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EXTRAPOLATION AND INTERPOLATION OF STATIONARY GAUSSIAN PROCESSES[1]

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CONTENTS

1. Introduction. The problem of extrapolation of a stationary Gaussian process $\mathfrak{x}$ using the whole past $\mathfrak{x}(t): t \leqq 0$ was solved about 1940 by A. N. Kolmogorov [15], M. G. Krein [16], and N. Wiener [27]. The purpose of this paper is to present the mathematical tools needed to solve the extrapolation problem if only part of the past is available. M. G. Krein [20] did this by applying the solution of the inverse spectral problem as initiated by Gel'fand–Levitan [10] and perfected by himself [17–19]. Dym–McKean [8] used a second method involving spaces of integral (entire) functions, of the kind introduced and extensively studied by L. de Branges [5, 6].

Both methods take advantage of the fact that the study of the short process $\mathfrak{x}(t): |t| \leqq T+$ is isomorphic, via the map $\mathfrak{x}(t) \to \exp(i\gamma t)$, to the study of the class of

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integral functions f of exponential type[2] $\leqq T$ belonging to $\mathbf{Z}(W) = \mathbf{L}^2(dW, R^1)$. W stands for the spectral weight which goes with $\mathfrak{x}$, and it is supposed for the moment that dW is non-singular with density Δ (relative to Lebesgue measure) and that

$$\int (1+\gamma^2)^{-1} \lg \Delta(\gamma)\, d\gamma > -\infty$$

which makes each of these function classes a closed subspace of $\mathbf{Z}(W)$. The projections upon these spaces form a spectral resolution which can be expressed by means of eigendifferential expansions similar to the usual Fourier sine and cosine transform for $\mathbf{L}^2(d\gamma, R^1)$. The solution of the extrapolation problem using only part of the past may be expressed in this language. By way of introduction, the solution of the classical problem of extrapolation off a half-line is sketched.

An account of the interpolation problem is included in view of its close relation to the above. To the best of our knowledge the recipe presented below is the first proposal for a full solution of that problem.

The paper is offered as an outline only. The proofs of new material will be presented elsewhere. The bibliography makes no pretensions to completeness, but it is hoped that all the major contributions to the field have been cited, expressly leaving aside chains (integral time) and several-dimensional processes. The reader will find an account of these and other topics not touched upon in Rozanov [26]. Proofs and further developments will be the subject of a joint book of the authors to be published at a later date.

2. Gaussian processes. Consider a 1-dimensional, centered, stationary Gaussian process with sample paths $t \to \mathfrak{x}(t) \in R^1$ and let W be its spectral weight, so that[3]

$$\mathbf{E}[\mathfrak{x}(a)\mathfrak{x}(b)] = \int e^{i\gamma(b-a)}\, dW(\gamma).$$

The measure dW splits into two pieces, a Lebesgue piece and a singular piece:

$$dW = \Delta\, d\gamma + dW^s.$$

Because $\mathfrak{x}$ is a real process dW must be even, esp., Δ is even, as well as nonnegative and summable.

The map $\mathfrak{x}(t) \to \exp(i\gamma t)$ provides an isomorphism between the closure $\mathbf{H}$ of finite sums $\mathfrak{y} = \sum c_j \mathfrak{x}(t_j)$ under the norm

$$\|\mathfrak{y}\| = (\mathbf{E}\,|\mathfrak{y}|^2)^{\frac{1}{2}}$$

and the closure in $\mathbf{Z}(W) = \mathbf{L}^2(dW, R^1)$ of the trigonometric sums $f = \sum c_j \exp(i\gamma t_j)$ under the norm

$$\|f\| = (\int |f|^2\, dW)^{\frac{1}{2}}.$$

The discussion will be carried out mostly in this second (trigonometrical) language, still employing probabilistic names if appropriate: for instance,

$$\mathbf{Z}^{-\infty}(W) = \bigcap_{T \leqq 0} [\text{the span in } \mathbf{Z}(W) \text{ of the functions } e^{i\gamma t} : t \leqq T]$$

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[2] This means that $\limsup_{R\to\infty} R^{-1} \max_{0 \leqq \theta \leqq 2\pi} \lg |f(Re^{i\theta})| \leqq T$.

[3] $\mathbf{E}$ stands for expectation. $\int$ means integration over the whole line unless otherwise indicated.

is still called the remote past. The reader should keep it in mind that in the present (centered Gaussian) situation, statistical independence is the same as perpendicularity in $\mathbf{H}$ (or $\mathbf{Z}$). A second important geometrical point is that if $\mathbf{A}$ is a closed subspace of $\mathbf{H}$, if $\mathbf{B}$ is the smallest Borel field over which $\mathbf{A}$ is measurable, and if $\mathbf{C}$ is the biggest subspace of $\mathbf{H}$ which is measurable over $\mathbf{B}$, then the self-evident inclusion $\mathbf{C} \supset \mathbf{A}$ is actually an identity, and the projection upon $\mathbf{A} = \mathbf{C}$ is the conditional expectation $\mathfrak{y} \to \mathbf{E}(\mathfrak{y} \mid \mathbf{B})$.

3. Szegö's alternative. Define $\mathbf{Z}^-$ to be the "past":

$$\mathbf{Z}^-(W) = \text{the span in } \mathbf{Z}(W) \text{ of the functions } e^{i\gamma t} : t \leqq 0.$$

A very important fact is Szegö's alternative, as refined by Kolmogorov and Krein:[4]

$$\text{EITHER} \quad \mathbf{Z}^{-\infty}(W) = \mathbf{Z}(W) \quad \text{and} \quad \int (1+\gamma^2)^{-1} \lg \Delta(\gamma)\, d\gamma = -\infty$$

$$\text{OR} \quad \mathbf{Z}^{-\infty}(W) = \mathbf{Z}(W^s) \quad \text{and} \quad \int (1+\gamma^2)^{-1} \lg \Delta(\gamma)\, d\gamma > -\infty.$$

The first possibility is the case of perfect prediction of the future knowing the whole past. In the second case, the process splits into the sum of a perfectly predictable piece corresponding to dW^s and an independent piece corresponding to $d(W-W^s) = \Delta\, d\gamma$. Because of this it is natural for the classical prediction problem to impose the so-called Hardy condition:

$$\int (1+\gamma^2)^{-1} \lg \Delta(\gamma)\, d\gamma > -\infty,$$

in order to exclude the first possibility, and simply to assume that W has no singular part $[dW = \Delta\, d\gamma]$. These two assumptions will be in force until Section 10. To emphasize the latter we shall write $\mathbf{Z}(\Delta)$ in place of $\mathbf{Z}(W)$, etc. Szegö's alternative implies that in the Hardy case

$$\mathbf{Z}^{-\infty}(\Delta) = 0 \quad \text{and} \quad \mathbf{Z}^-(\Delta) \neq \mathbf{Z}(\Delta).$$

The reader should notice that $\Delta = \exp(-|\gamma|)$ is the approximate dividing line between the Hardy and the non-Hardy cases: for this weight, the polynomials[5]

$$(i\gamma)^n = \partial^n \exp(i\gamma t) \text{ evaluated at } t = 0$$

belong to $\mathbf{Z}^-$ and already span the whole of $\mathbf{Z}$, so that also

$$\mathbf{Z}^{-\infty} = \bigcap_{T \leqq 0} \exp(i\gamma T)\mathbf{Z}^- = \mathbf{Z},$$

as it should.

4. Hardy functions. To proceed, you have to know something about the space $\mathbf{H}^{2+}$ of Hardy functions on the upper half-plane. This is the class of functions $h = h(\gamma) = h(a+ib)$ which are analytic on the upper half-plane $R^{2+} = (a+ib : b > 0)$ and satisfy the growth condition

$$\|h\|^2 = \sup_{b>0} \int |h(a+ib)|^2\, da < \infty.$$

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[4] The proof may be found in [2:263]; see also [14] and [22:115].

[5] ∂ stands for differentiation with regard to time.

For such h, the function $h_b = h(\bullet + ib)$ converges in $\mathbf{L}^2(d\gamma, R^1)$ as $b \downarrow 0$ to a function h_{0+}, and the map $h \to h_{0+}$ is 1:1, so you can permit yourself to identify h with its boundary value. The Fourier transform

$$h \to \hat{h}(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h(\gamma) e^{-i\gamma t} d\gamma$$

of every element h in $\mathbf{L}^2(d\gamma, R^1)$ arising this way vanishes on the left half-line, and conversely. That is to say, with a self-evident notation,

$$(\mathbf{H}^{2+})^\wedge = \mathbf{L}^2(d\gamma, [0, \infty)).$$

Another important point is that if $0 \leqq \Delta \in \mathbf{L}^1(d\gamma, R^1)$ satisfies

$$\int (1+\gamma^2)^{-1} \lg \Delta > -\infty,$$

then $\Delta = |h|^2$ for some $h \in \mathbf{H}^{2+}$ and vice versa (provided $h \not\equiv 0$), so that you have a simple test for the modulus of a function from $\mathbf{H}^{2+}$. The above facts go back to Paley–Wiener [24].

The structure of $\mathbf{H}^{2+}$ is further refined by the notion of inner and outer functions introduced by Beurling [3]. A function $h \not\equiv 0$ of class $\mathbf{H}^{2+}$ always satisfies

$$\lg |h(a+ib)| \leqq \frac{b}{\pi} \int \frac{\lg |h(\gamma)|}{(\gamma-a)^2+b^2} d\gamma$$

for $b > 0$ and is called *outer* if this is actually an identity. The reader will notice that such an outer function is root-free on R^{2+}. Given $k \in \mathbf{H}^{2+}$, you can find an outer function $h \in \mathbf{H}^{2+}$ with the same modulus as k on the line, namely,

$$h(\gamma) = \exp\left[\frac{1}{\pi i} \int \frac{1+\gamma x}{x-\gamma} \lg |k(x)| \frac{dx}{1+x^2}\right].$$

This outer function is completely determined by $|k|$ up to a multiplicative constant of modulus 1, and the ratio k/h is an *inner* function, meaning that its modulus is $= 1$ almost everywhere on the line and $\leqq 1$ above. To sum up, any function $k \in \mathbf{H}^{2+}$ can be factored in (essentially) only one way into the product of an inner and outer function.

A second way of expressing the fact that $h \in \mathbf{H}^{2+}$ is an outer function is to say that $\exp(i\gamma t)h : t \geqq 0$ spans $\mathbf{H}^{2+}$, which is the same as to say that $\hat{h}(\bullet - t) : t \geqq 0$ spans $\mathbf{L}^2[0, \infty)$. A third test states that if $h \in \mathbf{H}^{2+}$ is an outer function then

$$\int_0^t |\hat{h}(s)|^2 ds \geqq \int_0^t |k(s)|^2 ds$$

for every $t \geqq 0$ and every $k \in \mathbf{H}^{2+}$ with the same modulus as h on the line. This may be interpreted as saying that the associated filter $e \to \hat{h} * e = \int \hat{h}(t-s) e(s) ds$ responds to the input signal e as rapidly as possible.

The reader will find nice proofs of everything but the third test for outer functions in K. Hoffman [12]. The latter is due to Robinson [25].

5. Kolmogorov–Wiener extrapolation. The extrapolation (prediction) problem of Kolmogorov [15] and Wiener [27] can now be solved following K. Karhunen [13] who was the first to make explicit use of outer functions in this connection; see also [26: 109–129].

Given $T > 0$, the problem is to find the best approximation to $\mathfrak{x}(T)$ from the span of the past $\mathfrak{x}(t): t \leqq 0$, assuming that the remote past is trivial. As pointed out in Section 3, this is the same as to say that W is non-singular and satisfies the Hardy condition:

$$\int (1+\gamma^2)^{-1} \lg \Delta > -\infty.$$

This best approximation is the projection upon the past, or what is the same, the conditional expectation $\mathbf{E}(\mathfrak{x}(T) \mid \mathbf{A})$, $\mathbf{A}$ being the smallest Borel field over which the past is measurable.

$\mathbf{H}^{2+}$ comes in as follows. Because Δ is a Hardy weight, you can express it as $|h|^2$ for some $h \in \mathbf{H}^{2+}$, and since Δ is even, you can also impose the reality condition $h^*(\gamma) = h(-\gamma)$ for $\gamma \in R^1$.[6] This makes $\hat{h}$ real. Given a standard white noise $\mathfrak{e}$, the process $\mathfrak{x}_1(t) = \int \hat{h}(t-s)\mathfrak{e}(s)\, ds$ is centered and Gaussian, and since

$$\begin{aligned} \mathbf{E}[\mathfrak{x}_1(a)\mathfrak{x}_1(b)] &= \int \hat{h}(a-s)\hat{h}(b-s)\, ds \\ &= \int e^{-i\gamma a} h(\gamma)\, e^{i\gamma b} h^*(\gamma)\, d\gamma \\ &= \int e^{\, i\gamma(b-a)}\, \Delta(\gamma)\, d\gamma, \end{aligned}$$

it is identical in law to the original process $\mathfrak{x}$. This permits you to identify $\mathfrak{x}$ and $\mathfrak{x}_1$. Notice that this identification depends only upon the modulus of h on the line, especially, *h may be assumed to be an outer function.* The field $\mathbf{A}$ of the past of $\mathfrak{x}$ is now part of the field $\mathbf{B}$ of the past of the white noise $\mathfrak{e}$, and it is easy to prove that these two fields match if and only if h is an outer function. But this can just as well be assumed to begin with, as noted above. Then the expectation of $\mathfrak{x}(T)$ conditional upon $\mathbf{A}$ is the same as the expectation conditional upon $\mathbf{B}$:

$$\mathbf{E}(\mathfrak{x}(T) \mid \mathbf{B}) = \int_{-\infty}^{0} \hat{h}(T-t)\, \mathfrak{e}(t)\, dt$$

and the extrapolation problem is solved. The formula for the mean square prediction error is

$$\mathbf{E}\left[\int_0^T \hat{h}(T-t)\, \mathfrak{e}(t)\, dt\right]^2 = \int_0^T |\hat{h}(t)|^2\, dt.$$

WARNING. From now on, we reserve the letter h for the outer function in $\mathbf{H}^{2+}$ for which $|h|^2 = \Delta$ on the line.

6. Interpolation. The problem of interpolation originates with Karhunen [14], Kolmogorov [15], and Wiener [27]. The idea is to predict $\mathfrak{x}$ inside the interval $|t| < T$, knowing it on the whole complement $|t| \geqq T$. The problem is trivial in the non-Hardy case in view of Szegö's alternative, so it is natural to take

$$\int (1+\gamma^2)^{-1} \lg \Delta > -\infty.$$

[6] * means complex conjugation.

The condition for *perfect* interpolation is that

$$\int |f|^2 \Delta^{-1} = \infty$$

for every integral function of exponential type $\leqq T$. This fact is due to Karhunen [14]. For additional discussion see Section 22 below. An alternative condition for *imperfect* interpolation is that $\exp(2i\gamma T)h^*/h$ should agree on the line with the ratio h^+/h^- of a function $h^+ \in \mathbf{H}^{2+}$ and a function h^- from the class $\mathbf{H}^{2-} = (\mathbf{H}^{2+})^{\#}$ [7] of Hardy functions on the lower half-plane. The proof can be found in [1]. A recipe for the effective solution of the interpolation problem is presented later in Section 22. The special case of a rational weight Δ is treated in [28]; see also [1] and [26: 129–135].

7. Past, future, and splitting field. The subject of the present section is the degree of dependence between the past

$$\mathbf{Z}^-(\Delta) = \text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } e^{i\gamma t}: t \leqq 0,$$

and the future

$$\mathbf{Z}^+(\Delta) = \text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } e^{i\gamma t}: t \geqq 0$$

under the Hardy condition

$$\int (1+\gamma^2)^{-1} \lg \Delta > -\infty.$$

This may be measured by the projection $\mathbf{Z}^{+/-}$ of the future upon the past. The probabilistic meaning is that the field $\mathbf{B}^{+/-}$ of the projection of the future upon the past is the smallest subfield of the past, conditional upon which past and future are independent. $\mathbf{B}^{+/-}$ is called the *splitting field* [22: 101–102]. $\mathbf{Z}^{+/-}$ provides a way of saying how big this field is, and $\mathfrak{x}$ may be said to be more or less Markovian accordingly. For example, $\mathfrak{x}$ would be highly Markovian if $\dim \mathbf{Z}^{+/-} < \infty$ and highly non-Markovian if $\mathbf{Z}^{+/-} = \mathbf{Z}^-$. In any case $\mathbf{Z}^{+/-} \supset \mathbf{Z}^- \cap \mathbf{Z}^+$. The following facts from Levinson–McKean [22: 103–105] are apropos of this idea:

(a) $\mathbf{Z}^- \neq \mathbf{Z}^{+/-}$ iff h^*/h agrees on the line with the ratio of 2 inner functions.
(b) $\mathbf{Z}^{+/-} = \mathbf{Z}^- \cap \mathbf{Z}^+$ iff h^*/h agrees on the line with the reciprocal of an inner function.
(c) $\dim \mathbf{Z}^- \cap \mathbf{Z}^+ = 1$ iff h is the only outer function with phase h^*/h.
(d) $\dim \mathbf{Z}^{+/-} < \infty$ iff Δ is a rational function.

The so-called *germ*

$$\mathbf{Z}^{0+}(\Delta) = \bigcap_{T>0} [\text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } e^{i\gamma t}: |t| \leqq T]$$

is also of importance in this connection. $\mathbf{Z}^{0+}$ is a subspace of $\mathbf{Z}^- \cap \mathbf{Z}^+ \subset \mathbf{Z}^{+/-}$ and may be identified as the space of integral functions f of minimal exponential type [8]

[7] $h^{\#}(\gamma) = h^*(\gamma^*)$.
[8] This means that $\limsup_{R\to\infty} R^{-1} \max_{0\leq\theta\leq 2\pi} |f(Re^{i\theta})| = 0$.

which belong to $\mathbf{Z}$ [22: 111–115]. $\mathbf{Z}^{0+}$ describes the germ of the process. The principal facts about it are as follows:

(a) $\mathbf{Z}^{+/-} = \mathbf{Z}^{0+}$ iff $1/h$ is an integral function of minimal exponential type [22: 121–123].

(b) $\mathbf{Z}^{-} \cap \mathbf{Z}^{+} = \mathbf{Z}^{0+}$ if Δ^{-1} is locally summable [22: 115–118].

(c) $\mathbf{Z}^{0+}$ contains only functions of genus 0 if $\int (1+\gamma^2)^{-1} \lg^+ |\gamma| \lg \Delta > -\infty$ [22: 118–120].

$\mathbf{Z}^{0+} = \mathbf{Z}^{+/-}$ expresses the condition that *the process splits over its germ*, that is to say that the splitting field $\mathbf{B}^{+/-}$ coincides with the germ field:

$$\mathbf{B}^{0+} = \bigcap_{T>0} \mathbf{B}^T, \qquad \mathbf{B}^T = \text{field } [\mathfrak{x}(t): |t| \leqq T].$$

A final point of interest, amplifying the above, is the statement of [8: 340] that, for any $T > 0$, *the process splits over the field of* $\mathfrak{x}(t)$: $-2T \leqq t \leqq 0$ iff $[\exp(i\gamma T)h]^{-1}$ *is an integral* function of exponential type $\leqq T$.

8. Rational weights. The spectral density Δ is rational iff $\dim \mathbf{Z}^{+/-} < \infty$, as noted above. This is a case of particular importance from the electrical point of view. The following material is adapted from [22: 120–121], though the actual facts have been known for a long time; see, for example, [11], [26], [27], and [28].

A rational weight Δ satisfies

$$\int (1+\gamma^2)^{-1} \lg \Delta > -\infty$$

automatically, and the corresponding outer function h is rational and root-free on the open upper half-plane. The roots of h in the closed lower half-plane control the disposition of the subspaces

$$\mathbf{Z}^{+/-} \supset \mathbf{Z}^{-} \cap \mathbf{Z}^{+} \supset \mathbf{Z}^{0+}:$$

(a) $\mathbf{Z}^{0+} = \mathbf{Z}^{-} \cap \mathbf{Z}^{+}$ iff h is root-free on the line.

(b) $\mathbf{Z}^{-} \cap \mathbf{Z}^{+} = \mathbf{Z}^{+/-}$ iff h is root-free on the open lower half-plane.

(c) $\mathbf{Z}^{0+} = \mathbf{Z}^{-} \cap \mathbf{Z}^{+} = \mathbf{Z}^{+/-}$ iff h has no roots at all.

The third statement may be amplified by the following alternative conditions:

(d) $\Delta = |p|^{-2}$ for some polynomial p with no roots in the closed upper half-plane.

(e) $D[\mathfrak{x}] =$ a standard white noise for some differential operator with constant coefficients $[D = (2\pi)^{-\frac{1}{2}} p(-i\partial)]$.[9]

(f) $\mathbf{B}^{+/-} = \text{field } [\partial^k \mathfrak{x}(0): k < n]$ for some $n (= \text{degree of } p) < \infty$.

The last statement (f) expresses the fact that the n-dimensional process $\mathfrak{y} = (\mathfrak{x}, \partial \mathfrak{x}, \cdots, \partial^{n-1} \mathfrak{x})$ is Markovian in the customary sense.

9. Germ and gap. An important part of the germ $\mathbf{Z}^{0+}$ is the span of the polynomials:

$$\mathbf{Z}_\infty(\Delta) = \text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } (i\gamma)^k: k < n,$$

[9] ∂ stands for differentiation with regard to time.

in which $\int \gamma^{2n}\Delta$ is the first (even) moment of Δ to diverge ($n \leqq \infty$). This corresponds to the field of $\partial^k \mathfrak{x}(0): k < n$, and it is an open problem to find an effective test for deciding if $\mathbf{Z}^{0+} = \mathbf{Z}_\infty$ or not. The factor space $\mathbf{Z}^{0+}/\mathbf{Z}_\infty$ is called the *gap*. The reader should not think that just because each function in $\mathbf{Z}^{0+}$ is integral and can be expressed by a nice power series that such a gap cannot be present: the power series may not converge in $\mathbf{Z}$! Be that as it may, the gap is *either* absent *or* infinite-dimensional. The gap *is* absent if Δ is the reciprocal of an (even) power series with nonnegative coefficients; see Dym–McKean [8: 320–325] for proofs and additional information.

Levinson–McKean [22: 130–133] found a Hardy weight $\Delta = |h|^2$ with $1/h$ integral and of minimal exponential type, $\dim \mathbf{Z}_\infty = \infty$, and a gap. A simpler example with a gap is presented below. Both examples depend upon the fact that if Δ has a lot of spikes placed not too far apart, then the distance

$$\|f\| = (\int |f|^2 \Delta)^{\frac{1}{2}}$$

will discriminate very delicately between integral functions of small exponential type and make polynomial approximation very difficult. Carlson's theorem [4: 153] which states that an integral function of small exponential type with roots at the integers 0, 1, 2, etc., can only be identically 0, is closely related to this kind of thing.

The example is made as follows: for each $n \geqq 1$, Δ has a sharp rectangular spike centered about the point n^2, of height $n^{-3}\exp(\pi n)$, and width $n\exp(-\pi n)$, while between the nth and $(n+1)$st spike it drops to the level $n^{-3}\exp(-2\pi n)$. Δ is extended to the left so as to be even after making it $=1$ between 0 and spike no. 1. The reader will easily check that

(a) $\int \Delta < \infty$,

(b) $\int (1+\gamma^2)^{-1} \lg \Delta > -\infty$,

(c) $\int \gamma^2 \Delta = \infty$,

(d) $\int |\sin \pi(\gamma^{\frac{1}{2}}) \operatorname{sh}(\pi(\gamma)^{\frac{1}{2}})|^2 \Delta < \infty$.

(a) and (b) are self-evident, and (c) states that $\mathbf{Z}_\infty$ contains only constants, but by (d) $\mathbf{Z}^{0+}$ contains a non-constant integral function: $f = \sin(\pi(\gamma)^{\frac{1}{2}}) \operatorname{sh}(\pi(\gamma)^{\frac{1}{2}})$.

The probabilistic significance of the gap is unclear, except that it describes the part of the germ which is independent of the differential coefficients $\partial^k \mathfrak{x}(0)$: $k < n (\leqq \infty)$ but naturally is *still local*! The reader should note that for $1/h$ integral and of minimal exponential type, the possibly ∞-dimensional process ($\mathfrak{x}, \partial\mathfrak{x}, \partial^2\mathfrak{x}$, etc.) is Markovian iff there is no gap.

An interesting problem is to decide (in the non-Hardy case) if $\mathbf{Z}^{0+}(\Delta) = \mathbf{Z}(\Delta)$ or not. Levinson–McKean [22: 123–124] proved that this happens if

$$\int_1^\infty \gamma^{-2} \lg [\int_\gamma^\infty \Delta |k|^2] = -\infty$$

for some $k \in \mathbf{H}^{2+}$ whose modulus is a decreasing function on the half-line. If Δ itself is decreasing on the half-line, the condition is the same as

$$\int (1+\gamma^2)^{-1} \lg \Delta = -\infty.$$

10. Extrapolation off a bounded segment of the past. The problem is to project $\mathfrak{x}(R)$ for fixed $R > 0$ upon the span of $\mathfrak{x}(t): -2T \leqq t \leqq 0$, or what is the same after a self-explanatory shift of the time scale, to project $\exp[i\gamma(T+R)]$ upon

$$\mathbf{Z}^T(\Delta) = \text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } e^{i\gamma t}: |t| \leqq T.$$

This space is closely related (and sometimes identical) to the class of integral functions of exponential type $\leqq T$ which belong to $\mathbf{Z}$; see [20], [22: 135–142], and for a simpler proof [8: 319–320]. M. G. Krein [20] computed this projection by identifying the spectral density Δ of the process $\mathfrak{x}$ with the derivative of the spectral function of a generalized second-order differential operator

$$G: f \to df^+/dm,$$

the idea being to express the projections by means of the eigendifferential expansions for G. From this point of view, the fundamental problem is to determine G from its spectral function. This so-called inverse spectral problem was initiated by Gel'fand–Levitan [10] and perfected by Krein himself [17–19]. In our discussion of this problem below, we shall denote the spectral function (weight) which goes with G by W, and, following Krein, impose only the condition

$$\int (1+\gamma^2)^{-1}\, dW < \infty.$$

Another way of computing the projection upon $\mathbf{Z}^T(\Delta)$ is by identifying it as a space of integral functions of the kind introduced by L. de Branges [5, 6]. This was done by Dym–McKean [8] under extra technical assumptions. The case of a rational spectral density is done by counting roots and poles in Rozanov [26: 135–142]. The purpose of the remainder of this paper is to elucidate the deep structure of the spaces $\mathbf{Z}^T(W)$ and the allied spaces

$$\mathbf{Z}^{T+}(W) = \bigcap_{R>T} \mathbf{Z}^R(W)$$

following M. G. Krein [20], as supplemented with finer details from the standpoint of Dym–McKean [8].

11. The inverse spectral problem. The principal tools of Krein's investigations on the inverse spectral problem [17–19] are explained below in a form adapted to the present needs.

A *weighted string* is described by a nonnegative mass distribution m loaded up on a closed interval $0 \leqq x \leqq \mathfrak{l} \leqq \infty$ having mass at or near both $x = 0$ and $x = \mathfrak{l}$, and subject to the proviso that $m(\mathfrak{l}) = 0$ unless both $\mathfrak{l} < \infty$ and $m[0, \mathfrak{l}) < \infty$.

This data is supplemented by the choice of a nonnegative number $k \leqq \infty$ subject also to provisos as indicated in the second column of the accompanying table.

Each permissible choice of k determines a self-adjoint non-positive differential operator G acting in $\mathbf{Q} = \mathbf{L}^2(dm, [0, \mathfrak{l}])$. The domain $\mathbf{D}(G)$ of G is the class of all functions $f \in \mathbf{Q}$ which have 1-sided slopes f^- and f^+ satisfying

(a) $f^+(0) - f^-(0) = Gf(0)m(0)$,

(b) $f^+(b) - f^+(a) = \int_{a+}^{b+} Gf\, dm$ for $0 \leqq a < b < \mathfrak{l}$,

(c) $f^+(\mathfrak{l}) - f^-(\mathfrak{l}) = Gf(\mathfrak{l})m(\mathfrak{l})$

TABLE 1

$\int_0^\ell m\,dx$*	$\int_0^{\ell-} x\,dm$	$m(\ell)$	k
$< \infty$	$< \infty$	$0 \leqq m < \infty$	$0 \leqq k \leqq \infty$ $k = 0$ is permitted only if $m = 0$
$< \infty$	$= \infty$	$= 0$	$= 0$
$= \infty$	$< \infty$	$= 0$	$= \infty$
$= \infty$	$= \infty$	$= 0$	////////////////

* $\int_0^\ell m\,dx$ stands for $\int_0^\ell m[0, x]\,dx$.

for some $Gf \in \mathbf{Q}$, subject to the boundary conditions

(d) $f^-(0) = 0$,

(e) $f(\ell)+kf^+(\ell) = 0$.

The map

$$f \to Gf = df^+/dm$$

is well defined by this neat-looking recipe, but a lot of stuff is concealed by means of tricky conventions, and this must now be explained in detail.

To begin with, (b) implies that for $0 < b < \ell$, $f^+(b-)$ exists and agrees with the left-hand slope $f^-(b)$:

$$f^+(b-) = \lim_{a\uparrow b}(b-a)^{-1}\int_a^b f^+(x)\,dx = \lim_{a\uparrow b}\frac{f(b)-f(a)}{b-a} = f^-(b).$$

Notice also that $f^+ = f^-$ except perhaps at the (countably many) jumps of m:

$$f^+(b)-f^-(b) = \lim_{a\uparrow b}[f^+(b)-f^+(a)] = \lim_{a\uparrow b}\int_{a+}^{b+} Gf\,dm = Gf(b)m(b).$$

This formula holds for $0 < b < \ell$ and is extended to $b = 0$ and to $b = \ell$ by (a) and (c) respectively. The latter are to be regarded as *defining* $f^-(0)$ and $f^+(\ell)$ with the understanding that $Gfm = 0$ if $m = 0$. (d) is now self-explanatory, but (e) still needs some clarification.

CASE 1. $\int_0^\ell m\,dx = \int_0^{\ell-} x\,dm = \infty$. k is not specified and neither (c) nor (e) is imposed, as the shaded box in the table is meant to indicate.

CASE 2. $\int_0^\ell m\,dx = \infty > \int_0^{\ell-} x\,dm$. $f^-(\ell)$ is defined as $\lim_{x\to\ell} f^+(x)$ which exists in view of (b) and the fact that Gf is summable:

$$\left(\int_0^{\ell-} |Gf|\,dm\right)^2 \leqq \int_0^{\ell-} |Gf|^2\,dm \int_0^{\ell-} dm < \infty.$$

$m(\ell)$ is now 0, (c) means that $f^+(\ell) = f^-(\ell)$, $k = \infty$, and (e) says that $f^-(\ell) = 0$.

CASE 3. $\int_0^{\ell} m\,dx < \infty = \int_0^{\ell-} x\,dm$. k is now 0 as is $m(\ell)$, (c) means that $f^+(\ell) = f^-(\ell)$ as before, and (e) says that $f(\ell) = f(\ell-) = 0$.

CASE 4. $\int_0^{\ell} m\,dx$ and $\int_0^{\ell-} x\,dm < \infty$. $f^-(\ell)$ exists in the usual sense, (e) is self-explanatory if $0 < k < \infty$, and if $k = \infty$, it means that

$$f^+(\ell) = f^-(\ell) + Gf(\ell)m(\ell) = 0,$$

while if $k = 0$, in which case $m(\ell) = 0$, too, and $f^+(\ell) = f^-(\ell)$, it means that $f(\ell) = 0$.

G is now defined, and it turns out that the possibilities cited under (e) provide a complete list of all the non-positive self-adjoint contractions of G as defined by (a) through (d) on the half-open interval $0 \leqq x < \ell$.[10]

Given a fixed complex number γ, bring in the solution $A = A(x,\gamma)$ of $GA = -\gamma^2 A$ $(0 \leqq x < \ell)$, subject to $A(0,\gamma) = 1$ and $A^-(0,\gamma) = 0$, put $A(\ell,\gamma) = A(\ell-,\gamma)$ if $m(\ell) > 0$, and notice that A is an even cosine-like function of γ. An even nonnegative mass distribution dW on the line is a *spectral weight* for the weighted string corresponding to m if

$$\int (1+\gamma^2)^{-1}\, dW < \infty$$

and if the (cosine) transform

$$f \to \hat{f}(\gamma) = \pi^{-\frac{1}{2}} \int_0^{\ell+} A(x,\gamma) f(x)\, dm(x)$$

provides a partial isometry between $\mathbf{Q} = \mathbf{L}^2(dm, [0,\ell])$ and $\mathbf{Z} = \mathbf{L}^2(dW, R^1)$ expressed by a Plancherel formula:

$$\|f\|_m^2 = \int_0^{\ell} |f|^2\, dm = \|\hat{f}\|_W^2 = \int |\hat{f}|^2\, dW$$

for any compact function $f \in \mathbf{Q}$.[11] W is a *special spectral weight* if this partial isometry extends to an isomorphism of $\mathbf{Q}$ onto the class of even functions belonging to $\mathbf{Z}$. By the Weyl–Kodaira eigendifferential expansion, as adapted to the present needs,[12] there is precisely one special spectral weight for each permissible choice of k, i.e., for each choice of G. Krein [17] turns this around and proves that every even nonnegative mass distribution dW subject to $\int (1+\gamma^2)^{-1}\, dW < \infty$ arises in this way as the special spectral weight of some string. The inverse spectral problem is to compute ℓ, m, and k from W. The present statement is in the nature of an existence proof only; for some hints on the actual computation of these quantities see Krein [18, 19], Levinson [21], and also the final section below.

Krein [17] actually proves much more: every non-special spectral weight of a weighted string is the special spectral weight of a longer string, i.e., a string with $\ell^* > \ell$ and $m^* = m$ for $x \leqq \ell$; esp., if either $\ell = \infty$ or $m[0,\ell) = \infty$, the string has only one spectral weight, to wit, its special spectral weight. A string for which W is a (non)-special spectral weight will be called a (short) long string for W.

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[10] The proof is made by imposing the condition $G \leqq 0$ upon the list of all merely self-adjoint candidates provided by W. Feller [9:491].

[11] A compact function is one which vanishes near $x = \ell$ if either $\ell = \infty$ or $m(\ell) = \infty$.

[12] See McKean [23].

12. Green functions. The connection between a weighted string and the associated special spectral weights W may be clarified as follows. Given an admissible $0 \leqq k \leqq \infty$ and a positive number α^2, the Green function $K_\alpha = K_\alpha(x, y)$ for $\alpha^2 - G$ can be expressed as an eigendifferential expansion by means of the cosines A and the special spectral weight associated with k:

$$K_\alpha(x, y) = \int \frac{A(x,\gamma)A(y,\gamma)}{\pi(\alpha^2+\gamma^2)} dW(\gamma)$$

when x and y are points of growth of m. At the same time, the classical Green function recipe provides us with a second expression for K_α. Choose the positive increasing solution $A = A(\bullet, i\alpha)$ of $GA = \alpha^2 A$ $(0 \leqq x < \ell)$ which meets the left-hand boundary condition, and a positive decreasing solution $D = D(\bullet, i\alpha)$ of $GD = \alpha^2 D$ which meets the right-hand boundary condition and is subject also to the normalization $D^-(0, i\alpha) = -1$. Then the Wronskian

$$A^+D - AD^+ = A^-D - AD^- \equiv 1, \qquad \text{and}$$

$$K_\alpha(x, y) = A(x, i\alpha)D(y, i\alpha)$$

for $x \leqq y$. This gives the formula

$$A(x, i\alpha)D(y, i\alpha) = \frac{1}{\pi}\int \frac{A(x,\gamma)A(y,\gamma)}{\alpha^2+\gamma^2} dW(\gamma)$$

for points $x \leqq y$ of growth of m, esp., you can evaluate this at $x = y = 0$:

$$D(0, i\alpha) = \frac{1}{\pi}\int (a^2+\gamma^2)^{-1} dW,$$

since $x = 0$ is a point of growth of m. An important formula is easily deduced from this. D may be expressed as

$$D(x, i\alpha) = D(0, i\alpha)A(x, i\alpha) - C(x, i\alpha)$$

where

$$C(x, i\alpha) = A(x, i\alpha) \textstyle\int_0^x [A(y, i\alpha)]^{-2} dy$$

is the solution of $GC = \alpha^2 C$ subject to $C(0, i\alpha) = 0$ and $C^-(0, i\alpha) = 1$. Therefore,

$$\frac{1}{\pi}\int (\alpha^2+\gamma^2)^{-1} dW = D(0, i\alpha) = \frac{C(\ell, i\alpha)+kC^+(\ell, i\alpha)}{A(\ell, i\alpha)+kA^+(\ell, i\alpha)} \text{ or } \lim_{x\to\ell} \frac{C(x, i\alpha)+kC^+(x, i\alpha)}{A(x, i\alpha)+kA^+(x, i\alpha)}$$

according as $m(\ell) > 0$ or not, esp., for $\alpha^2 \downarrow 0$, you find the formula

$$\ell + k = \frac{1}{\pi}\int \gamma^{-2} dW.$$

k need not be specified if $\int_0^\ell m\,dx = \int_0^{\ell-} x\,dm = \infty$, as the expression for $D(0)$ is independent of k in that circumstance.

Krein [17] presents a similar description of the non-special spectral weights W associated with the short string in case both ℓ and $m[0, \ell)$ are $< \infty$. The long string associated with $\ell^* > \ell$ and m^* for which W is a special spectral weight corresponding to some permissible $0 \leqq k^* \leqq \infty$ may be broken up into 3 pieces: the short string associated with ℓ and m, a possible (open) gap between ℓ and $\ell_* > \ell$ containing no $*$ mass, and an extension string corresponding to $\ell^\circ = \ell^* - \ell_*$ and the restriction m° of m^* to $\ell_* \leqq x \leqq \ell^*$. Krein's formula states that

$$\frac{1}{\pi}\int(\alpha^2+\gamma^2)^{-1}\,dW = \frac{C(\ell, i\alpha)+KC^+(\ell, i\alpha)}{A(\ell, i\alpha)+KA^+(\ell, i\alpha)},$$

in which

$$K = \ell_* - \ell - \frac{D(\ell_*, i\alpha)}{D^-(\ell_*, i\alpha)} = k^\circ + \frac{1}{\pi}\int(\alpha^2+\gamma^2)^{-1}\,dW^\circ.$$

$0 \leqq k^\circ = \ell_* - \ell$ is the length of the gap and W° is the special spectral weight for the extension string subject to $f + k^* f^+ = 0$ at its upper end, esp., every function K of the above general form appears in this way for some extension.

The special case of a string with lumps of mass loaded up on a series of points $0 = x_0 < x_1 < x_2 <$ etc. $\uparrow \ell \leqq \infty$ is explained in detail by Krein [17]; it is of special interest as it subsumes the classical investigations of Stieltjes into the problem of moments and the associated continued fraction expansions; see Section 16.

13. Fourier transforms. Krein's investigations actually go much deeper than the preceding account indicates.

Consider the weighted string associated with m on $0 \leqq x \leqq \ell \leqq \infty$, pick an admissible $0 \leqq k \leqq \infty$, and let W be the corresponding special spectral weight. Then, as stated above, the transform

$$f \in L^2(dm, [0, \ell]) \to \hat{f}_{\text{even}} = \pi^{-\frac{1}{2}} \int_0^{\ell+} A(x, \gamma) f(x)\, dm^{13}$$

is an isomorphism onto the class of even functions belonging to $\mathbf{Z} = \mathbf{L}^2(dW, R^1)$, and there is a Plancherel formula:

$$\|f\|_m^2 = \int_0^{\ell+} |f(x)|^2\, dm = \|\hat{f}\|_W^2 = \int |\hat{f}(\gamma)|^2\, dW.$$

The next step is to develop the transform based upon the function $B = B(x, \gamma)$ defined by $A^+ = -\gamma B$. Because A is something like a cosine and $A^+ = -\gamma B$ is reminiscent of the familiar $d(\cos \gamma t)/dt = -\gamma \sin \gamma t$, you may expect B to imitate the customary sine, esp., B is an odd function of γ and $dB = +\gamma A\, dm$, which should remind the reader of the familiar $d(\sin \gamma t)/dt = \gamma \cos \gamma t$. The similarity is actually very deep, as will become plain. To be precise, you can define a transform for odd functions from $\mathbf{Z}$ which stands in the same relation to the customary sine transform for $\mathbf{L}^2(d\gamma, R^1)$ as the even transform $f \to \pi^{-\frac{1}{2}} \int Af\, dm$ does to the customary cosine transform: namely, the map

$$f \in \mathbf{L}^2(dx, [0, \ell]) \to \hat{f}_{\text{odd}} = \pi^{-\frac{1}{2}} \int_0^\ell B(x, \gamma) f(x)\, dx$$

[13] As for the customary cosine transform in $\mathbf{L}^2(d\gamma, R^1)$, the integral $\int Af\, dm$ has to be treated with due caution.

is an isomorphism onto the class of odd functions belonging to **Z**, and there is a corresponding Plancherel formula:

$$\|f\|_x^2 = \int_0^{\ell} |f(x)|^2\,dx = \|\hat{f}\|_W^2 = \int |\hat{f}(\gamma)|^2\,dW,$$

provided that you understand by $f \in \mathbf{L}^2(dx, [0,\ell))$ *that* f *is constant on any interval in which* $dm = 0$. The associated inverse transform formulas are easily derived:

$$f(x) = \pi^{-\frac{1}{2}} \int A(x,\gamma)\hat{f}_{\text{even}}(\gamma)\,dW$$

for even functions belonging to **Z**, and

$$f(x) = \pi^{-\frac{1}{2}} \int B(x,\gamma)\hat{f}_{\text{odd}}(\gamma)\,dW$$

for odd functions belonging to **Z**. As simple examples, the reader will note the transforms

$$\gamma^{-1}(A(x,\gamma)-1) = \int_0^x B(y,\gamma)\,dy$$

and

$$\gamma^{-1}B(x,\gamma) = \int_0^{x+} A(y,\gamma)\,dm,$$

and the resulting Plancherel formulas which hold at growth points x of m:

$$\|\gamma^{-1}(A-1)\|_W^2 = \pi x, \qquad \|\gamma^{-1}B\|_W^2 = \pi m[0,x].$$

14. Integral subspaces. Krein [20] uses the odd/even transform pairs to investigate integral subspaces of **Z**, that is to say closed subspaces populated by functions which can be extended off the line into the whole complex plane so as to be integral (entire). The present section is devoted to a statement of these results of Krein, esp., to the counterpart of the classical Paley–Wiener theorem for $\mathbf{L}^2(d\gamma, R^1)$. This is applied presently to solve the problem of extrapolation (prediction) off a bounded segment of the past, following Krein [20]. After a brief discussion of de Branges' spaces of integral functions, it will be possible to go still more deeply into this subject.

Given a number $x \leqq \ell$, let $\mathbf{B}^{x+} = \mathbf{B}^{x+}(W)$ be the class of functions $f \in \mathbf{Z}(W)$ for which both of the inverse transforms

$$f^{\vee}_{\text{even}} = \pi^{-\frac{1}{2}} \int Af\,dW \quad \text{and} \quad f^{\vee}_{\text{odd}} = \pi^{-\frac{1}{2}} \int Bf\,dW$$

vanish outside the closed interval $[0,x]$, subject to the proviso that $x = \ell$ is allowed iff both ℓ and $m[0,\ell)$ are $<\infty$. $\mathbf{B}^{x-} = \mathbf{B}^{x-}(W)$ is defined similarly except that the inverse transforms vanish outside the half-closed interval $[0,x)$. $\mathbf{B}^{x+}$ differs from $\mathbf{B}^{x-}$ iff $m(x) > 0$, and then by multiples of $A(x,\bullet)$ only. Moreover, each such space **B** is closed in **Z**, and it is easy to prove that it is populated by integral functions of exponential type not larger than[14]

$$T = \int_0^x (m'(y))^{\frac{1}{2}}\,dy,$$

[14] m' stands for the density of the Lebesgue part of m.

as follows from the fact that both $A(x, \cdot)$ and $B(x, \cdot)$ are integral functions of the precise exponential type T. These are the integral subspaces figuring in the section title.

Krein [20] deduces several simple but very important facts from this type formula. To begin with, it is plain that the class $\mathbf{Z}^{T+}(W)$ of integral functions of exponential type $\leqq T < \infty$ (which includes $\mathbf{B}^{x+}$ whenever $\int_0^x (m'(y))^{\frac{1}{2}} dy \leqq T$) is dense in $\mathbf{Z}$ iff

$$\int_0^l (m'(y))^{\frac{1}{2}} dy \leqq T,$$

and that in the opposite case $[T < \int_0^l (m')^{\frac{1}{2}}]$, such functions fill up a closed subspace of $\mathbf{Z}$ which may be identified as the space $\mathbf{B}^{x+}$, $x = x(T+)$ being the biggest root of $T = \int_0^x (m')^{\frac{1}{2}}$. This is the counterpart of the classical Paley–Wiener theorem for $\mathbf{L}^2(dy, R^1)$. Notice that the class of integral functions of minimal exponential type $\mathbf{Z}^{0+}(W)$ is dense in $\mathbf{Z}(W)$ iff m is singular with regard to Lebesgue measure.

The smallest root $x = x(T-)$ of $T = \int_0^x (m')^{\frac{1}{2}}$ also plays an important role in connection with

$$\mathbf{Z}^T(W) = \text{the span in } \mathbf{Z}(W) \text{ of the functions } \gamma^{-1}(e^{i\gamma t} - 1) : |t| \leqq T.$$

Krein [20] states that functions of type $< T$ span out the whole of $\mathbf{Z}(W)$ iff

$$\text{either } T > \int_0^l (m')^{\frac{1}{2}} \quad \text{or } T = \int_0^l (m')^{\frac{1}{2}},\ x(T-) = l, \text{ and } m(l) = 0,$$

and that in the opposite case, this span is the same as $\mathbf{Z}^T(W)$ and can be identified as the integral subspace $\mathbf{B}^{x-}$ for $x = x(T-)$. By this and the preceding identification,

$$\mathbf{Z}^{T+}(W) = \bigcap_{R>T} \mathbf{Z}^R(W)$$

as the notation is meant to suggest.

In case W is non-singular with a Hardy density Δ, $\mathbf{Z}^{T+}(\Delta)$ is a *proper* closed subspace of $\mathbf{Z}(\Delta)$; see Levinson–McKean [22: 135]. By the preceding remarks $\int_0^l (m')^{\frac{1}{2}} = \infty$ is a necessary condition for this to happen.

The reader will notice that for $T \leqq \int_0^l (m')^{\frac{1}{2}}$, a gap exists between $x(T-)$ and $x(T+)$ iff m is singular on the intervening interval. This phenomenon is of the same nature as the gap between $\mathbf{Z}_\infty$ and $\mathbf{Z}^{0+}$ discussed before. The latter is reflected in the possible existence of an interval of singular mass following immediately upon the series of mass lumps which account for $\mathbf{Z}_\infty$, as described in Section 16.

15. Extrapolation off a bounded segment of the past: Krein's solution. The problem of extrapolation off a bounded segment of the past, posed in Section 10, can now be solved. Given $T < t$, the problem is to project the function $f = \exp(i\gamma t)$[15] upon the space $\mathbf{Z}^T(\Delta) = \mathbf{B}^{x-}(\Delta)$. Here Δ, the spectral density, is Hardy and *summable*, x stands for the smallest solution $x(T-)$ of $T = \int_0^x (m')^{\frac{1}{2}}$, and neither the situation

$$T > \int_0^l (m')^{\frac{1}{2}}$$

[15] Δ is summable, so $f \in \mathbf{L}^2(\Delta, R^1)$.

nor the situation

$$T = \int_0^{\ell} (m')^{\frac{1}{2}}, \qquad x(T-) = \ell, \qquad \text{and } m(\ell) > 0$$

prevails, as the projection is trivial otherwise. Krein's even and odd transforms are precisely the tools for this: the projection of

$$f = \pi^{-\frac{1}{2}} \int_0^{\ell+} A f^{\vee}_{\text{even}}\, dm + \pi^{-\frac{1}{2}} \int_0^{\ell} B f^{\vee}_{\text{odd}}\, dx$$

upon $\mathbf{B}^{x-}$ is simply

$$\pi^{-\frac{1}{2}} \int_{x<x(T-)} A f^{\vee}_{\text{even}}\, dm + \pi^{-\frac{1}{2}} \int_{x<x(T-)} B f^{\vee}_{\text{odd}}\, dx,$$

and the prediction error is the sum of 2 parts:

$$\int_{x \geqq x(T-)} |f^{\vee}_{\text{even}}|^2\, dm + \int_{x \geqq x(T-)} |f^{\vee}_{\text{odd}}|^2\, dx$$
$$\equiv e(+) + e(-) \equiv e.$$

The additional technical assumption $\int \gamma^6 \Delta < \infty$ makes it easy to check that

$$\frac{\partial^2 e}{\partial t^2} = \frac{\partial^2 e(+)}{\partial x\, \partial m} + \frac{\partial^2 e(-)}{\partial m\, \partial x},$$

a fact of mysterious significance.[16]

The above recipe can be expressed in terms of white noise integrals for $\mathfrak{x}$, much as in the classical problem of extrapolation off a half-line; see [8: 339].

16. The investigation of Stieltjes. The mass distribution m often begins with a number of isolated lumps accounting for the space $\mathbf{Z}_{\infty}$; for example, if $\int \gamma^4\, dW < \infty$,

$$\frac{1}{\pi} \int dW = m(0)^{-1},$$

$$\frac{1}{\pi} \int m(0)^2 \gamma^2\, dW = (x_1 - x_0)^{-1},$$

$$\frac{1}{\pi} \int [1 + \gamma^2 (x_1 - x_0) m(0)]^2\, dW = \frac{1}{m(x_1)},$$

and m places no mass between $x_0 = 0$ and x_1. The case investigated by Stieltjes corresponds to a spectral weight with $\int \gamma^{2n}\, dW < \infty$ for all $n = 1, 2, 3$, etc. This is the case in which m begins with positive lumps of mass placed upon a series of points $0 = x_0 < x_1 < x_2 < \text{etc.} \uparrow x_{\infty} \leqq \ell \leqq \infty$; it may serve as a model for the whole investigation of Krein as it evidently did for Krein himself. If ℓ and $m[0, \ell)$ are $< \infty$, polynomials are dense in $\mathbf{Z}$ iff $x_{\infty} = \ell$ and $m(\ell) = 0$.

Consider, for simplicity, only this case. As a function of x, A (B) is a broken-line (step function) with corners (steps) at the jumps of m. $A(x_n, \bullet)$, is an even poly-

[16] Dym–McKean [8: 339].

nomial of exact degree $2n$, while $B(x_n,\bullet)$ is an odd polynomial of exact degree $2n+1$, and for any of the associated spectral weights W,

$$\pi^{-1}\|A(x_n,\bullet)\|_W^2 = m(x_n)^{-1}, \qquad \pi^{-1}\|B(x_n,\bullet)\|_W^2 = (x_{n+1}-x_n)^{-1},$$

for $n = 0, 1, 2$, etc.; indeed, at the jumps of $m, A\,(B)$ coincides with the even (odd) orthogonal polynomial for W of the appropriate degree. W can therefore be thought of as the solution of a Stieltjes' moment problem, and the most general solution can be pictured as coming from a longer string. The classical facts about Stieltjes' problem follow easily; for example, if $l < \infty$ and $k = 0$, the Stieltjes transform

$$\pi^{-1}\int(\alpha^2+\gamma^2)^{-1}\,dW$$

of the corresponding special spectral weight W can be expressed as an infinite continued fraction:

$$\frac{1\,|}{|\alpha^2 m(0)}+\frac{1\,|}{|x_1-x_0}+\frac{1\,|}{|\alpha^2 m(x_1)}+\frac{1\,|}{|x_2-x_1}+\frac{1\,|}{|\alpha^2 m(x_2)}+\text{etc.}$$

17. De Branges spaces. To probe more deeply into the spaces $\mathbf{B}^x$, you have to know something about Hilbert spaces of integral functions as introduced and extensively studied by L. de Branges. [6] contains the most up-to-date information on the subject; for proofs of the more elementary material below, [7] and [8] can also be consulted.

For the present purposes, a de Branges space is a class of integral functions f described in terms of an integral function E of exponential type $<\infty$ satisfying the following conditions:

(a) $E(0) = 1$.

(b) $E^\#(\gamma) = E(-\gamma)$.[17]

(c) $|E| > |E^\#|$ on the open upper half-plane.

(d) E is root-free on the line (and so in the closed upper half-plane in view of the line above).

(e) $\int(1+\gamma^2)^{-1}\lg^+|E| < \infty$.[18]

The associated de Branges space $\mathbf{B} = \mathbf{B}(E)$ is the class of integral functions f satisfying

$$\|f\|^2 = \int|f|^2|E|^{-2} < \infty,$$

together with a growth condition to the effect that $|f(\omega)|^2$ is bounded by a constant multiple of

$$\frac{|E(\omega)|^2-|E^\#(\omega)|^2}{4\pi b} \qquad \text{if } b \neq 0,$$

$$\pi^{-1}|E(\omega)|^2\,\theta' \qquad \text{if } b = 0,$$

[17] $E^\#(\gamma)$ stands for $E(\gamma^*)^*$.

[18] $\lg^+ x$ means $\lg x$ if $x \geqq 1$ and 0 otherwise.

in which b is the imaginary part of ω, and θ is minus the phase of E. The constant multiplier figuring in the growth conditions may be replaced by $\|f\|^2$, this fact being the key to the proof that $\mathbf{B}$ is complete and therefore a Hilbert space. The correspondence between $\mathbf{B}$ and E is 1:1, though it is noted for future use that if the condition $E(0) = 1$ is abandoned, then $\mathbf{B}(E) = \mathbf{B}(F)$ (norm and all) iff the odd and even parts of E and F are related according to the recipe:

$$F_{\text{even}} = kE_{\text{even}}, \qquad F_{\text{odd}} = k^{-1}E_{\text{odd}}$$

for some real number $k \neq 0$.[19]

Because of the growth condition, the point evaluation $f \to f(\omega)$ is a continuous application of $\mathbf{B}$ into the complex numbers. This means that $\mathbf{B}$ is endowed with a so-called reproducing kernel $J = J(\alpha, \beta) = J_\alpha(\beta)$ satisfying

(a) $J_\omega \in \mathbf{B}$,
(b) $f(\omega) = (f, J_\omega) = \int f J_\omega{}^* |E|^{-2}$,

for every complex number ω and every $f \in \mathbf{B}$. The explicit formula is

$$J(\alpha, \beta) = \frac{E^*(\alpha)E(\beta) - E(\alpha^*)E^\#(\beta)}{-2\pi i(\beta - \alpha^*)}$$

The reader may verify that for any non-real complex number ω,

$$J(\omega, \omega) = \frac{|E(\omega)|^2 - |E^\#(\omega)|^2}{4\pi b} = \sup\left[|f(\omega)|^2 : f \in \mathbf{B}, \|f\| \leqq 1\right].$$

A very striking fact about such spaces is the inclusion principle of de Branges [5 (1962): 44] which states that for any 2 spaces $\mathbf{B}_1$ and $\mathbf{B}_2$ sitting isometrically inside some $\mathbf{Z}(W) \equiv \mathbf{L}^2(dW, R^1)$, either $\mathbf{B}_1 \subset \mathbf{B}_2$ or $\mathbf{B}_2 \subset \mathbf{B}_1$, esp., the family of all such spaces $\mathbf{B}$ which sit (isometrically) inside a fixed space $\mathbf{Z}$ forms a tower under isometrical inclusion.

A simple example $[dW = d\gamma]$ described in the de Branges language will clarify the above. Bring in the functions $E^T = \exp(-i\gamma T)$ for $T > 0$ and the corresponding de Branges spaces $\mathbf{B}(E^T)$. Clearly $\mathbf{B}(E^T)$ is isometrically included in $\mathbf{Z} = \mathbf{L}^2(d\gamma, R^1)$, and since

$$J_\alpha(\beta) = \frac{e^{i\alpha^* T}e^{-i\beta T} - e^{-i\alpha^* T}e^{i\beta T}}{-2\pi i(\beta - \alpha^*)} = \frac{1}{2\pi}\int_{-T}^{T} e^{i(\alpha^* - \beta)t}\,dt,$$

it follows that $f \in \mathbf{Z}$ belongs to $\mathbf{B}(\exp(-i\gamma T))$ iff

$$f(\alpha) = (f, J_\alpha) = (2\pi)^{-\frac{1}{2}}\int_{-T}^{T}\left[(2\pi)^{-\frac{1}{2}}\int f(\beta)\,e^{i\beta t}\,d\beta\right]e^{-i\alpha t}\,dt$$

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[19] To check this, note that the formula for the reproducing kernel $J(\alpha, \beta)$ given below is unaffected by this transformation.

for every complex number α; in brief, $\mathbf{B}(\exp(-i\gamma T))$ can be identified with $\mathbf{L}^2(d\gamma, [-T, T])$ via the classical Fourier transform. By the evaluation,

$$J(\omega, \omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{2bt}\, dt = \frac{\sinh 2bT}{2\pi b}$$

for $\omega = a + ib$, every $f \in \mathbf{B}(\exp(-i\gamma T))$ satisfies

$$|f(\omega)|^2 \leqq \|f\|^2 \frac{\sinh 2bT}{2\pi b},$$

and you can think of the above as a variant of the classical Paley–Wiener theorem. Notice that $\bigcap_{T>0} \mathbf{B}(\exp(-i\gamma T)) = 0$, while $\bigcup_{T>0} \mathbf{B}(\exp(-i\gamma T))$ is dense in $\mathbf{Z}$, and therefore, by de Branges' inclusion principle, $\mathbf{B}(\exp(-i\gamma T))\colon T > 0$ is a complete list of all the de Branges' subspaces of $\mathbf{Z}$. Notice also that this scheme fits into Krein's plan with:

$$l = \infty, \qquad dm = dx, \qquad T = \textstyle\int_0^x (m'(y))^{\frac{1}{2}}\, dy = x$$

$$A(x, \gamma) = \cos \gamma x, \qquad B(x, \gamma) = \sin \gamma x,$$

permitting us to identify $\mathbf{B}(\exp(-i\gamma T))$ with Krein's space $\mathbf{B}^x$. The general connection between de Branges spaces and Krein's spaces is explained in the next section.

18. De Branges subspaces of $\mathbf{Z}(W)$. A bird's eye view of the de Branges subspaces isometrically included in $\mathbf{Z} = \mathbf{L}^2(dW, R^1)$ may now be obtained in case W is the special spectral weight of a weighted string. The discussion is adapted (in part) from Dym–McKean [8: 314–319].

The chief point is that $\mathbf{B}^{x+}$ is the de Branges space $\mathbf{B}(E)$ based upon the function $E = A(x, \cdot) - iB(x, \cdot)$ for every growth point x of m. To begin with, it is plain from the Plancherel formula that

$$J_\alpha(\beta) = \pi^{-1} \textstyle\int_0^{x+} A^*(\alpha, y) A(\beta, y)\, dm + \pi^{-1} \int_0^x B^*(\alpha, y) B(\beta, y)\, dy$$

$$= \frac{A^*(x, \alpha) B(x, \beta) - B^*(x, \alpha) A(x, \beta)}{\pi(\beta - \alpha^*)}$$

$$= \frac{E^*(\alpha) E(\beta) - E(\alpha^*) E^\#(\beta)}{-2\pi i(\beta - \alpha^*)}$$

is a reproducing kernel for $\mathbf{B}^{x+}$. This means that

$$f(\omega) = (f, J_\omega)_W$$

for $f \in \mathbf{B}^{x+}$. The conditions on E laid down at the beginning of the last section can now be verified, and from the fact that J is likewise the reproducing kernel of $\mathbf{B}(E)$ follows the identification of the 2 spaces (norms and all). A necessary step in this verification is to check

$$\textstyle\int (1 + \gamma^2)^{-1} |E|^{-2} < \infty;$$

see, for instance, Dym [7: (5.5)]. A side benefit is that $|E|^{-2}$ now appears as a (non-special) spectral weight for the short string associated with m to the left of $x+$. The long string, of which it is the special spectral weight, is obtained by placing the weight $dm^* = dx$ between $x+$ and $l^* = \infty$.

An application of Nevanlinna's representation formula[20] gives

$$\log|E(a+ib)| = \frac{b}{\pi}\int \frac{\log|E(x)|}{(x-a)^2+b^2}\,dx + bT$$

for $b > 0$, where T is the type of E. It follows that the function

$$h = e^{-i\gamma T}(1-i\gamma)^{-1}E^{-1}$$

is an outer function belonging to the Hardy class $\mathbf{H}^{2+}$. The factor $(1-i\gamma)^{-1}$ appearing in h is put in merely to insure that h belongs to $\mathbf{L}^2(d\gamma, R^1)$; if

$$\int |E|^{-2} = \pi m(0)^{-1} = \int dW < \infty,$$

it can be dropped.

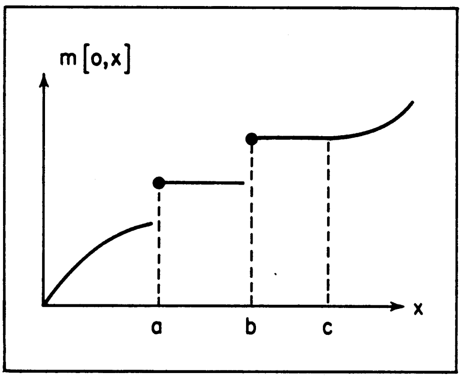

FIG. 1.

The same discussion applies to $\mathbf{B}^{x-}$ with the proviso that $B(x,\bullet) = B(x+,\bullet)$ has to be replaced by $B(x-,\bullet)$ throughout, and it follows easily from de Branges' inclusion principle that $\mathbf{B}^{x\pm}: x \leqq l$ is a complete list of all the de Branges spaces isometrically included in $\mathbf{Z}$.

To illustrate how complicated the picture can be, consider for example the mass distribution sketched in Figure 1.

In this situation $(m')^{\frac{1}{2}} = 0$ on the interval $[a, c]$, so there is no change of the type $T = \int_0^x (m')^{\frac{1}{2}}$ between a and c, although a, b and c are points of growth of m and all the inclusions

$$\mathbf{Z}^T = \mathbf{B}^{a-} \subset \mathbf{B}^{a+} \subset \mathbf{B}^{b-} \subset \mathbf{B}^{b+} \subset \mathbf{B}^{c-} = \mathbf{B}^{c+} = \mathbf{Z}^{T+}$$

[20] Boas [4:92].

are proper. The point is that the Krein spaces allow a finer decomposition of $\mathbf{Z}$ than is possible if one simply restricts attention to the spaces $\mathbf{Z}^T$ and $\mathbf{Z}^{T+}$. The reader can readily envision the complexity of the gap between $\mathbf{Z}^T$ and $\mathbf{Z}^{T+}$ which can arise upon allowing arbitrary singular mass distributions on the interval $[a, c]$ in place of the 2 jumps dealt with above. This circle of ideas is closely related to the discussion of germ and gap in Section 9, but unlike the gap between $\mathbf{Z}_\infty$ and $\mathbf{Z}^{0+}$, the gap between $\mathbf{Z}^T$ and $\mathbf{Z}^{T+}$ for positive T, may be finite- or infinite-dimensional.

Dym–McKean [8] discussed the inverse spectral problem for summable Hardy weights under the technical assumption that m is not singular on any subinterval of $0 \leqq x \leqq l$. The present discussion shows how far short of the general case this falls: for example, if you take *any* mass distribution m on $0 < x < l < \infty$, add a mass $m(0) = \pi^{-1}$ at $x = 0$, and extend by putting $dm = dx$ for $l \leqq x < \infty$, then $\Delta = |E(l, \bullet)|^{-2}$ is a summable Hardy weight; see Section 20 below.

19. Short strings and the sampling formula. Besides the weights W and $|E|^{-2}$, the short string associated with m to the left of $x+$ has its own special spectral weights corresponding to the choice of the number $0 \leqq k \leqq \infty$. Both x and $m[0, x]$ are $< \infty$, so each such weight W consists of an (even) series of mass lumps placed upon the roots

$$\omega = \pm\gamma_0, \pm\gamma_1, \pm\gamma_2, \text{etc.} \to \pm\infty$$

of $A(x, \omega) + kA^+(x, \omega) = 0$. This state of affairs reflects the fact that the differential operator G associated with the boundary condition $f(x) + kf^+(x) = 0$ has a simple point spectrum

$$0 \geqq -\gamma_0^2 > -\gamma_1^2 > -\gamma_2^2 > \text{etc.} \downarrow -\infty.$$

The corresponding Plancherel formula is easily computed:

$$\|f^\vee_{\text{even}}\|_m^2 + \|f^\vee_{\text{odd}}\|_x^2 = \|f\|_W^2 = \int |f|^2 \, dW = \sum |f(\omega)|^2 J(\omega, \omega)^{-1}.$$

This is the sampling formula referred to in the section title. The same formula can be found in de Branges [6: 55] in a slightly different language. The classical version (corresponding to $dm = dx$ and $k = \infty$) states that for integral functions f of exponential type $\leqq T$ belonging to $\mathbf{L}^2(d\gamma, R^1)$,

$$\|f\|^2 = \int |f|^2 = (\pi/T) \sum_{n=-\infty}^{\infty} |f(n\pi/T)|^2.$$

As in this case, the general series $0 \leqq \gamma_0 < \gamma_1 < \gamma_2 <$ etc. is (close to) arithmetic:

$$\gamma_n = n\pi[T^{-1} + o(1)] \quad \text{for} \quad n \uparrow \infty \quad \text{and} \quad T = \int_0^x (m')^{\frac{1}{2}}.$$

The reader may be mystified upon reflecting that $f \in \mathbf{L}^2(dW, R^1)$, which is merely a series of numbers $f(\omega)$, is supposed to be an integral function of exponential type $\leqq T$, but all this means is that the condition $\|f\| < \infty$ implies that the series $f(\omega)$ can be uniquely *interpolated* by such a function.

20. Hardy weights. Let $\Delta\, d\gamma$ be the Lebesgue part of dW, as usual. The purpose of this section is to investigate the connection between the Hardy condition

$$\int(1+\gamma^2)^{-1}\lg\Delta > -\infty$$

and the mass distribution m of the associated long string for which W is the special spectral weight. By Levinson–McKean [22: 135] or Dym–McKean [8: 319–320]. $x(T+) < \mathfrak{l}$ for any type $T < \infty$ whenever Δ is Hardy, since $\mathbf{Z}^{T+}(W)$ is then a proper subspace of $\mathbf{Z}(W)$. But this means that

$$\infty = \int_0^{\mathfrak{l}}(m'(y))^{\frac{1}{2}}\,dy \leqq \mathfrak{l}^{\frac{1}{2}}\,m[0,\mathfrak{l})^{\frac{1}{2}},$$

esp., either $\mathfrak{l} = \infty$ or $m[0,\mathfrak{l}) = \infty$, and the string has precisely one spectral weight. This will be assumed for the rest of the present section.

The principal conclusion of this section is that Δ is Hardy iff

$$Q = \int_0^{\mathfrak{l}}[k^2\,dx + k^{-2}\,dm - 2(m')^{\frac{1}{2}}\,dx] < \infty$$

in which $k^2 = -D^+/D$ and $D = D(x,i)$ is the positive decreasing solution of $GD = D$ with $D^-(0) \equiv -1$ introduced in connection with the Green function. Notice that

$$k^2\,dx + k^{-2}m'\,dx - 2(m')^{\frac{1}{2}}\,dx = (k - k^{-1}(m')^{\frac{1}{2}})^2\,dx \geqq 0,$$

so that the integrand of Q is nonnegative. A brief sketch of the proof is presented below as it clarifies several interesting points.

Define $e = e(x,\gamma) = kA - ik^{-1}B$ for fixed $x < \mathfrak{l}$. e defines the same de Branges space $\mathbf{B}(e) = \mathbf{B}^{x+}$ as $E = A - iB$ for any real number $k \neq 0$. The special choice of k made above is the only one for which both

$$\int|e|^2\,dW^\bullet = 1 \qquad \text{and}$$

$$\int\frac{|e|^{-2}\,d\gamma}{\pi(1+\gamma^2)} = \int dW^\bullet$$

Here, $dW^\bullet$ stands for the jacked-up weight $\pi(1+\gamma^2)^{-1}dW$. This choice has the additional advantage (of which the formula just above is an instance) that $\pi^{\frac{1}{2}}(1-i\gamma)e$ stands in the same relation to $W^\bullet$ as e does to W; namely, for $0 \leqq x < \mathfrak{l}$, $\mathbf{B}[\pi^{\frac{1}{2}}(1-i\gamma)\,e(x\pm,\gamma)]$ runs through a complete list of the Krein subspaces of $\mathbf{Z}(W^\bullet) = \mathbf{L}^2(dW^\bullet, R^1)$. This fortunate circumstance leads to the evaluation

$$\int|e^{i\gamma T(x_2)}\,e(x_2,\gamma) - e^{i\gamma T(x_1)}\,e(x_1,\gamma)|^2\,dW^\bullet = 2\left[1 - \frac{e^{-T(x_1)}\,e(x_1,i)}{e^{-T(x_2)}\,e(x_2,i)}\right]$$

for $x_2 > x_1$, esp., $\exp(-T(x))\,e(x,i)$ is an increasing function of x, and hence

$$f = \exp[i\gamma T(x)]\,e(x,\gamma)$$

converges in $\mathbf{Z}(W^\bullet)$ as $x \uparrow l$ iff $\exp(-T(x))e(x,i)$ tends to a finite limit as $x \uparrow l$, or equivalently iff

$$\lim_{x\to l} \lg[e^{-2T(x)}e(x,i)^2] = \lg e(0-,i)^2 + \int_0^{l-} d\lg[e^{-2T(x)}e(x,i)^2]$$
$$= -\lg D(0) + \int_{\text{non-jumps of } m}[k^2\,dx + k^{-2}\,dm - 2(m')^{\frac{1}{2}}\,dx]$$
$$+ \sum\nolimits_{\text{jumps of } m} \lg(1+k^{-2}m) < \infty,$$

and this is the same as to say that $Q < \infty$ in view of the elementary estimate

$$x/2 \leqq \lg(1+x) \leqq x \quad \text{for small} \quad x \geqq 0.$$

The actual limit of f^{-1} is now identified in the convergent case as the outer Hardy function $h \in (1-i\gamma)\mathbf{H}^{2+}$ that figures in the factorization $\Delta = |h|^2$, leading to the final formula:

$$\int_{\text{non-jumps of } m}[k^2\,dx + k^{-2}\,dm - 2(m')^{\frac{1}{2}}\,dx] + \sum\nolimits_{\text{jumps of } m}\lg(1+k^{-2}m)$$
$$= \lg\int\frac{dW}{\pi(1+\gamma^2)} - \int\frac{\lg\Delta\,d\gamma}{\pi(1+\gamma^2)}.$$

A simple example is provided by the scale $dx = (T+1)^2\,dT$ and the loading $dm = (T+1)^2\,dT$ for $0 \leqq T < \infty$. The reader is invited to verify that

$$A(T,\gamma) = (T+1)\cos\gamma T - \gamma^{-1}\sin\gamma T, \qquad B(T,\gamma) = (T+1)^{-1}\sin\gamma T,$$
$$dW = d\gamma + \text{a mass of magnitude } \pi \text{ placed at } 0.$$
$$k^{-1} = [(T+1)(T+2)]^{\frac{1}{2}},$$
$$\lim_{x\to l} e^{i\gamma T(x)}e(x,\gamma) = 1 \quad \text{if} \quad \gamma \neq 0,$$
$$= 0 \quad \text{if} \quad \gamma = 0,$$

and

$$Q = \int_0^\infty \frac{dT}{(T+1)(T+2)} = \lg 2 = \lg\int\frac{dW}{\pi(1+\gamma^2)} - \int\frac{\lg 1}{\pi(1+\gamma^2)}\,d\gamma.$$

The reader should note the formula of Dym–McKean [8: 315] for the function e in case W is non-singular with a Hardy density $\Delta = |h|^2$: for any $0 \leqq T < \infty$ and $x = x(T+)$,

> e = the projection in $\mathbf{Z}(W^\bullet)$ of $[\exp(i\gamma T)h]^{-1}$ upon the class of integral functions of exponential type $\leqq T$, divided by the $W^\bullet$ norm of this projection.

21. De Branges subspaces of $\mathbf{Z}(W^\bullet)$. The function e is of special interest in view of the fact, cited above under the special assumption $\int_0^l (m')^{\frac{1}{2}} = \infty$ that for $0 \leqq x < l$, $\mathbf{B}[\pi^{\frac{1}{2}}(1-i\gamma)e(x\pm,\gamma)]$ runs through the complete list of the Krein subspaces of $\mathbf{Z}(W^\bullet)$. The corresponding string may be identified by noticing that

$$\pi^{\frac{1}{2}}(1-i\gamma)\,e = \pi^{\frac{1}{2}}(kA - \gamma k^{-1}B) - i\pi^{\frac{1}{2}}(k^{-1}B + \gamma kA)$$

defines the same de Branges space as

$$(A-\gamma k^{-2}B)-i\pi(B+\gamma k^2 A) = A^\bullet - iB^\bullet$$

and checking the formulas

$$dA^\bullet = -\gamma B^\bullet\, dx^\bullet, \qquad dB^\bullet = +\gamma A^\bullet\, dm^\bullet$$

in which the new scale $x^\bullet$ is defined by $dx^\bullet = \pi^{-1}k^{-4}\,dm$[21] and the new mass distribution $m^\bullet$ by $dm^\bullet = \pi k^4\, dx$.

The new scale $dx^\bullet$ may jump and it may have flat stretches, so you have to modify things a little to conform to Krein's scheme. A jump is to be pictured as a $\bullet$-mass-free interval on which $B^\bullet$ is constant and $A^\bullet$ is of constant slope. A flat stretch is to be collapsed to a point and pictured as a jump of $m^\bullet$ at which $A^\bullet$ has a corner and $B^\bullet$ jumps. To make the (left $\bullet$) slope of $A^\bullet$ vanish at $x = 0$, you have to place there an extra $\bullet$-mass

$$m^\bullet(0) = \pi k^2(0-) = \pi D(0)^{-1},$$

in agreement with the general formula $\int dW = \pi m(0)^{-1}$:

$$\int dW^\bullet = D(0) = k^{-2}(0-).$$

At $x = \ell$, the situation is more complicated. To begin with, the (possible) jump of the $\bullet$-scale at $x = \ell$ is pictured, in agreement with the above recipe, as a $\bullet$-mass-free interval and so does not contribute to the actual length of the $\bullet$ string:

$$\ell^\bullet = \int_0^{\ell-} dx^\bullet = \int_0^{\ell-} \pi^{-1}k^{-4}\, dm.$$

The recipe also says that you should count the total mass of $dm^\bullet = \pi k^4\, dx$ placed between $x = \ell$ and the last preceding point of growth of m as a lump placed at the end of the $\bullet$ string (such a lump can only be $< \infty$). A boundary condition is imposed at the end iff both

$$\ell^\bullet = \int_0^{\ell-} \pi^{-1}k^{-4}\,dm = \pi^{-1}\int_0^{\ell-} (dk^{-2}+dx) \qquad \text{and}$$

$$m^\bullet[0,\ell^\bullet] = \int_0^{\ell} \pi k^4 dx = \pi\int_0^{\ell}(dk^2+dm)$$

are $< \infty$, and this happens only if $\ell < \infty$, $m[0,\ell) < \infty$ and $0 < k^{-2}(\ell+) < \infty$. The appearance of $k^{-2}(\ell+)$ has to do with the fact that $D(\ell)+k^{-2}(\ell+)D^+(\ell) = 0$, to wit, $k^{-2}(\ell+)$ is the old number k which specifies the boundary condition for the string having W as its special spectral weight. The actual boundary condition for the $\bullet$ string is specified by the number

$$0 \leqq k^\bullet = \pi^{-1}[k^{-2}(\ell+)-k^{-2}(\ell-)] \leqq \infty,$$

[21] $k^{-4}dm$ stands for $[k(x+)k(x-)]^{-2}dm$ at a jump of m. The formulas $dk^2+dm = k^4\,dx$ and $dk^{-2}+dx = k^{-4}\,dm$ are helpful for the verification.

in agreement with the general formula $\int \gamma^{-2}\,dW = \pi(\ell+k)$:

$$\begin{aligned}\int \gamma^{-2}\,dW^{\bullet} &= \int \frac{dW}{\pi\gamma^2} - \int \frac{dW}{\pi(1+\gamma^2)} \\ &= \ell + k^{-2}(\ell+) - k^{-2}(0-) \\ &= \int_{0-}^{\ell+} (dx + dk^{-2}) \\ &= \pi \int_{0-}^{\ell+} dx^{\bullet} \\ &= \pi\ell^{\bullet} + [k^{-2}(\ell+) - k^{-2}(\ell-)].\end{aligned}$$

The remarkable feature of this development is that, aside from the factors $(\pi k^4)^{\pm 1}$, the roles of x and m are reversed.

22. A fresh look at the interpolation problem. At this stage it is possible to take a much deeper look at the interpolation problem, as advertised in Section 6. Recall that the problem is to project $f = \exp(i\gamma t)$ for fixed $|t| < T$ upon the subspace

$$^{T}\mathbf{Z}(\Delta) = \text{the span in } \mathbf{Z}(\Delta) \text{ of the functions } e^{i\gamma t}\colon |t| \geqq T,$$

naturally, the spectral density Δ is assumed to be even, summable, and Hardy, so that $\Delta = |h|^2$ with an outer function $h \in \mathbf{H}^{2+}$ subject to $h^*(\gamma) = h(-\gamma)$ when γ is real. Also, it suffices to compute the co-projection, i.e., the projection upon the annihilator $(^{T}\mathbf{Z})^{\circ}$.

The key to the whole problem is to notice that $\Delta(^{T}\mathbf{Z})^{\circ}$ can be identified as the class $\mathbf{Z}^{T+}(\Delta^{-1})$ of integral functions of exponential type $\leqq T$ belonging to $\mathbf{Z}(\Delta^{-1})$. The point is that $f \in (^{T}\mathbf{Z})^{\circ}$ iff $f\Delta$ belongs to $\mathbf{Z}(\Delta^{-1}) \subset \mathbf{L}^1(d\gamma, R^1)$ and can be extended off the line to an integral function of exponential type $\leqq T$ as the reader can easily check using the standard Paley–Wiener theorem for $\mathbf{L}^1(d\gamma, R^1)$. The condition for perfect interpolation mentioned earlier, namely, that $\mathbf{Z}^{T+}(\Delta^{-1}) = 0$, should now be clear. $\mathbf{Z}^{0+}(\Delta^{-1}) = 0$, and you can prove much as in Dym–McKean [8: 319–320] that for $T > 0$, $\mathbf{Z}^{T+}(\Delta^{-1})$ is the de Branges space based upon the function

> $e =$ the projection in $\mathbf{Z}[(\Delta^{-1})^{\bullet}]$ of $\exp(-i\gamma T)h$ upon the class of integral functions of exponential type $\leqq T$, divided by the $(\Delta^{-1})^{\bullet}$ norm of this projection.

This suggests that you should look for a connection with strings as in the inverse spectral problem, though the situation is necessarily more complicated. To begin with

$$\infty = \left(\int (1+\gamma^2)^{-\frac{1}{2}}\right)^2 \leqq \int \Delta \int [(1+\gamma^2)\Delta]^{-1}$$

so Δ^{-1} cannot be a spectral weight in Krein's sense. Additional discouragement is provided by the example $\Delta = \gamma^{-2}\sin^2\gamma$ for which $\mathbf{Z}^{T+}(\Delta^{-1}) = 0$ for $T \leqq 1$ and $\dim \mathbf{Z}^{T+}(\Delta^{-1}) = \infty$ for $T > 1$,[22] the chief point being that an integral function f

[22] L. Pitt [private communication].

of type <1 cannot have enough roots to make $\int |f|^2\Delta^{-1} < \infty$.[23] This is bad news, since in any picture similar to Krein's, the type differential should look like $dT = (m')^{\frac{1}{2}}\,dx$ so that T cannot suddenly jump up to $T = 1$.

A picture similar to Krein's can be verified using the methods of Dym–McKean [8] under additional technical assumptions; for example, it is enough to suppose that for every $T > 0$

(a) $\mathbf{Z}^{T+}(\Delta^{-1}) \neq 0$,
(b) e is root-free on the line,
(c) $\mathbf{Z}^{T+}(\Delta^{-1})$ is spanned by functions of type $<T$;

any rational weight which is root-free on the line meets these conditions. The corresponding string is extended between $x = -\infty$ and $x = l \leqq \infty$ and $\int_{-\infty}^{0} x\,dm$ is $> -\infty$ so that you have the usual boundary condition at the left $[f^+(-\infty) = 0]$, while $\int_0^l (m')^{\frac{1}{2}} = \infty$ which means that there is only one spectral weight. $T = \int_0^x (m')^{\frac{1}{2}}$ as usual and is nowhere flat. The relation between the present x and m and those for the string associated with Δ is not known. A simple example is provided by $\Delta = (1+\gamma^2)^{-1}$:

$$\frac{dx}{dT} = \frac{dT}{dm} = (2\pi)^{-1}\coth^2 T.$$

De Branges' methods [5, 6] should help to eliminate the conditions (a), (b), (c) above, but this lies in the future. An interesting problem is to find out what it means for h that the interpolation should be imperfect for every $T > 0$.

23. Effective solution of the inverse spectral problem. The following method adapted from Levinson [21] solves the inverse spectral problem in the neighborhood of a known solution, Levinson treated only weights close to $\Delta = 1[dx = dm = dT(0 \leqq T < \infty)]$. The basic idea is due to Gel'fand–Levitan [10] and has been extended by Krein in the course of his investigations [17–19]. The plan is to pick a spectral weight W° for which the corresponding (long) string with scale x°, mass distribution m°, and boundary condition number k° is known and to try to express the cosines A for the (long) string having a neighboring spectral weight W by means of the cosines A° for the $\circ$ string. More explicitly, the proposal is that

$$A = LA^\circ + KA^\circ$$

for an appropriate function $L = L(x)$ and (triangular) integral operator

$$K : f \to \int_0^x K(x, y) fy\,dm^\circ,$$

provided that W is sufficiently close to W°. The computations below are merely formal but can be justified under extra technical assumptions.

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[23] An alternative proof (including the case $T = 1$) is obtained from the sampling formula

$$\int |f|^2 = \pi \sum_{n=-\infty}^{\infty} |f(n\pi)|^2$$

for functions of type $\leqq 1$.

The relation $A = (L+K)A^\circ$ can be inverted to express $A^\circ(x, \gamma)$ as a superposition of the functions $A(y, \gamma)$ for $y \leqq x$. But $\int A(x, \gamma)\, A(y, \gamma)\, dW = 0$ for $x \neq y$, so the (formal) transform

$$\int A^\circ(y, \gamma) A(x, \gamma)\, dW$$

should also vanish for $x > y$, and substituting the expression $A = (L+K)A^\circ$, you find that

$$K + LQ + KQ = 0$$

for $x \geqq y$, in which Q is the symmetric operator

$$Q : f \to \int Q(x, y) f y\, dm^\circ$$

based upon the (known) kernel

$$Q(x, y) = \int A^\circ(x, \gamma) A^\circ(y, \gamma)(dW - dW^\circ).$$

The problem $f + Qf = 0$ is easily proved to have only the trivial solution on any subinterval, so K is completely determined in accordance with the Fredholm alternative, by the (known) kernel Q and the (unknown) function L.

At the same time, A is supposed to satisfy $dA^+ = -\gamma^2 A\, dm$, in which x and m characterize the string having W as special spectral weight. By (formal) differentiation of $A = (L+K)A^\circ$, you find that this can be guaranteed only if

(a) $dx = L^2\, dx^\circ$,
(b) $dm = L^{-2}\, dm^\circ$
(c) $d\left[\dfrac{dL + J\, dm^\circ}{dx}\right] + dJ\dfrac{dm^\circ}{dx} = 0$,

in which J is short for $K(x, x)$. L, x, and m can now be determined by means of these identities and the (known) relation between L and K.

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