \phi_n(t) &= E\, e^{itT_n} \\ &= \prod_j \phi_{nj}(t) \qquad (3)\\ &= \exp(\sum_j (f_{nj}(t)-1)) \\ &= \exp\{\sum_j (\int_{-\infty}^{\infty} (e^{itx}-1)\, dF_{nj}(x))\}. \end{aligned}$$

Clearly, the distributions of $\{T_n\}$ are infinitely divisible.

Now let $\phi_0(t)$ be the characteristic function of another infinitely divisible random variable T_0 with mean zero and finite variance. By Kolmogorov's representation, we can write, for $n = 0, 1, 2, \cdots$

$$\log \phi_n(t) = \int_{-\infty}^{\infty} (e^{itx} - 1 - itx)x^{-2}\, dG_n(x),$$

where $G_n(x)$ is a non-decreasing function of x with $G_n(\infty) - G_n(-\infty) = D(T_n)$. It is readily verified from (3) that, for $n \geqq 1$,

$$dG_n(x) = \sum_j x^2\, dF_{nj}(x). \qquad (4)$$

We assume throughout that S_n converges in law as $n \to \infty$ to T_0, for which a necessary and sufficient condition is that the accompanying laws T_n also converge in law to T_0 as $n \to \infty$ (Theorem 1, page 317 of [3]). An equivalent necessary and sufficient condition is that

$$G_n(x) \to G_0(x) \quad \text{weakly} \qquad \text{as} \quad n \to \infty. \qquad (5)$$

The proof of the last fact is contained in the proofs of Theorem 3, page 319 and Theorem, page 312 of [3], where the required uniform boundedness of total variations of $G_n(x)$ is provided by (2). The additional property $\lim_{n\to\infty}(G_n(\infty) - G_n(-\infty)) = G_0(\infty) - G_0(-\infty)$ is equivalent to $\lim_{n\to\infty} D(T_n) = D(T_0)$.

Our basic assumptions throughout are therefore (1), (2) and (5).

For r a positive integer, the rth cumulant of a random variable $X, K_r(X)$, is defined to be $i^{-r}\{(d/dt)^r \log E \exp(itX)\}_{t=0}$, provided this derivative exists. Consider the rth cumulants (assumed to exist) of the accompanying laws. For $r = 2, 3, \cdots$ and $n = 1, 2, \cdots$

$$\begin{aligned} K_r(T_n) &= \sum_j K_r(Y_{nj}) \\ &= \sum_j EX_{nj}^r \qquad (6)\\ &= \sum_j \int_{-\infty}^{\infty} x^r\, dF_{nj}(x) \\ &= \int_{\infty}^{\infty} x^{r-2}\, dG_n(x) \qquad \text{from (4).} \qquad (7) \end{aligned}$$

The relation (7) also holds for arbitrary infinitely divisible random variables whose rth moments are bounded; i.e. for $n = 0$. It is now convenient to define the rth *absolute* cumulant $B_r(Y)$ of an infinitely divisible random variable Y as the $(r-2)$th absolute moment of its canonical measure $G(\cdot)$, i.e.

$$B_r(T_n) = \int_{-\infty}^{\infty} |x|^{r-2}\, dG_n(x) \tag{8}$$

for $n = 0, 1, 2, \cdots$; and, for $n \geqq 1$,

$$\begin{aligned} B_r(T_n) &= \sum_j E|X_{nj}|^r \\ &= \sum_j B_r(Y_{nj}). \end{aligned} \tag{9}$$

Furthermore, it follows from Lemma 3 (below) that the weak convergence of G_n to G_0, together with the uniform boundedness of absolute moments $\limsup_{n\to\infty} \int_{-\infty}^{\infty} |x|^{k-2}\, dG_n(x) < \infty$ for some integer $k > 2$, entails

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^r\, dG_n(x) = \int_{-\infty}^{\infty} x^r\, dG_0(x) \quad \text{and} \tag{10}$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |x|^r\, dG_n(x) = \int_{-\infty}^{\infty} |x|^r\, dG_0(x) \tag{11}$$

for all integers $r < k-2$. In addition, if (11) holds for $r = k-2$, then (10) also holds for $r = k-2$. Combining these observations with equations (6)–(9) gives

LEMMA 1. *If for some* $k = 3, 4, \cdots$, $\limsup_{n\to\infty} B_k(T_n) = \limsup_{n\to\infty} \sum_j E|X_{nj}|^k < \infty$, *then*

$$\lim_{n\to\infty} B_r(T_n) = \lim_{n\to\infty} \sum_j E|X_{nj}|^r = B_r(T_0) \tag{12}$$

and

$$\lim_{n\to\infty} K_r(T_n) = \lim_{n\to\infty} \sum_j EX_{nj}^r = K_r(T_0) \tag{13}$$

for all integers $r < k$. *If in addition* (12) *holds for* $r = k$, *then* (13) *holds for* $r = k$.

LEMMA 2. *There exists a constant* $M_k < \infty$ *such that*

$$\begin{aligned} |K_r(T_n)| &\leqq B_r(T_n) \\ &\leqq M_k(B_k(T_n))^{(r-2)/(k-2)} \end{aligned}$$

for all $n = 1, 2, \cdots$ *and positive integers* r, k *with* $2 \leqq r \leqq k$.

PROOF. Express the cumulants $K_r(T_n)$ in terms of the moments of the (normalized) distribution function $G_n(x)/K_2(T_n)$, apply elementary moment inequalities, and use (2).

Finally, we state the following routine lemma for completeness (see Loève [5] page 184).

LEMMA 3. *Let* U_n, $n = 1, 2, \cdots$ *be a sequence of random variables which converges in distribution to a random variable* U *as* $n \to \infty$, *and let* $\limsup_{n\to\infty} E|U_n|^a < \infty$ *for some positive* a. *Then* $\lim_{n\to\infty} E|U_n|^b = E|U|^b$ *and* $\lim_{n\to\infty} EU_n^{\,b} = EU^b$ *for all positive* $b < a$. *Furthermore if* $\lim_{n\to\infty} E|U_n|^a = E|U|^a < \infty$, *then* $\lim_{n\to\infty} EU_n^{\,a} = EU^a$.

3. Behavior of moments.

THEOREM 1. *If* $E|S_n|^k < \infty$ *for some* $k = 4, 5, \cdots$, *and all* n, *then*

$$|K_k(S_n) - K_k(T_n)| = o(B_{k-2}(T_n)) = o(\sum_j E|X_{nj}|^{k-2})$$

as $n \to \infty$. (*For* $k = 1, 2$, *or* 3, *the* LHS *is identically zero.*)

PROOF. A relation between the moments and cumulants is given by

$$\begin{aligned} K_k(S_n) &= \sum_j K_k(X_{nj}) \\ &= \sum_j (EX_{nj}^k + \sum_{Q_k} (k!(-1)^{l-1}/l) \prod_{r=1}^l (EX_{nj}^{w_r}/w_r!)) \end{aligned} \tag{14}$$

where $Q_k = \{(w_1, \cdots, w_l)$: each integer $w_r \geqq 2$, $1 \geqq 2$, and $w_1 + \cdots + w_l = k\}$, i.e. Q_k is the set of ordered partitions of k into integers > 1.

Thus $|K_k(S_n) - K_k(T_n)| \leqq \sum_j \sum_{Q_k} (k!/l) |\prod_{r=1}^l EX_{nj}^{w_r}/w_r!|$.
To estimate the RHS we need the following

LEMMA 4. *For any random variable* X *and* $(w_1, \cdots, w_l) \in Q_k$, $|\prod_{r=1}^l EX^{w_r}| \leqq EX^2 \cdot E|X|^{k-2}$.

PROOF. From Hölder's inequality, for $2 \leqq w \leqq k-2$,

$$E|X|^w \leqq (EX^2)^{1-(w-2)/(k-4)} (E|X|^{k-2})^{(w-2)/(k-4)};$$

giving

$$\begin{aligned} |\prod_{r=1}^l EX^{w_r}| &\leqq (EX^2)^{1+(l-2)(k-2)/(k-4)} (E|X|^{k-2})^{1-2(l-2)/(k-4)} \\ &\leqq EX^2 \cdot E|X|^{k-2} \end{aligned}$$

by employing the inequality $E(X^2)^{(l-2)(k-2)/(k-4)} \leqq (E|X|^{k-2})^{2(l-2)/(k-4)}$.

Now applying Lemma 4 to (14), there exists a constant C_k, depending only on k, for which

$$\begin{aligned} |K_k(S_n) - \sum_j EX_{nj}^k| &\leqq C_k \sum_j EX_{nj}^2 \cdot E|X_{nj}|^{k-2} \\ &\leqq C_k \max_{j \leqq j_n} DX_{nj} \sum_j E|X_{nj}|^{k-2} \\ &= o(B_{k-2}(T_n)), \qquad \text{from (1) and (9).} \end{aligned}$$

COROLLARY 1. *If* $E|S_n|^4 < \infty$ *for all* n, *then* $\lim_{n\to\infty} |K_4(S_n) - K_4(T_n)| = 0$.

PROOF. This follows immediately from Theorem 1 and the assumption that $\sum_j D(X_{nj}) \leqq C$.

COROLLARY 2. *If* $E|S_n|^k < \infty$ *for some* $k = 3, 4, \cdots$, *and all* n, *and if*

$$\limsup_{n\to\infty} B_{k-2}(T_n) = \limsup_{n\to\infty} \sum_j E|X_{nj}|^{k-2} < \infty \tag{15}$$

then

$$\lim_{n\to\infty} |K_r(S_n) - K_r(T_n)| = 0 \tag{16}$$

for all integers $r \leqq k$.

PROOF. The inequality of Lemma 2 implies that $\limsup_{n\to\infty} B_{r-2}(T_n) < \infty$ for all integers r such that $2 \leqq r \leqq k$. Applying the Theorem, it follows that (16) is true for all $r \leqq k$.

COROLLARY 3. *If* $E|S_n|^k < \infty$ *for some* $k = 4, 5, \cdots$, *and all* n, *then*

$$|K_r(S_n) - K_r(T_n)| = o(B_k(T_n))^{(r-4)/(k-2)}$$

as $n \to \infty$, *for all integers* r *with* $4 \leqq r \leqq k$.

PROOF. The proof follows immediately from the inequality of Lemma 2.

THEOREM 2. *For fixed* $k = 1, 2, \cdots$, *the two conditions*

$$\limsup_{n\to\infty} ES_n^{2k} < \infty, \quad \textit{and} \tag{17}$$

$$\limsup_{n\to\infty} B_{2k}(T_n) < \infty \tag{18}$$

are equivalent.

Further, either (17) *or* (18) *implies that*

$$\lim_{n\to\infty} |K_r(S_n) - K_r(T_n)| = 0 \tag{19}$$

for all integers $r \leqq 2k$.

PROOF. By Lemma 3, (17) implies that $\lim_{n\to\infty} ES_n^r = ET_0^r < \infty$, for all integers $r < 2k$. Then, since cumulants of order $2k$ may be expressed in terms of moments of order $\leqq 2k$ (cf. equation (14)), it follows that $\limsup_{n\to\infty} K_r(S_n) < \infty$ for all integers $r \leqq 2k$, and in particular for $r = 2, 4, \cdots, 2k$. Therefore, by Theorem 1, $\limsup_{n\to\infty} B_{2j-2}(T_n) < \infty$ implies that $\limsup_{n\to\infty} B_{2j}(T_n) < \infty$, for $j = 1, 2, \cdots, k$. (18) then follows by induction, starting from the initial assumption, (2), that $\limsup_{n\to\infty} B_2(T_n) < \infty$.

Conversely (18) implies that $\limsup_{n\to\infty} |K_r(T_n)| < \infty$ for all $r \leqq 2k$, by Lemma 2. Then by Corollary 3 of Theorem 1, $\limsup_{n\to\infty} |K_r(S_n)| < \infty$ for all integers r with $2 \leqq r \leqq 2k$, and (17) follows.

Finally, (19) follows from (18) by Corollary 3 of Theorem 1.

THEOREM 3. *Let* $E|T_0|^k < \infty$ *for some* $k = 2, 3, \cdots$ *The condition*

$$\lim_{n\to\infty} B_k(T_n) = \lim_{n\to\infty} \textstyle\sum_j E|X_{nj}|^k = B_k(T_0)$$

is necessary and sufficient for

$$\lim_{n\to\infty} ES_n^k = ET_0^k \tag{20}$$

when k *is even; sufficient for* (20) *when* k *is odd; and it also implies that*

$$\lim_{n\to\infty} ES_n^r = ET_0^r, \quad \textit{for all integers} \quad r \leqq k. \tag{21}$$

PROOF. If $\lim_{n\to\infty} B_k(T_n) = B_k(T_0)$, then Lemma 1 shows that $\lim_{n\to\infty} B_r(T_n) = B_r(T_0)$, and $\lim_{n\to\infty} K_r(T_n) = K_r(T_0)$, for all integers $r \leqq k$. Therefore, by Theorem 1, $\lim_{n\to\infty} K_r(S_n) = K_r(T_0)$, for all integers $r \leqq k$, and equation (20) follows.

Conversely, let $\lim_{n\to\infty} ES_n^{\ k} = ET_0^{\ k}$ when k is even. Then by Lemma 3, $\lim_{n\to\infty} ES_n^{\ r} = ET_0^{\ r}$, for all integers $r \leqq k$, and therefore

$$\lim_{n\to\infty} K_k(S_n) = K_k(T_0) = B_k(T_0) < \infty.$$

But condition (20) implies that $\limsup_{n\to\infty} ES_n^{\ k} < \infty$, so that $\limsup_{n\to\infty} B_k(T_n) < \infty$, by Theorem 2.

Equation (21) now follows by applying Lemma 3.

COROLLARY 4. *Suppose that* $E|T_0|^{2k} < \infty$ *for some* $k = 1, 2, \cdots$. *Then a necessary and sufficient condition that* $\lim_{n\to\infty} ES_n^{\ 2k} = ET_0^{\ 2k}$ *is that* $\lim_{n\to\infty} ET_n^{\ 2k} = ET_0^{\ 2k}$.

PROOF. The Corollary follows immediately from the Theorem.

REMARK. Theorem 3 and its Corollary state that the even integer moments of the sequence of row sums converge to the moments of the limit law if and only if the corresponding moments of the accompanying laws converge to the moments of the limit law; or alternatively, if and only if the corresponding cumulants of the accompanying laws converge to the cumulants of the limit law. In the particular case of convergence to the normal or Poisson distribution, these necessary and sufficient conditions take the following form:

COROLLARY 5. *If* T_0 *is normally distributed with zero mean and variance* σ^2, *then* $\lim_{n\to\infty} E(S_n^{\ 2k}) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} x^{2k} \exp(-x^2/2\sigma^2)\, dx$, *where* k *is a positive integer, if and only if*

$$\lim_{n\to\infty} \sum_j E|X_{nj}|^{2k} = 0, \qquad \textit{for } k \geqq 2,$$
$$= \sigma^2, \qquad \textit{for } k = 1.$$

COROLLARY 6. *If* T_0 *has a Poisson distribution with parameter* λ, *then* $\lim_{n\to\infty} E(S_n^{\ 2k}) = \sum_{x=0}^{\infty} x^{2k} \lambda^x e^{-\lambda}/x!$, *where* k *is a positive integer, if and only if* $\lim_{n\to\infty} \sum_j E|X_{nj}|^{2k} = \lambda$.

Furthermore, the condition $K_{2k}(T_n) = \int_{\infty}^{\infty} x^{2k-2}\, dG_n(x) \to 0$ as $n \to \infty$ implies that

$$\int_{|x|>\varepsilon} dG_n(x) = \sum_j \int_{|x|>\varepsilon} x^2\, dF_{nj}(x) \to 0$$

as $n \to \infty$ for all $\varepsilon > 0$, so that T_0 is normally distributed. Similarly, if

$$\sum_j E(X_{nj}^2(X_{nj}-1)^{2k-2}) = \sum_{r=0}^{2k-2} \binom{2k-2}{r} (-1)^r K_{r+2}(T_n) \tag{22}$$
$$= \int_{-\infty}^{\infty} (x-1)^{2k-2}\, dG_n(x) \to 0 \qquad \text{as } n \to \infty,$$

then

$$\int_{|x-1|>\varepsilon} dG_n(x) = \sum_j \int_{|x-1|>\varepsilon} x^2\, dF_{nj}(x) \to 0 \qquad \text{as } n \to \infty$$

for all $\varepsilon > 0$; and T_0 has a Poisson distribution. These observations lead to the following improved versions of Corollary 5 and Corollary 6.

COROLLARY 7. (see also [2]). *The condition*

$$\lim_{n\to\infty} \sum_j EX_{nj}^{2k} = 0, \qquad \textit{for} \quad k \geqq 2,$$
$$= \sigma^2, \qquad \textit{for} \quad k = 1,$$

where k is an integer $\geqq 1$, is necessary and sufficient for the convergence in law as $n \to \infty$ of S_n to a normally distributed random variable T_0, and $\lim_{n\to\infty} E{S_n}^{2k} = E{T_0}^{2k}$.

COROLLARY 8. *The conditions*

$$\lim_{n\to\infty} \sum_j EX_{nj}^{2k} = \lambda \qquad \textit{and}$$
$$\lim_{n\to\infty} \sum_j EX_{nj}^2(1-X_{nj})^{2k-2} = 0,$$

where k is an integer $\geqq 1$, are necessary and sufficient for the convergence in law as $n \to \infty$ of S_n to a Poisson random variable T_0, and $\lim_{n\to\infty} E{S_n}^{2k} = E{T_0}^{2k}$.

REMARK. Since $K_j(S_n) = K_j(T_n)$ for $j = 2, 3$ and all n, while $\lim_{n\to\infty} |K_4(S_n) - K_4(T_n)| = 0$ (Corollary 1) it follows that $\lim_{n\to\infty} K_4(S_n) = 0$ implies $\lim_{n\to\infty} K_4(T_n) = 0$ and hence that S_n converges in law as $n \to \infty$ to a normally distributed random variable. Similarly,

$$\lim_{n\to\infty} (K_4(S_n) - 2K_3(S_n) + K_2(S_n)) = 0$$

implies

$$\lim_{n\to\infty} (K_4(T_n) - 2K_3(T_n) + K_2(T_n)) = 0,$$

and hence (from (22) with $k = 2$) that S_n converges in law to a Poisson random variable. These results are stronger than those obtained by Pierre [6].

4. Remarks. The principal features of our results seem to us to be the following:

(i) The behavior of the cumulants $K_k(S_n)$ is closely linked to that of the cumulants $K_k(T_n)$ of the accompanying laws. In fact, the difference $K_k(S_n) - K_k(T_n)$ is identically zero for $k = 1, 2, 3$, is $o(1)$ as $n \to \infty$ for $k = 4$, and thereafter ($k \geqq 5$) is $o(1)$ as $n \to \infty$ if the $B_{k-2}(T_n)$ are uniformly bounded in n.

(ii) Necessary and sufficient conditions are obtained for the convergence of the moments $E{S_n}^{2k}$ to $E{T_0}^{2k}$ as $n \to \infty$, when S_n converges in law to T_0 as $n \to \infty$. These conditions are in terms of the moments of the accompanying laws, or, more usefully, in terms of their cumulants. The conditions involving cumulants are easy to work with, because of the relation $K_{2k}(T_n) = \sum_j EX_{nj}^{2k}$. They lead, for example, to simple necessary and sufficient conditions for convergence of moments to those of the limit law T_0 when T_0 has either a normal or Poisson distribution. In the case of the normal, the condition is also sufficient for the convergence in law itself.

(iii) The requirement of uniform boundedness in n of $B_{k-2}(T_n)$ is a key one. Apart from ensuring the asymptotic property for kth order cumulants, $|K_k(S_n) - K_k(T_n)| = o(1)$ as $n \to \infty$ (see (i)), it also governs the behavior of *lower* order moments and cumulants in that it forces their convergence as $n \to \infty$ to those of the limit law T_0, by Lemma 1 and Theorem 3.

Note also that when $k = 4$, it reduces to the basic assumption (2), and in that sense, our initial assumptions act as a prototype for Corollary 2.

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