The Annals of Mathematical Statistics
1971, Vol. 42, No. 1, 12–34

POWER BOUNDS AND ASYMPTOTIC MINIMAX RESULTS FOR ONE-SAMPLE RANK TESTS

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1. Introduction and summary. Let $X_1, \cdots, X_N$ be identically independently distributed with the common continuous distribution function F, and let $r_1 < r_2 < \cdots < r_S$ denote the ordered ranks of the positive X's among $|X_1|, \cdots, |X_N|$. Two problems are considered. The first is the location problem where the null hypothesis H_0 that F is symmetric about zero (or more generally $F(x) \geqq 1-F(-x)$ for all x) is tested against the alternative that X is stochastically larger than $-X$ (i.e. $F(x) \leqq 1-F(-x)$ for all x with strict inequality for some x). Departure from the null hypothesis will be measured in terms of the distribution difference $D(x) = [1-F(-x)]-F(x)$, i.e. the difference of the distributions of $-X$ and X. We let $\delta(F) = \sup_x |D(x)|$ denote the Kolmogorov distance between these distributions and consider the class $\Omega(\Delta) = \{F: D(x) \geqq 0 \text{ and } \delta(F) \geqq \Delta\}$ of one-sided alternatives with this distance at least Δ. Lower bounds on the power of monotone rank tests are given for F in $\Omega(\Delta)$ (Theorem 2.1), and similarly, upper bounds on the power of monotone rank tests are found for F in $\bar{\Omega}(\Delta) = \{F: \delta(F) \leqq \Delta\}$ (see Theorem 2.2, Corollary 2.2 and Corollary 2.3). $\Omega(\Delta)$ and $\bar{\Omega}(\Delta)$ are of interest for paired comparison and one-sample experiments in which the location of F is the main concern. Note that the above location alternatives are not necessarily shift alternatives.

The second problem considered is the symmetry problem where the hypothesis that F is symmetric about zero is to be tested against the alternative that it is skewed to the right. Lower (Theorem 3.1) and upper (Corollary 3.2) bounds on the power of monotone rank tests are found for F in $\Omega_s(\Delta) = \{F: F \in \Omega(\Delta) \text{ and } F \text{ has median } 0\}$ and $\bar{\Omega}_s(\Delta) = \{F: \delta(F) \leqq \Delta \text{ and } F \text{ has median } 0\}$, respectively.

Hoeffding (1951) and Ruist (1954) considered alternative classes of the type $\Gamma(q) = \{F: F \text{ is symmetric and } (F(0)-\frac{1}{2}) \leqq q\}$ and found that the Sign test is minimax (maximizes the minimum power, i.e., minimizes the maximum risk = (1-power)) for $\Gamma(q)$. For the location alternative, we consider the problem of maximizing the minimum power over $\Omega(\Delta)$ and find (Theorem 4.1) that for a class of statistics of the form $\sum_{i=1}^{S} J_N(r_i/(N+1))$, a solution is asymptotically given by the statistics $T(\Delta) = W + [(\frac{1}{2}\Delta(N-1)-1)/(N+1)]S$, where S is the number of positive X's and where $W = \sum_{i=1}^{S} r_i/(N+1)$ is the one-sample Wilcoxon statistic. This result contrasts with the two-sample result of [6] in which the Wilcoxon statistic is asymptotically the uniformly (in Δ) unique minimax solution. For the symmetry problem, a class of statistics of the form $\sum_{i=1}^{S} J(r_i/(N+1))$ is considered,

Received March 25, 1969; revised September 3, 1970.

[1] Prepared with the partial support of the National Science Foundation, Grant GP-7454.

[2] Prepared with the partial support of the Office of Naval Research under Contract Nonr-241. Contribution No. 2303 from Woods Hole Oceanographic Institution.

and it is shown (Theorem 4.2) that $V = W - \frac{1}{2}S$ asymptotically maximizes the minimum power over $\Omega_s(\Delta)$.

$T(\Delta)$ and V are functions of the one-sample Wilcoxon statistic W and the Sign statistic S. Such statistics have also been considered by Ruist (1954). See also Hodges and Lehmann (1962, page 495). V has been considered by Gross (1966, page 76) and is asymptotically equivalent to the statistic considered by Gupta (1967).

The power bounds are tabulated exactly or estimated using Monte Carlo methods for sample sizes 10 and 20 in Section 2B. In particular, the minimum power over $\Omega(\Delta)$ of the statistics W, $T^{(1)} = T(\Delta_1)$ and $T^{(2)} = T(\Delta_2)$, where $\Delta_1 = (1.645/(3N)^{\frac{1}{2}})^{\frac{1}{2}}$ and $\Delta_2 = (2.326/(3N)^{\frac{1}{2}})^{\frac{1}{2}}$, is given or estimated using Monte Carlo power methods. These choices of Δ are discussed in Section 4, Remark 4.3. The results show that for $N = 10$, $T^{(1)}$ and $T^{(2)}$ do not improve on the minimum power of the Wilcoxon statistic W, while for $\alpha = .05$ and $N = 20$, the asymptotic results are in effect in the sense that $T^{(1)}$ and $T^{(2)}$ are improvements on W.

The statistic V is similarly compared with the statistic [Gross (1966)] $T_G = \sum_{i=1}^{S} r_i^2/(N+1)^2 - (\frac{1}{3})S$ and it is found (Section 3) that in terms of minimum power over $\Omega_s(\Delta)$, V is much better than T_G already for sample sizes $N = 10$ and 20.

Power bounds similar to the ones obtained in this paper have been obtained by Birnbaum (1953) and Chapman (1958) for the goodness-of-fit problem, by Bell, Moser and Thompson (1966) for the two-sample problem, and by Bell and Doksum (1967) for the independence problem.

In Section 5, tables of the null distributions of $T^{(1)}$, V and T_G are given for $N \leqq 10$, and the Monte Carlo powers of $T^{(1)}$, $T^{(2)}$ and W are compared for normal, double exponential and logistic shift alternatives for $N = 10$ and 20. For the double exponential distribution, $T^{(1)}$ and $T^{(2)}$ are slightly better than W when $N = 20$. For the normal distribution, W is slightly better than $T^{(1)}$ and $T^{(2)}$. Thus $T^{(1)}$ and $T^{(2)}$ appear to be better than W for "heavy" tail shift alternatives, while the opposite holds for "light" tail shift alternatives. Although there is essentially no difference in the power of $T^{(1)}$ and $T^{(2)}$ for the various models considered, $T^{(1)}$ is recommended because the normal approximation to the rejection limits of $(N+1)T^{(1)} = (N+1)W + [.487(N-1)N^{-\frac{1}{4}} - 1]S$ are closer to the true limits.

2. Power bounds for monotone tests in the location problem.

2A. *Theoretical results.* The alternative classes of distributions and the tests considered in this section are both of interest in the paired comparison problem in which the sample $X_1, \cdots, X_N$ consists of differences and under the null-hypothesis the distribution of the X's is symmetric about zero (or more generally, X is stochastically no larger than $-X$), while under the alternative X is stochastically larger than $-X$. It will be shown in this section that (i) for rank tests, the distributions $F_{\Delta,a}$ of Figures 1 and 2 below are least favorable for the location problem and (ii) upper bounds on the power of rank tests for the location problem are given by the formulas in Corollaries 2.2 and 2.3.

A test (function) $\varphi = \varphi(x_1, \cdots, x_N)$ is said to be *monotone* if $\varphi(x_1', \cdots, x_N') \leqq \varphi(x_1, \cdots, x_N)$ whenever $x_i' \leqq x_i$, $i = 1, \cdots, N$. Thus the test that rejects H_0 when

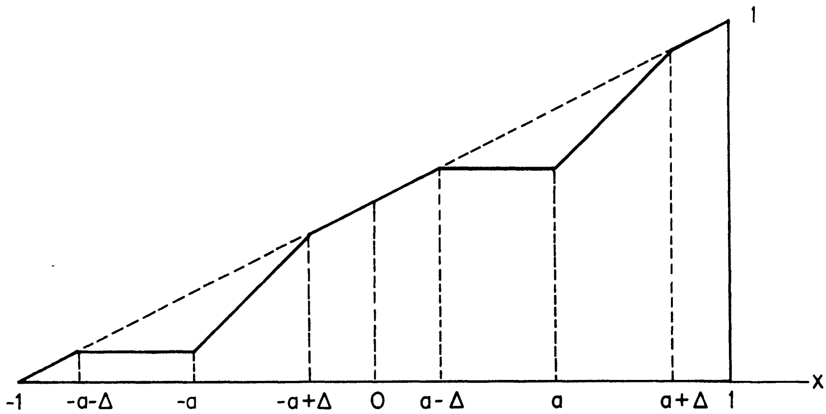

FIG. 1. $F_{\Delta,a}$ for $\Delta \leqq a \leqq 1-\Delta$.

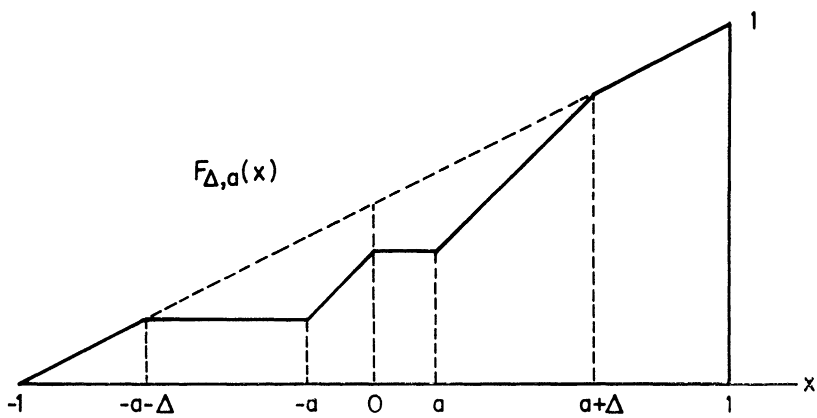

FIG. 2. $F_{\Delta,a}$ for $0 \leqq a \leqq \Delta$.

$\bar{X}$ exceeds a constant is monotone. However, the t-test which rejects H_0 when the t statistic exceeds a constant is not monotone. The rank tests commonly used in paired comparison experiments are monotone, i.e., if $J_N(k/(N+1))$ is a nonnegative function non-decreasing in $k = 1, 2, \cdots, N$ and if φ_{J_N} is of the form

$$\begin{aligned} \varphi_{J_N} &= 1 \quad \text{if } \textstyle\sum_{i=1}^{S} J_N(r_i/(N+1)) \geqq k_N, \\ &= 0 \quad \text{otherwise} \end{aligned}$$

for some constant k_N, then φ_{J_N} is a monotone test.

Monotone tests have monotone power, i.e.,

LEMMA 2.1. *If $F(x) \leqq G(x)$ for all x, then $E_F(\varphi) \geqq E_G(\varphi)$ for each monotone test φ.*

PROOF.* Let $U_1, \cdots, U_N$ be a random sample from the uniform distribution on (0, 1). Define $F^{-1}(u) = \inf\{x: F(x) \geqq u\}$ and G^{-1} similarly, then $F^{-1}(U_i)$ and $G^{-1}(U_i)$ have distributions F and G and they satisfy $F^{-1}(U_i) \geqq G^{-1}(U_i)$. Applying the definition of monotone, we have

$$(2.1) \qquad \varphi(F^{-1}(U_1), \cdots, F^{-1}(U_N)) \geqq \varphi(G^{-1}(U_1), \cdots, G^{-1}(U_N)).$$

The result follows upon taking expectations in this inequality.

COROLLARY 2.1. *All monotone rank tests are unbiased for testing* $H_0: F(x) \geqq 1-F(-x)$ *against* $H_1: F(x) \leqq 1-F(-x)$.

PROOF.* Define the distribution average $A(x) = \frac{1}{2}[F(x)+1-F(-x)]$. Then $A(x)$ is a distribution function symmetric about zero. Note that under H_0, $F(x) \geqq A(x)$, while under H_1, $F(x) \leqq A(x)$. The result now follows from Lemma 2.1.

The ranks are invariant under odd and increasing transformations of the X's, such as

$$(2.2) \qquad H(x) = F(x) - F(-x).$$

Thus if we define

$$(2.3) \qquad X_i' = H(X_i) \quad \text{and} \quad F_1(t) = P(X_i' \leqq t),$$

then for rank tests φ,

$$(2.4) \qquad E_F(\varphi) = E_{F_1}(\varphi).$$

Note also that $|H(x)| = H(|x|)$.

Let $\Omega_1 = \{F: F(t) \leqq \frac{1}{2}(t+1) \text{ for } -1 \leqq t \leqq 1, \text{ and } F(t)-F(-t) = t \text{ for } 0 < t < 1\}$; let $\Omega_1(\Delta) = \{F: F \in \Omega_1 \text{ and } \sup_{-1 \leqq t \leqq 1}[\frac{1}{2}(t+1)-F(t)] \geqq \frac{1}{2}\Delta\}$, and $\bar{\Omega}_1(\Delta) = \{F: F \in \Omega_1 \text{ and } \sup_{-1 \leqq t \leqq 1}[\frac{1}{2}(t+1)-F(t)] \leqq \frac{1}{2}\Delta\}$.

LEMMA 2.2. *If* φ *is a rank test, then*

$$\text{(i)} \quad \inf_{F \in \Omega(\Delta)} E_F(\varphi) = \inf_{F \in \Omega_1(\Delta)} E_F(\varphi), \qquad \textit{and}$$

$$\text{(ii)} \quad \sup_{F \in \bar{\Omega}(\Delta)} E_F(\varphi) = \sup_{F \in \bar{\Omega}_1(\Delta)} E_F(\varphi).$$

PROOF. (i) Note that $\Omega_1(\Delta) \subset \Omega(\Delta)$. Thus it is sufficient to show that the second infimum in (i) is no larger than the first. This is done by showing that if $F \in \Omega(\Delta)$, then $F_1 \in \Omega_1(\Delta)$; and by using (2.4). To see that $F_1 \in \Omega_1(\Delta)$; note that:

(a) $F_1(t) - F_1(-t) = P(|X_i'| \leqq t) = P(|H(X_i)| \leqq t) = P(H(|X_i|) \leqq t) = t$

for $0 < t < 1$ since H is the distribution of $|X_i|$;

$$\begin{aligned}\text{(b)} \quad \sup_x[1-F(-x)-F(x)] &= 2\sup_x\{\tfrac{1}{2}[F(x)+(1-F(-x))]-F(x)\} \\ &= 2\sup_x[A(x)-F(x)] \\ &= 2\sup_{1 \leqq t \leqq 1}[AA^{-1}(\tfrac{1}{2}(t+1)) - FA^{-1}(\tfrac{1}{2}(t+1))] \\ &= 2\sup_{1 \leqq t \leqq 1}[\tfrac{1}{2}(t+1) - F_1(t)]\end{aligned}$$

* The result is well known, but the proof is needed for later reference.

since

$$F_1(t) = P_F(F(X) - F(-X) \leqq t) = P_F(A(X) \leqq \tfrac{1}{2}(t+1)) = FA^{-1}(\tfrac{1}{2}(t+1));$$

(c) $\frac{1}{2}[1-F(-x)-F(x)] \geqq 0$ implies $A(x) \geqq F(x)$, thus $A^{-1}(u) \leqq F^{-1}(u)$ and $FA^{-1}(\frac{1}{2}(t+1)) \leqq \frac{1}{2}(t+1)$, i.e., $F_1(t) \leqq \frac{1}{2}(t+1)$, $0 < t < 1$.

The proof for (ii) is similar.

The next result is that a set of least favorable distributions for $\Omega(\Delta)$, $0 < \Delta < 1$, is $\{F_{\Delta,a} : 0 \leqq a \leqq 1-\Delta\}$, where $F_{\Delta,a}(x)$ is defined for $0 \leqq a \leqq 1-\Delta$, $-1 \leqq x \leqq 1$, by (see Figures 1 and 2).

$$\begin{aligned} F_{\Delta,a}(x) &= \tfrac{1}{2}(-a-\Delta+1) && \text{for } x \in (-a-\Delta, -a], \\ &= x + \tfrac{1}{2}(a-\Delta+1) && \text{for } x \in (-a, -b), \\ &= \tfrac{1}{2}(a-\Delta+1) && \text{for } x \in (b, a], \\ &= x + \tfrac{1}{2}(-a-\Delta+1) && \text{for } x \in (a, a+\Delta), \\ &= \tfrac{1}{2}(x+1) && \text{elsewhere,} \end{aligned}$$

where $b = \max\{0, a-\Delta\}$.

We can now show that the minimum power of monotone rank tests over $\Omega(\Delta)$ is attained at $F_{\Delta,a}$ for some a.

THEOREM 2.1. *If φ is a monotone rank test, then*

$$\inf_{F \in \Omega(\Delta)} E_F(\varphi) = \inf_{0 \leqq a \leqq 1-\Delta} E_{F_{\Delta,a}}(\varphi).$$

PROOF. Since $F_{\Delta,a} \in \Omega(\Delta)$, we need only show that the second infimum is no larger than the first. According to Lemmas 2.1 and 2.2, we need only show that for each F on $[-1, 1]$ for which $\sup_x[\frac{1}{2}(x+1) - F(x)] \geqq \frac{1}{2}\Delta$, $\frac{1}{2}(x+1) - F(x) \geqq 0$ on $[-1, 1]$, and $F(x) - F(-x) = x$ on $(0, 1)$, there exists $0 \leqq a \leqq 1-\Delta$ such that $F_{\Delta,a}(x) \geqq F(x)$. Since $F(x) - F(-x) = x$, there exists $x_0 \geqq 0$ such that $\frac{1}{2}(x_0+1) - F(x_0) \geqq \frac{1}{2}\Delta$; moreover, since $F(x) = x + F(-x)$ and $F(-x)$ is non-increasing, the largest $F(x)$ can be for $x \in [x_0, x_0+\Delta]$ is $x + \frac{1}{2}(1-\Delta-x_0)$. Thus $F_{\Delta,a}(x)$ with $a = x_0$ satisfies $F_{\Delta,a}(x) \geqq F(x)$ on $[x_0, x_0+\Delta]$. Since $F(x) \leqq \frac{1}{2}(x+1)$, the same holds for x in $(x_0+\Delta, 1]$. For $x_0 - \Delta \leqq x \leqq x_0$, the largest $F(x)$ can be is $F_{\Delta,a}(x_0)$. Since $\frac{1}{2}(x+1) \geqq F(x)$ and $F(x) - F(-x) = x$, $F_{\Delta,a}(x) \geqq F(x)$ also holds on $[-1, x_0-\Delta)$. The above $a = x_0$ satisfies $a \geqq 0$; since $\frac{1}{2}(-x_0+1) - F(-x_0) \geqq \frac{1}{2}\Delta$ must hold, we find $-x_0 \geqq -1+\Delta$, or $a \leqq 1-\Delta$.

The maximum power over $\overline{\Omega}(\Delta)$ can be given in terms of simpler expressions. We need the uniform distribution function $G_\Delta(x) = \frac{1}{2}(x+1-\Delta)$ on $(-1+\Delta, 1+\Delta)$, and $\overline{G}_\Delta$ defined by

$$\begin{aligned} \overline{G}_\Delta(x) &= x && \text{for } x \in (1-\Delta, 1], \\ &= 1 && \text{for } x \in (1, 1+\Delta), \\ &= G_\Delta(x) && \text{elsewhere.} \end{aligned}$$

Similarly to, but more simply than Theorem 2.1, we find the maximum alternatives.

THEOREM 2.2. *If φ is a monotone rank test, then*

$$\sup_{F \in \bar{\Omega}(\Delta)} E_F(\varphi) = E_{\bar{G}_\Delta}(\varphi) = E_{G_\Delta}(\varphi).$$

An expression for the maximum power itself can be given now. For each test φ, let $A(\varphi, N, k)$ denote the number of rank orderings leading to rejection when it is known that the k largest ranks correspond to positive X's.

COROLLARY 2.2. *If φ is a monotone rank test, then*

$$\sup_{F \in \bar{\Omega}(\Delta)} E_F(\varphi) = \sum_{k=0}^{N} \binom{N}{k} \Delta^k (1-\Delta)^{N-k} A(\varphi, N, k) 2^{k-N}.$$

PROOF.

$$\begin{aligned} E_{G_\Delta}(\varphi) &= \sum_{k=0}^{N} E_{G_\Delta}(\varphi \mid k\ X\text{'s exceed } 1-\Delta) P(k\ X\text{'s exceed } 1-\Delta), \\ &= \sum_{k=0}^{N} \binom{N}{k} \Delta^k (1-\Delta)^{N-k} E_{G_\Delta}(\varphi \mid k\ X\text{'s exceed } 1-\Delta). \end{aligned}$$

Next note that

$$\begin{aligned} &E_{G_\Delta}(\varphi \mid k\ X\text{'s exceed } 1-\Delta) \\ &\quad = E_{G_\Delta}(\varphi \mid r_i = i, i = N+1-k, \cdots, N \text{ and the remaining} \\ &\qquad (N-k)\ X\text{'s are no greater than } 1-\Delta), \qquad k = 1, \cdots, N. \end{aligned}$$

Furthermore, X given $X \leqq 1-\Delta$ has a uniform distribution on $(-1+\Delta, 1-\Delta)$, so each possible rank ordering of the $N-k$ remaining X's has probability 2^{k-N}, which completes the proof.

In order to find an upper bound on the power of any monotone rank test (which is independent of the test), one needs to maximize the maximum power. This is done by the test

$$\begin{aligned} \varphi_0 &= 1 \qquad \text{if } B = \textstyle\sum_{i=1}^{S} 2^{r_i - 1} \geqq k_{N\alpha}; \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

where $k_{N\alpha}$ is such that $E_F \varphi_0 = \alpha$ when F is symmetric about zero. Tests φ which satisfy $E_F \varphi \leqq \alpha$ for F symmetric about zero are said to be of *level* α. φ_0 is a generalization of a test considered by Savage (1959) which consists of rejecting H_0 if too many of the positive X's exceed all of the negative ones in absolute value. It follows from the results of Savage that φ_0 is a uniformly most powerful level α rank test for uniform shift (G_Δ vs. G_0). This is also clear from Corollary 2.2. Moreover, if $[t]$ denotes the largest integer no larger than t, then Corollary 2.2 yields

COROLLARY 2.3. *An upper bound on the power of all level α monotone rank tests for $F \in \bar{\Omega}(\Delta)$ is given by*

$$B_{\Delta,\alpha} = 1 - \sum_{k=0}^{r} \binom{N}{k} \Delta^k (1-\Delta)^{N-k} (1-\alpha 2^k),$$

where $r = [-\log_2 \alpha]$.

PROOF. Note that if $k \leqq -\log_2 \alpha$, then $2^{N-k} \geqq \alpha 2^N$. Suppose α is of the form $l/2^N (l = 0, \cdots, 2^N)$. Then $2^{N-k} \geqq \alpha 2^N$ implies $A(\varphi_0, N, k) = l = \alpha 2^N$. On the other hand, $k > -\log_2 \alpha$ implies $A(\varphi_0, N, k) = 2^{N-k}$, thus

$$E_{G_\Delta}(\varphi_0) = \sum_{k=0}^{r} \binom{N}{k} \Delta^k (1-\Delta)^{N-k} \alpha 2^k + \sum_{k=1+r}^{N} \binom{N}{k} \Delta^k (1-\Delta)^{N-k} = B_{\Delta,\alpha}.$$

If α is not of the form $l/2^N$, the upper bound still holds.

Note that $B_{\Delta,\alpha}$ is the power of φ_0 for the alternative G_Δ. Thus $B_{\Delta,\alpha}$ is the best possible upper bound.

2B. *Numerical results for the location problem.* The results of Section 2A can be used to obtain numerical bounds on the power functions of rank tests. Some such results are tabled below.

The upper bound $B_{\Delta,\alpha}$ is given in Table 2.1 for various values of $5 \leqq N \leqq 50$, for $\alpha = .1, .05$ and $.01$, and for Δ as multiples of $\frac{1}{8}$. The rest of the tables concern the tests that reject H_0 for large values of the following statistics.

$$S = \text{No. of positive observations (sign)},$$

$$W = \sum_{i=1}^{S} r_i/(N+1) \quad \text{(Wilcoxon)},$$

$$B = \sum_{i=1}^{S} 2^{r_i - 1} \quad \text{(Binary)},$$

$$T^{(1)} = W + [\tfrac{1}{2}(N-1)(1.645/(3N)^{\frac{1}{2}})^{\frac{1}{2}} - 1]S/(N+1) \quad \text{(minimax (1))},$$

$$T^{(2)} = W + [\tfrac{1}{2}(N-1)(2.326/(3N)^{\frac{1}{2}})^{\frac{1}{2}} - 1]S/(N+1) \quad \text{(minimax (2))}.$$

B is called the Binary statistic since if it is written as a binary number, then it determines a rank ordering by identifying positive (negative) x's with the 1's (0's) in the binary expansion. Thus 10011 determines $(r_1, r_2, r_3) = (1, 2, 5)$ when $N = 5$ and $B = 19$ (base 10) $= 2^0 + 2^1 + 2^4$ (base 10) $= 10011$ (base 2). $T^{(1)}$ and $T^{(2)}$ are discussed in Remark 4.3. For $N = 10$ and the significance levels α considered here, $T^{(1)}$ and $T^{(2)}$ are equivalent.

Table 2.2 gives the upper bounds on the power of the tests based on S, W, $T^{(1)}$, $T^{(2)}$ and B for $N = 10$ and $\alpha = .05, .01$. B is included to give a comparison with the bound in Table 2.1. Table 2.3 gives Monte Carlo estimates of these upper power bounds for $N = 20$ and $\alpha = .05, .01$. The Monte Carlo power in this paper is based on $1000 = 250 \times 4$ antithetic samples (see Hammersley and Hanscomb (1964, page 60)). In some special cases, it was possible to obtain the exact power. These results are marked with "*".

Tables 2.4–2.5 give exact values and Monte Carlo estimates of the power of W, S, $T^{(1)}$ and $T^{(2)}$ for the least favorable distributions $F_{\Delta,a}$ for selected values of a. From considerations of asymptotic power (Section 4), we decided to use $a = 0$, Δ, $\frac{1}{2}$ and $1-\Delta$.

Note that for the alternatives $F_{\Delta,a}$ with $a = \Delta$, $\frac{1}{2}$ and $1-\Delta$, the Sign test obviously has power α and from Table 2.5 the Wilcoxon test has the highest power among S, W, $T^{(1)}$ and $T^{(2)}$. However, when $a = 0$, Table 2.4 shows that the Sign test has the highest power and the Wilcoxon test has the lowest power. This agrees with the asymptotic results (see Remark 4.1).

TABLE 2.1
The upper bound $B_{\Delta,\alpha}$ on the power of monotone rank tests

	α					
Δ	.100	.050	.010	.100	.050	.010
		$N=5$			$N=6$	
.125	.17950	.09008	.01802	.20067	.10126	.02027
.250	.29424	.15200	.03052	.35071	.18756	.03815
.375	.43810	.24129	.04915	.53008	.31509	.06758
.500	.59687	.36094	.07594	.70781	.47891	.11391
.625	.75163	.50933	.11331	.85423	.66076	.18413
.750	.88193	.67827	.16413	.95012	.83102	.28723
		$N=7$			$N=8$	
.125	.22342	.11369	.02281	.24757	.12740	.02566
.250	.40921	.22835	.04767	.46792	.27356	.05948
.375	.61600	.39571	.09263	.69250	.47832	.12570
.500	.79609	.59336	.16867	.86211	.69473	.24145
.625	.91903	.78168	.28878	.95695	.86753	.41860
.750	.98046	.92035	.46528	.99277	.96543	.64871
		$N=9$			$N=10$	
.125	.27290	.14242	.02886	.29921	.15873	.03246
.250	.52530	.32215	.07398	.58015	.37294	.09154
.375	.75808	.55864	.16754	.81261	.63343	.21800
.500	.90918	.77842	.32928	.94150	.84380	.42603
.625	.97790	.92342	.55475	.98898	.95747	.67919
.750	.99744	.98594	.79453	.99913	.99457	.89096
		$N=12$			$N=14$	
.125	.35388	.19504	.04101	.40991	.23576	.05167
.250	.67913	.47646	.13675	.76131	.57594	.19572
.375	.89193	.75920	.33969	.94032	.85034	.47504
.500	.97703	.92784	.61780	.99149	.96918	.77294
.625	.99744	.98809	.85772	.99945	.99698	.94673
.750	.99991	.99928	.97538	.99999	.99992	.99554
		$N=16$			$N=18$	
.125	.46581	.28008	.06479	.52033	.32705	.08066
.250	.82646	.66570	.26685	.87628	.74270	.34675
.375	.96818	.91115	.60603	.98352	.94922	.71951
.500	.99698	.98762	.87749	.99897	.99527	.93898
.625	.99989	.99929	.98249	.99998	.99984	.99480
.750	1.00000	.99999	.99931	1.00000	1.00000	.99991
		$N=20$			$N=30$	
.125	.57250	.37565	.09952	.78111	.61230	.23944
.250	.91328	.80613	.43107	.98786	.96312	.79075
.375	.99166	.97189	.80948	.99979	.99896	.98385
.500	.99966	.99826	.97156	1.00000	.99999	.99969

TABLE 2.1 (*continued*)

	α					
Δ	.100	.050	.010	.100	.050	.010
	$N=40$			$N=50$		
.125	.90045	.79057	.43138	.95836	.89812	.62263
.250	.99862	.99470	.94853	.99986	.99936	.99038
.375	1.00000	.99997	.99924	1.00000	1.00000	.99997

TABLE 2.2
Least upper bounds for $F \in \bar{\Omega}(\Delta)$ on the power of S, W, $T^{(1)}$, $T^{(2)}$ and B

	Δ					
	.125	.250	.375	.500	.625	.750
			$N=10$, $\alpha=.05$			
S	.10496	.19529	.32675	.49556	.68314	.85473
W	.14051	.30445	.52359	.74282	.90298	.98037
$T^{(1)}, T^{(2)}$	.13340	.28153	.48321	.69634	.86803	.96578
B	.15873	.37294	.63343	.84380	.95747	.99457
			$N=10$, $\alpha=.01$			
S	.02596	.05952	.12267	.22976	.39273	.61034
W	.03241	.08993	.20784	.39744	.63383	.85112
$T^{(1)}, T^{(2)}$	.03227	.08715	.19602	.37249	.60201	.82761
B	.03246	.09154	.21800	.42603	.76919	.89096

TABLE 2.3
Monte Carlo estimates of the least upper bound for $F \in \bar{\Omega}(\Delta)$ on the power of S, W, $T^{(1)}$ and $T^{(2)}$

	Δ						
	0	.1	.2	.3	.4	.5	.75
				$N=20$, $\alpha=.05$			
S	.051	.106	.218	.377	.571	.765	.985
W	.048	.178	.400	.653	.867	.963	1.000
$T^{(1)}$	.047	.159	.361	.604	.825	.954	.999
$T^{(2)}$	.047	.160	.360	.598	.819	.951	.999
				$N=20$, $\alpha=.01$			
S	.011	.032	.067	.146	.283	.471	.986
W	.006	.035	.129	.325	.595	.822	1.000
$T^{(1)}$	.007	.030	.112	.286	.540	.765	.999
$T^{(2)}$	.007	.030	.112	.280	.522	.754	.999

TABLE 2.4

Exact values ($N = 10$) *and Monte Carlo estimates* ($N = 20$) *of the power of* W, S, $T^{(1)}$ *and* $T^{(2)}$ *for the alternative* $F_{\Delta,0}$

	Δ						
	0	.1	.2	.3	.4	.5	.75
			$N = 10$, $\alpha = .05$				
W	.0500	.0589	.0765	.1086	.1637	.2514	.6392
S	.0500	.0914	.1544	.2429	.3579	.4955	.8547
$T^{(1)}, T^{(2)}$	.0500	.0658	.0885	.1220	.1749	.2584	.6396
			$N = 10$, $\alpha = .01$				
W	.0100	.0130	.0181	.0255	.0381	.0627	.2635
S	.0100	.0219	.0433	.0805	.1402	.2298	.6104
$T^{(1)}, T^{(2)}$	.0100	.0147	.0230	.0344	.0498	.0740	.2655
			$N = 20$, $\alpha = .05$				
W	.048	.059	.082	.120	.189	.296	.776
S	.051	.106	.215	.380	.567	.766	.985
$T^{(1)}$	.047	.068	.101	.168	.244	.400	.830
$T^{(2)}$	.047	.069	.105	.175	.253	.418	.880
			$N = 20$, $\alpha = .01$				
W	.006	.010	.015	.026	.062	.110	.560
S	.011	.026	.067	.147	.278	.476	.940
$T^{(1)}$	.007	.013	.023	.041	.088	.169	.588
$T^{(2)}$	.007	.015	.026	.044	.100	.192	.629

TABLE 2.5

Monte Carlo estimates of the power of W, $T^{(1)}$ *and* $T^{(2)}$ *for the alternative* $F_{\Delta,a}$ *with* $a = \Delta$, $1-\Delta$, *and* $\frac{1}{2}$

	Δ					
	0	0.1	0.2	0.3	0.4	0.5
			$N = 10$, $\alpha = .05$, $a = \Delta$			
W	.049	.055	.079	.110	.198	.382
$T^{(1)}, T^{(2)}$	.049	.054	.064	.074	.101	.156
			$N = 10$, $\alpha = .01$, $a = \Delta$			
W	.010	.010	.013	.021	.035	.052
$T^{(1)}, T^{(2)}$	.011	.012	.014	.018	.030	.044
			$N = 20$, $\alpha = .05$, $a = \Delta$			
W	.048	.057	.092	.169	.313	.578
$T^{(1)}$	.047	.049	.075	.132	.222	.422
$T^{(2)}$	.047	.048	.074	.122	.214	.422

TABLE 2.5 (*continued*)

	Δ					
	0	0.1	0.2	0.3	0.4	0.5
		$N=20,\quad \alpha=.01,\quad a=\Delta$				
W	.006	.008	.017	.046	.094	.252*
$T^{(1)}$	.007	.008	.012	.024	.044	.132*
$T^{(2)}$	.007	.009	.011	.021	.043	.132*
		$N=10,\quad \alpha=.05,\quad a=1-\Delta$				
W	.049	.053	.069	.099	.179	.382
$T^{(1)}, T^{(2)}$	.049	.054	.064	.085	.113	.156
		$N=10,\quad \alpha=.01,\quad a=1-\Delta$				
W	.010	.010	.011	.015	.034	.052
$T^{(1)}, T^{(2)}$	.011	.011	.013	.014	.024	.044
		$N=20,\quad \alpha=.05,\quad a=1-\Delta$				
W	.048	.053	.082	.155	.309	.578
$T^{(1)}$	.047	.051	.075	.123	.238	.422
$T^{(2)}$	.047	.050	.074	.118	.218	.422
		$N=20,\quad \alpha=.01,\quad a=1-\Delta$				
W	.006	.006	.009	.033	.098	.252*
$T^{(1)}$	.007	.007	.009	.023	.055	.132*
$T^{(2)}$	.007	.007	.010	.022	.053	.132*
		$N=10,\quad \alpha=.05,\quad a=\frac{1}{2}$				
W	.050	.051	.066	.111	.188	.382
$T^{(1)}, T^{(2)}$	.048	.050	.062	.084	.112	.156
		$N=10,\quad \alpha=.01,\quad a=\frac{1}{2}$				
W	.009	.010	.010	.025	.034	.050
$T^{(1)}, T^{(2)}$	.011	.011	.012	.019	.026	.044
		$N=20,\quad \alpha=.05,\quad a=\frac{1}{2}$				
W	.055	.060	.082	.164	.307	.568
$T^{(1)}$	.055	.058	.075	.141	.247	.432
$T^{(2)}$	.055	.059	.075	.137	.237	.432
		$N=20,\quad \alpha=.01,\quad a=\frac{1}{2}$				
W	.011	.011	.020	.038	.104	.252*
$T^{(1)}$	.012	.013	.018	.027	.073	.132*
$T^{(2)}$	.012	.013	.018	.027	.068	.132*

TABLE 2.6

Monte Carlo estimates of the minimum over $a \in \{0, \Delta, \frac{1}{2}, 1-\Delta\}$ of the power of W, $T^{(1)}$ and $T^{(2)}$ for the alternative $F_{\Delta,a}$

	Δ					
	0	.1	.2	.3	.4	.5
			$N = 10$,	$\alpha = .05$		
W	.049	.051	.066	.099	.164	.247
$T^{(1)}, T^{(2)}$	.048	.050	.062	.074	.101	.156
			$N = 10$,	$\alpha = .01$		
W	.009	.010	.010	.015	.034	.050
$T^{(1)}, T^{(2)}$	.010	.011	.012	.014	.024	.044
			$N = 20$,	$\alpha = .05$		
W	.048	.053	.082	.120	.189	.296
$T^{(1)}$	.047	.049	.075	.123	.222	.400
$T^{(2)}$	.047	.048	.074	.118	.214	.400
			$N = 20$,	$\alpha = .01$		
W	.006	.006	.009	.026	.062	.110
$T^{(1)}$	.007	.007	.009	.023	.044	.132
$T^{(2)}$	.007	.007	.010	.021	.043	.132

The results of Tables 2.4 and 2.5 are summarized in Table 2.6 in which we consider the minimum over $a \in \{0, \Delta, \frac{1}{2}, 1-\Delta\}$ of the power of W, $T^{(1)}$ and $T^{(2)}$ for the alternative $F_{\Delta,a}$. Asymptotically (see Section 4), the minimum power of W is attained at $a = 0$, while for $T^{(1)}$ and $T^{(2)}$ the minimum is attained for each $a \in [\Delta, 1-\Delta]$. Tables 2.4 and 2.5 show that for W, the minimum power is not attained at $a = 0$ when $N = 10$, while for $N = 20$, the tables indicate that the asymptotic result is in effect and the minimum power is attained at $a = 0$. As far as $T^{(1)}$ and $T^{(2)}$ are concerned, the tables for $N = 10$ and 20 do not indicate any disagreement with the asymptotic results of Section 4. Keeping this in mind, Table 2.6 indicates that for $N = 10$, the asymptotic results have not taken effect and W is better than $T^{(1)}$ and $T^{(2)}$, while for $\alpha = .05$ and $N = 20$, $T^{(1)}$ and $T^{(2)}$ are better than W. Note that $T^{(1)}$ and $T^{(2)}$ have about the same power.

3. Power bounds for monotone tests in the symmetry problem. In this section, one assumes that the population median (see Remark 3.1) has been subtracted from the observations, and the null hypothesis of symmetry about zero (or skewness to the left) is to be tested against skewness to the right. A test φ is said to be *s-monotone* if $\varphi(t_1, \cdots, t_N) \leqq \varphi(x_1, \cdots, x_N)$ whenever $t_i \leqq x_i$ and $t_i x_i > 0$ $(i = 1, \cdots, N)$. Thus all monotone tests are s-monotone. If $J_N(k/(N+1))$ is nondecreasing in k, then the test that rejects for large values of $\sum_{i=1}^{S} J_N(r_i/(N+1))$ is s-monotone. Note that such tests are not necessarily monotone if J_N can take on negative values.

s-monotone tests also have monotone power, i.e., if the smallest median of F is $\inf_x \{F(x) = \frac{1}{2}\}$, then

LEMMA 3.1. *If $F(x) \leqq G(x)$ for all x and the smallest medians of F and G equal zero, then $E_F(\varphi) \geqq E_G(\varphi)$ for each s-monotone test φ.*

The proof proceeds as in Lemma 2.1. Note that $F^{-1}(U_i) > 0$ if and only if $G^{-1}(U_i) > 0$.

COROLLARY 3.1. *Suppose that F has unique median zero. Then all s-monotone rank tests are unbiased for testing $H_0: F(x) \geqq 1-F(-x)$ against $H_1: F(x) \leqq 1-F(-x)$.*

The proof is the same as for Corollary 2.1, using the fact that $A(x) = \frac{1}{2}[F(x)+1-F(-x)]$ has unique median zero.

One can now proceed as in Section 2 and one finds that the minimum power is attained at $F_{\Delta,a}$; however the restriction $F(0) = \frac{1}{2}$ requires $a \geqq \Delta$. Thus

THEOREM 3.1. *If φ is an s-monotone rank test, then for $0 < \Delta < \frac{1}{2}$,*

$$\inf_{F \in \Omega_s(\Delta)} E_F(\varphi) = \inf_{\Delta \leqq a \leqq 1-\Delta} E_{F_{\Delta,a}}(\varphi).$$

For the maximum power, one needs the distribution H_Δ defined by

$$\begin{aligned} H_\Delta(x) &= \bar{G}_\Delta(x) \quad \text{for} \quad x \notin (-\Delta, \Delta), \\ &= x+\tfrac{1}{2} \quad \text{for} \quad x \in (-\Delta, 0], \\ &= \tfrac{1}{2} \quad \text{for} \quad x \in (0, \Delta]. \end{aligned}$$

From the results of Section 2 and this section, it is clear that

COROLLARY 3.2. *If φ is an s-monotone rank test, then for $0 < \Delta < \frac{1}{2}$, $\sup_{F \in \bar{\Omega}_s(\Delta)} E_F(\varphi) = E_{H_\Delta}(\varphi)$.*

REMARK 3.1. In this paper, we only consider the problem of testing for symmetry when the population median is assumed known. If it is not known, one could (a) subtract an estimate of the median based on $X_1, \cdots, X_N$ from the observations (see Gross (1966) and Gupta (1967)) or (b) subtract an estimate of the median which is independent of $X_1, \cdots, X_N$ from the observations, e.g., use a preliminary sample to estimate the median. It is hoped that the results of this paper will in some sense carry over to tests based on the procedures (a) and (b) above.

Table 3.1 below gives Monte Carlo estimates of the power of

$$V = W - \tfrac{1}{2}S \quad \text{and} \quad T_G = \textstyle\sum_{i=1}^{S} [(r_i/(N+1))^2 - \tfrac{1}{3}] \tag{3.1}$$

for the alternative $F_{\Delta,a}$ with $a = \Delta$, $a = \frac{1}{2}$ and with $a = 1-\Delta$. Gross (1966) has compared these statistics in terms of Pitman efficiency. Asymptotically (see Section 4B), the minimum power of T_G is attained for $a = \Delta$, while for V, the minimūm power is attained for each $a \in [\Delta, 1-\Delta]$. The results of Table 3.1 indicate that the same is true for finite sample sizes. Moreover, if one considers the minimum over $a \in \{\Delta, \frac{1}{2}, 1-\Delta\}$ of the power of V and T_G for the alternative $F_{\Delta,a}$, then one finds that V is much better than T_G.

4. Asymptotic minimax results.

4A. *The location problem.* In this sub-section, statistics that maximize the minimum power over $\Omega(\Delta)$ are obtained asymptotically for certain classes of

TABLE 3.1

Monte Carlo estimates of the power of V and T_G for the alternative $F_{\Delta,a}$ with $a = \Delta$, $1-\Delta$, and $\frac{1}{2}$

	Δ					
	0	0.1	0.2	0.3	0.4	0.5
			$N=10$, $\alpha=.05$, $a=\Delta$			
V	.052	.076	.112	.241	.506	.968
T_G	.053	.061	.093	.165	.371	.934
			$N=10$, $\alpha=.01$, $a=\Delta$			
V	.012	.017	.037	.101	.332	.912
T_G	.013	.012	.023	.074	.281	.806
			$N=20$, $\alpha=.05$, $a=\Delta$			
V	.052	.069	.151	.402	.813	1.000
T_G	.055	.062	.094	.235	.622	.998
			$N=20$, $\alpha=.01$, $a=\Delta$			
V	.008	.012	.038	.164	.569	.993*
T_G	.007	.010	.020	.085	.375	.979*
			$N=10$, $\alpha=.05$, $a=1-\Delta$			
V	.052	.073	.116	.243	.508	.968
T_G	.053	.072	.128	.292	.607	.934
			$N=10$, $\alpha=.01$, $a=1-\Delta$			
V	.012	.017	.036	.103	.333	.912
T_G	.013	.016	.038	.100	.308	.806
			$N=20$, $\alpha=.05$, $a=1-\Delta$			
V	.052	.069	.151	.402	.813	1.000
T_G	.055	.081	.215	.575	.910	.998
			$N=20$, $\alpha=.01$, $a=1-\Delta$			
V	.008	.012	.038	.164	.569	.993*
T_G	.007	.011	.066	.292	.757	.979*
			$N=10$, $\alpha=.05$, $a=\frac{1}{2}$			
V	.052	.064	.110	.264	.526	.968
T_G	.052	.061	.105	.237	.486	.934
			$N=10$, $\alpha=.01$, $a=\frac{1}{2}$			
V	.012	.018	.042	.109	.352	.912
T_G	.011	.014	.037	.096	.318	.806
			$N=20$, $\alpha=.05$, $a=\frac{1}{2}$			
V	.048	.058	.138	.388	.816	1.000
T_G	.048	.064	.142	.378	.778	1.000
			$N=20$, $\alpha=.01$, $a=\frac{1}{2}$			
V	.014	.014	.048	.170	.576	.993*
T_G	.012	.014	.045	.167	.520	.979*

statistics. Govindarajulu (1960), Gross (1966) and others considered the class of statistics of the form

(4.1) $$T_N(J) = \sum_{i=1}^{S} J(r_i/(N+1)),$$

where J is a function on (0, 1). It turns out that there is a statistic $T(\Delta)$ that has a higher minimum power than each member of this class (see Theorem 4.1 and Remark 4.1). This statistic is defined by

(4.2) $$T(\Delta) = W + [\tfrac{1}{2}(N-1)\Delta - 1]S/(N+1),$$

where $W = \sum_{i=1}^{S} r_i/(N+1)$ and S = Number of positive observations. For this reason, we will consider a larger class of test statistics. Let $\mathscr{T}$ denote the class of level α tests of the form

(4.3) $$\begin{aligned} \varphi_N = \varphi_N(J, d_N, b_N) &= 1 \quad \text{if} \quad V_N(J) = T_N(J) + d_N \Delta S + b_N N^{-1} S \geqq k_{N\alpha}, \\ &= 0 \quad \text{otherwise}, \end{aligned}$$

where $\{d_N\}$ and $\{b_N\}$ are sequences of constants converging to d and b respectively, and J is a continuously differentiable function satisfying the conditions of Gross (1966, page 40).

When $\Delta > 0$ is fixed and $N \to \infty$, then the minimum power over $\Omega(\Delta)$ of the Wilcoxon, Sign and other consistent statistics tends to one, while if one considers sequences $\{\Delta_N\}$ such that

(4.4) $$\Delta_N N^{\frac{1}{2}} \to c, \qquad \text{some} \quad c \in [0, \infty],$$

then it turns out that for $0 < c < \infty$, the limit of the minimum power depends on the alternative sequence $\{\Delta_N\}$. Thus we restrict attention to the sequences (4.4). Note that the trivial cases $\Delta_N = 0$ and $\Delta_N = \Delta > 0$ are included in (4.4).

Gross (1966) has shown that $T_N(J)$ properly normalized can be written as a sum of N independent random variables plus a remainder term that tends uniformly in the underlying distribution to zero in probability. Since S is a sum of N independent random variables, it follows that if $V_N(J)$ is a statistic of the form

(4.5) $$V_N(J) = T_N(J) + d_N \Delta S + b_N N^{-1} S,$$

then it has a distribution that can be approximated uniformly by the normal distribution in the sense that if (4.4) holds with $c < \infty$ and Φ denotes the standard normal distribution function, then

$$\sup_{0 \leqq a \leqq 1-\Delta_N} \left| P_{\Delta_N, a}\left(\frac{V_N(J) - \mu_{\Delta_N, a}(J)}{\sigma_N(J)} \leqq t \right) - \Phi(t) \right| \to 0 \quad \text{as} \quad N \to \infty,$$

where

(4.6) $$\mu_{\Delta_N, a}(J) = N\{\textstyle\int_0^\infty J(t)\, dF_{\Delta_N, a}(t) + \tfrac{1}{2}(1 + \Delta_N - a)(d\Delta_N + bN^{-1})\}$$

when $0 \leqq a \leqq \Delta_N$,

$$= N\{\textstyle\int_0^\infty J(t)\, dF_{\Delta_N, a}(t) + \tfrac{1}{2}(d\Delta_N + bN^{-1})\}$$

when $\Delta_N \leqq a \leqq 1-\Delta_N$,

$$\sigma_N^2(J) = \tfrac{1}{4}\textstyle\sum_{i=1}^{N}[J(i/(N+1))+d_N\Delta_N+b_N N^{-1}]^2 \sim \tfrac{1}{4}\sum J^2(i/(N+1))$$
$$\sim \tfrac{1}{4}N\textstyle\int_0^1 J^2(u)\,du.$$

Here, $P_{\Delta_N,a}$ denotes probabilities computed under $F_{\Delta_N,a}$, and $\gamma_N \sim \lambda_N$ means $\gamma_N/\lambda_N \to 1$ as $N \to \infty$. For some statistics such as S, W and $T(\Delta)$, better approximations can be obtained by replacing $\mu_{\Delta_N,a}(J)$ by the exact expected value. Applying (4.6) to the power $E_{\Delta_N,a}(\varphi_N(J, d_N, b_N)) = P_{\Delta_N,a}(V_N(J) \geqq k_{N\alpha})$, one obtains the following approximation to the minimum power of $\varphi_N(J, d_N, b_N)$ for J nonnegative and nondecreasing, and $d > 0$ (or $d = b = 0$).

$$(4.7)\quad \left|\inf_{F\in\Omega(\Delta_N)} E_F(\varphi_N(J, d_N, b_N)) - \inf_{0\leqq a\leqq 1-\Delta_N}\Phi\left(z_\alpha + \frac{\mu_{\Delta_N,a}(J)-\mu_{0,0}(J)}{\sigma_N(J)}\right)\right| \to 0$$

as $N \to \infty$, where

$$z_\alpha = \Phi^{-1}(\alpha) \quad\text{and}\quad \mu_{0,0}(J) = N\{\textstyle\int_0^\infty J(t)\tfrac{1}{2}dt + \tfrac{1}{2}(d\Delta_N + bN^{-1})\}.$$

Let $\varphi(\Delta)$ denote the level α test

$$(4.8)\qquad \begin{aligned} \varphi(\Delta) &= 1 \quad \text{if}\quad T(\Delta) \geqq k_{N,a}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

LEMMA 4.1. *For sequences* $\{\Delta_N\}$ *satisfying* (4.4), *the minimum power of* $\varphi(\Delta_N)$ *over* $\Omega(\Delta_N)$ *has the limit*

$$(4.9)\qquad \lim_{N\to\infty}[\inf_{F\in\Omega(\Delta_N)} E_F(\varphi(\Delta_N))] = \Phi(z_\alpha + 3^{\frac{1}{2}}c^2).$$

To see this, compute

$$E_{\Delta,a}(T(\Delta)) - E_{0,0}(T(\Delta))$$
$$= [\tfrac{1}{2}N(N-1)\Delta^2 + \tfrac{1}{4}N(N-1)(a\Delta - a^2)]/(N+1), \quad\text{when}\quad 0 \leqq a \leqq \Delta;$$
$$= \tfrac{1}{2}N(N-1)\Delta^2/(N+1), \quad\text{when}\quad \Delta \leqq a \leqq 1-\Delta.$$
$$\sigma_N^2(T(\Delta)) = \tfrac{1}{4}\textstyle\sum(i + \tfrac{1}{2}(N-1)\Delta - 1)^2/(N+1)^2$$
$$\sim \tfrac{1}{24}N(2N+1)/(N+1) \quad\text{when}\quad \Delta = \Delta_N \to 0.$$

Now the result follows upon substituting these quantities and $\Delta_N = cN^{-\frac{1}{4}}$ into the right-hand side of (4.7). Note that the result is immediate when Δ_N does not tend to zero or $c = \infty$.

THEOREM 4.1. $\varphi(\Delta)$ *is asymptotically minimax over* $\Omega(\Delta)$ *and* $\mathscr{T}$ *in the sense that*

$$(4.10)\qquad \lim_{N\to\infty}[\inf_{F\in\Omega(\Delta_N)} E_F(\varphi(\Delta_N))] \geqq \limsup_{N\to\infty}[\inf_{F\in\Omega(\Delta_N)} E_F(\varphi_N)]$$

for all tests $\varphi_N \in \mathscr{T}$ *and all sequences* $\{\Delta_N\}$ *satisfying* (4.4).

PROOF. Using Lemma 4.1, we need only show that there is $F \in \Omega(\Delta_N)$ such that the limit of the minimum power of $\varphi_N(J, d_N, b_N) \in \mathscr{T}$ is bounded above by the right-hand side of (4.9). When $c = \infty$, (4.10) trivially holds in view of Lemma 4.1; we thus assume $c < \infty$. Using (4.6), we find that for the distribution $F_{\Delta_N, a} \in \Omega(\Delta_N)$ with $a > 0$ fixed,

$$(4.11) \qquad \lim_{N\to\infty} E_{\Delta_N, a}(\varphi_N(J, d_N, b_N)) = \Phi\left(z_\alpha + \frac{J'(a)c^2}{(\int_0^1 J^2(x)\,dx)^{\frac{1}{2}}}\right), \qquad 0 < a < 1.$$

If $J'(a) \leqq 0$ for some $0 < a < 1$, the result follows. If $J'(a) > 0$ for all $0 < a < 1$, then $J = H^{-1}$ for some distribution function H (say) with density h and the right-hand side of (4.11) becomes

$$(4.12) \qquad \Phi(z_\alpha + c^2/h(J(a))(\int x^2 h(x)\,dx)^{\frac{1}{2}}).$$

Among all densities h with $\int x^2 h(x)\,dx$ fixed, the uniform density has the smallest supremum $\sup_x h(x)$, namely $[3\int x^2 h(x)\,dx]^{-\frac{1}{2}}$. This follows from the arguments of Capon (1965, pages 851 and 852) (see also [6]). Thus there exists an $a_0 \in (0, 1)$ such that $h(J(a_0)) \geqq [3\int x^2 h(x)\,dx]^{-\frac{1}{2}}$. This inequality together with (4.11) and (4.12) yields the result.

REMARK 4.1. The above proof shows that no test based on a statistic of the form $\sum_1^S J(r_i/(N+1))$ can be asymptotically minimax. For if it were, then we would have to have $J(u) = \gamma u$ for some $\gamma > 0$ and all $u \in (0, 1)$. The test would then be equivalent to the Wilcoxon test; however, the asymptotic minimum power of the Wilcoxon test φ_W is attained at $F_{\Delta_N, 0}$ and equals

$$(4.13) \qquad \lim_{N\to\infty} [\inf_{F\in\Omega(\Delta_N)} E(\varphi_W)] = \Phi(z_\alpha + 3^{\frac{1}{2}}c^2/2).$$

On the other hand, the statistic $W + \varepsilon S$ is of the form $\sum_1^S J(r_i/(N+1))$ with $J(u) = u + \varepsilon$, and the test based on it has an asymptotic minimum power attained at F_{Δ_N, Δ_N} which can be made arbitrarily close to the upper bound $\Phi(z_\alpha + 3^{\frac{1}{2}}c^2)$ by taking ε small enough. Note the discontinuity (see (4.13)) at $\varepsilon = 0$.

REMARK 4.2. The asymptotic minimax solution $\varphi(\Delta)$ is not asymptotically unique in $\mathscr{T}$. In fact, from the proof of Lemma 4.1, it is clear that all statistics of the form

$$(4.14) \qquad W + [\tfrac{1}{2}d(N-1)\Delta - 1]S/(N+1)$$

are asymptotically minimax provided $d \geqq 1$. In $T(\Delta)$ we have chosen $d = 1$ since this value minimizes the *exact* null variance of (4.14). Asymptotically, the null variance is independent of d.

REMARK 4.3. When applying $T(\Delta_N)$ one has to choose one specific sequence $\{\Delta_N\}$ in order to be able to carry out the asymptotic minimax test $\varphi(\Delta_N)$. Suppose $\alpha = .05$ and one chooses $\Delta_N^2 = (\Delta_N^*)^2 = 1.645/(3N)^{\frac{1}{2}}$, then $c^2 = c_*^2 = 1.645/3^{\frac{1}{2}}$ and the right-hand side of (4.9) becomes $\frac{1}{2}$. It is clear from Remark 4.2 that $\varphi(\Delta_N^*)$ is not only asymptotically minimax for $\{\Delta_N^*\}$, but uniformly in all sequences $\{\Delta_N\}$ with $\lim_{N\to\infty} \Delta_N/\Delta_N^* \leqq 1$. Since $\Phi(z_{.05} + 3^{\frac{1}{2}}c_*^2) = \frac{1}{2}$, these are all the sequences for which the limit of the minimum power of all tests in $\mathscr{T}$ is bounded below by $\frac{1}{2}$.

By choosing Δ_N larger than Δ_N^*, we could obtain a larger class of $\{\Delta_N\}$ for which $\varphi(\Delta_N)$ if optimal. However, the null variance of $T(\Delta_N)$ would then be increased and the limiting minimum power (4.9) used in proving the minimax result would be a poorer approximation (an "overestimate") to the finite sample size power function.

The statistics $T(\Delta_N^*)$ and $T(\Delta_N')$ with $(\Delta_N')^2 = 2.326/(3N)^{\frac{1}{2}}$ are investigated in terms of Monte Carlo power in Section 2B with $N = 10$ and 20. The results there indicate that the asymptotic results are in effect for $N \geqq 20$ and $\alpha = .05$ in the sense that $T(\Delta_N^*)$ and $T(\Delta_N')$ are improvements on the Wilcoxon statistic W, but that they are not in effect when $N = 10$ (see Table 2.6).

4B. *The symmetry problem.* For the class of alternatives $\Omega_s(\Delta)$, we consider the class $\mathscr{T}_s$ of level α tests of the form

$$\begin{aligned} \varphi_N^{(s)} = \varphi_N^{(s)}(J) &= 1 \qquad \text{if } \textstyle\sum_1^S J(r_i/(N+1)) \geqq k_{N\alpha}, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

where J is a continuously differentiable function satisfying the conditions of Gross (1966, page 47). Let φ_V be the test that rejects for large values of $V = W - \frac{1}{2}S = \sum_1^S (r_i/(N+1) - \frac{1}{2})$, then $E_{\Delta_N,a}(V) = \frac{1}{2}N(N-1)\Delta^2/(N+1)$ for $\Delta \leqq a \leqq 1-\Delta$, and $\operatorname{Var}(V \mid H_0) = \frac{1}{4}\sum(i/(N+1)-\frac{1}{2})^2 \sim N/48$. From this, Theorem 3.1, and the arguments of Section 4A, one concludes that

$$\lim_{N\to\infty}[\inf_{F\in\Omega_s(\Delta)} E_F(\varphi_V)] = \Phi(z_\alpha + 2\,(3^{\frac{1}{2}})c^2). \tag{4.15}$$

Similarly, we find that for $\varphi_N^{(s)}(J) \in \mathscr{T}_s$,

$$\lim_{N\to\infty} E_{\Delta_N,a}(\varphi_N^{(s)}) = \Phi(z_\alpha + J'(a)c^2(\textstyle\int_0^1 J^2(x)\,dx)^{-\frac{1}{2}}) \tag{4.16}$$

provided (4.4) holds with $c < \infty$, and $0 < a < 1$. As in the proof of Theorem 4.1, we find that (4.16) is bounded by (4.15), moreover, since J does not depend on N, the only function that reaches the upper bound (4.15) is $J(u) = u - \frac{1}{2}$. We have

THEOREM 4.2. *φ_V is asymptotically minimax over $\Omega_s(\Delta)$ and $\mathscr{T}_s$ in the sense that*

$$\lim_{N\to\infty}[\inf_{F\in\Omega(\Delta_N)} E_F(\varphi_V)] \geqq \limsup_{N\to\infty}[\inf_{F\in\Omega(\Delta_N)} E_F(\varphi_N^{(s)})] \tag{4.17}$$

for all tests $\varphi_N^{(s)} \in \mathscr{T}_s$ and all sequences $\{\Delta_N\}$ satisfying (4.4). *Moreover, φ_V is unique, i.e., it is the only test in $\mathscr{T}_s$ asymptotically minimax in the above sense.*

The minimum power of the test based on T_G (see (3.1)) has the limit α when $0 \leqq c < \infty$ since here $J(u) = u^2 - \frac{1}{3}$ and $\inf_{0<a<1} J'(a) = 0$ (see (4.16)). The results of Section 3 show that already for sample size 10, V has a considerably larger minimum power than T_G.

5. Rejection limits for V, $T^{(1)}$ and T_G, and Monte Carlo power for shift alternatives. Tables 5.1–5.3 below give the rejection limits for V, $T^{(1)}$ and T_G for $4 \leqq N \leqq 10$ and for significance levels close to $\alpha = .01, .025, .05$ and $.1$. In Table 5.4, the rejection limits obtained from the normal approximation to the null-distributions are given, and the significance levels which would result if these limits were used

TABLE 5.1

Upper percentage points v for the null distribution of the $(N+1)V$ statistic for sample sizes $N = 4(1)10$ and significance levels bracketing $\alpha = .01, .025, .050,$ and $.100$. $P(v) = \Pr\{(N+1)V \geqq v\}$

		Significance levels α							
		.010		.025		.050		.100	
$N=4$	N		2.0		2.0		2.0	2.0	1.5
	$P(v)$		.06250		.06250		.06250	.06250	.18750
$N=5$	v		3.0		3.0		3.0	3.0	3.0
	$P(v)$		.06250		.06250		.06250	.06250	.18750
$N=6$	v		4.5	4.5	4.0	4.0	3.5	3.0	2.5
	$P(v)$		.01563	.01563	.04688	.04688	.06250	.09375	.15625
$N=7$	v		6.0	6.0	5.0	5.0	4.0	4.0	3.0
	$P(v)$		.01563	.01563	.04688	.04688	.09375	.09375	.18750
$N=8$	v	8.0	7.5	6.5	6.0	6.0	5.5	4.5	4.0
	$P(v)$	.00391	.01172	.02344	.03906	.03906	.05469	.09766	.13281
$N=9$	v	10.0	9.0	8.0	7.0	7.0	6.0	6.0	5.0
	$P(v)$	.00391	.01172	.02344	.04688	.04688	.08203	.08203	.12891
$N=10$	v	10.5	10.0	9.0	8.5	8.0	7.5	6.5	6.0
	$P(v)$	.00977	.01367	.02441	.03321	.04492	.05762	.08789	.10938

TABLE 5.2

Upper percentage points t for the null distribution of the $(N+1)T^{(1)}$ statistic for sample sizes $N = 4(1)10$ and significance levels bracketing $\alpha = .010, .025, .050,$ and $.100$.
$P(t) = \Pr\{(N+1)T^{(1)} \geqq t\}$

		Significance levels α							
		.010		.025		.050		.100	
$N=4$	t		10.135		10.135		10.135	10.135	9.101
	$P(t)$		.06250		.06250		.06250	.06250	.12500
$N=5$	t		16.517		16.517	16.517	15.214	14.214	13.214
	$P(t)$		.03125		.03125	.03125	.06250	.09375	.12500
$N=6$	t		24.340	24.340	22.783	21.783	20.783	19.784	19.227
	$P(t)$		.01563	.01563	.03125	.04688	.06250	.09375	.10938
$N=7$	t	33.582	31.785	30.785	29.785	28.785	27.987	26.785	25.987
	$P(t)$	.00781	.01563	.02344	.03125	.04688	.05469	.08594	.10156
$N=8$	t	42.197	41.197	39.197	38.197	37.197	36.197	34.169	33.169
	$P(t)$	.00781	.01172	.02344	.03125	.04297	.05469	.08984	.11328
$N=9$	t	51.005	50.754	48.005	47.754	45.504	44.754	41.754	41.504
	$P(t)$	.00977	.01172	.02344	.02734	.04688	.05469	.09375	.10352
$N=10$	t	62.195	61.729	58.195	57.729	54.729	54.263	50.729	50.263
	$P(t)$	.00879	.01074	.02441	.02832	.04980	.05469	.09668	.10645

TABLE 5.3

Upper percentage points t for the null distribution of the T_G statistic for sample sizes $N = 4(1)10$ and significance levels bracketing $\alpha = .010, .025, .050,$ and $.100$. $P(t) = \Pr\{T_G \geqq t\}$

		Significance levels α							
		.010		.025		.050		.100	
$N=4$	t		.33333		.33333		.33333	.33333	.30667
	$P(t)$		.06250		.06250		.06250	.06250	.12500
$N=5$	t		.47222		.47222	.47222	.38889	.36111	.27778
	$P(t)$		.03125		.03125	.03125	.06250	.09375	.12500
$N=6$	t		.57823	.57823	.57143	.42857	.42177	.39456	.32653
	$P(t)$		.01563	.01563	.03125	.04688	.06250	.09375	.10938
$N=7$	t	.71875	.66146	.63542	.57813	.48958	.46875	.40104	.39063
	$P(t)$	.00781	.01563	.02344	.03125	.04688	.05469	.09375	.10156
$N=8$	t	.81481	.72840	.67901	.61728	.55556	.54321	.43210	.41975
	$P(t)$	.00781	.01172	.02344	.02734	.04688	.05078	.09375	.10156
$N=9$	t	.81000	.79333	.70000	.69667	.59000	.57667	.46000	.45667
	$P(t)$	.00977	.01172	.02344	.02539	.04883	.05078	.09961	.10156
$N=10$	t	.86501	.86226	.76033	.73829	.63361	.62810	.47934	.47658
	$P(t)$	.00977	.01074	.02414	.02539	.04980	.05078	.09961	.10156

TABLE 5.4

Normal approximate rejection limits and resultant actual significance levels for $N = 10$ and desired levels $\alpha = .01, .025,$ and $.05$ for $(N+1)V$, T_G, $(N+1)T^{(1)}$, and $(N+1)T^{(2)}$

	desired α	.01	.025	.050
$(N+1)V$	limit	10.563	8.901	7.471
	actual level	.00586	.02441	.05762
T_G	limit	.89706	.74398	.61224
	actual level	.00781	.02441	.06078
$(N+1)T^{(1)}$	limit	62.543	58.182	54.418
	actual level	.00781	.02441	.04980
$(N+1)T^{(2)}$	limit	66.467	61.856	57.888
	actual level	.00684	.02344	.04980

are listed. This table shows that the normal approximation is very good for $\alpha = .025$ and $.050$; in fact for $T^{(1)}$, the normal and exact (see Table 5.2) rejection limits are equivalent.

Next, the powers of functions of W, $T^{(1)}$ and $T^{(2)}$ are considered for shift alternatives, i.e., it is assumed that the distribution of the X's is of the form $F(x) = G(x-\theta)$, $\theta > 0$, for some continuous distribution G symmetric about zero. For these alternatives, W, $T^{(1)}$ and $T^{(2)}$ are asymptotically equivalent in the sense that their relative Pitman efficiencies are one (assuming conditions on G under

TABLE 5.5

Monte Carlo estimates of the power of W, $T^{(1)}$ and $T^{(2)}$ for normal, double exponential and logistic shift alternatives

Normal shift

	θ = 0	.25	.50	.75	1.00	1.25	1.50
			$N = 10$, $\alpha = .05$				
W	.049	.184	.400	.692	.893	.967	.992
$T^{(1)}, T^{(2)}$	.049	.172	.395	.672	.882	.962	.992
			$N = 10$, $\alpha = .01$				
W	.010	.049	.157	.358	.621	.829	.951
$T^{(1)}, T^{(2)}$	.010	.046	.151	.356	.606	.823	.946
			$N = 20$, $\alpha = .05$				
W	.048	.280	.678	.939	.987	.999	
$T^{(1)}$	.047	.278	.662	.934	.985	.999	
$T^{(2)}$	.047	.275	.661	.931	.985	.999	
			$N = 20$, $\alpha = .01$				
W	.006	.079	.370	.757	.960	.998	
$T^{(1)}$	.007	.085	.349	.733	.950	.998	
$T^{(2)}$	.007	.081	.341	.728	.948	.998	

Double exponential shift

	θ = 0	.25	.50	.75	1.00	1.25	1.50
			$\alpha = .05$, $N = 10$				
W	.049	.170	.336	.559	.731	.855	.922
$T^{(1)}, T^{(2)}$	.050	.166	.340	.557	.732	.851	.924
			$\alpha = .01$, $N = 10$				
W	.010	.046	.134	.270	.423	.581	.716
$T^{(1)}, T^{(2)}$	.010	.044	.129	.269	.422	.578	.716
			$\alpha = .05$, $N = 20$				
W	.055	.239	.540	.821	.948	.983	.997
$T^{(1)}$	.055	.250	.577	.839	.953	.985	.998
$T^{(2)}$	.055	.251	.579	.840	.953	.998	.998
			$\alpha = .01$, $N = 20$				
W	.006	.073	.264	.555	.782	.918	.976
$T^{(1)}$	.007	.072	.284	.574	.808	.933	.983
$T^{(2)}$	.007	.072	.284	.574	.808	.934	.981

TABLE 5.5 *(continued)*

Logistic shift							
	θ						
	0	.25	.50	.75	1.00	1.25	1.50
			$\alpha = .05$,	$N = 10$			
W	.050	.115	.213	.344	.510	.661	.792
$T^{(1)}, T^{(2)}$	.048	.108	.209	.337	.512	.648	.792
			$\alpha = .01$,	$N = 10$			
W	.009	.027	.063	.128	.233	.343	.481
$T^{(1)}, T^{(2)}$	.011	.026	.060	.122	.229	.341	.475
			$\alpha = .05$,	$N = 20$			
W	.055	.147	.346	.568	.782	.916	.971
$T^{(1)}$	.055	.158	.352	.587	.782	.918	.973
$T^{(2)}$	.055	.159	.349	.588	.783	.918	.973
			$\alpha = .01$,	$N = 20$			
W	.011	.036	.112	.283	.483	.711	.863
$T^{(1)}$	.012	.041	.113	.286	.490	.711	.868
$T^{(2)}$	.012	.041	.113	.282	.487	.713	.866

which the efficiencies can be computed). This is because the part of $T^{(1)}$ and $T^{(2)}$ containing the sign statistic is of a smaller order of magnitude than the part containing the Wilcoxon statistic. This holds in general for each $T(\Delta_N)$ when Δ_N satisfies (4.4) with $0 < c < \infty$. Thus for this model one can only compare power functions for finite sample sizes. Monte Carlo estimates of the power of W, $T^{(1)}$ and $T^{(2)}$ are given in Table 5.5 for normal shift alternatives $F(x) = \Phi(x-\theta)$, double exponential shift alternatives $F(x) = K(x-\theta)$, where $k(x) = \frac{1}{2}\exp(-|x|)$, and logistic shift alternatives $F(x) = L(x-\theta)$, where $L(x) = [1+\exp(-x)]^{-1}$. For normal shift, W is slightly better than $T^{(1)}$ and $T^{(2)}$, while for double exponential shift there is no difference in the power functions when $N = 10$, while when $N = 20$, $T^{(1)}$ and $T^{(2)}$ are slightly better than W. W is locally most powerful for logistic shift alternatives, however Table 5.5 shows that it is not better than $T^{(1)}$ and $T^{(2)}$ for $N = 20$ and the values of θ given. Table 5.5 is computed using the normal approximations to the rejection limits of W, $T^{(1)}$ and $T^{(2)}$ when $N = 20$.

Acknowledgment. We wish to thank the M.I.T. Electrical Engineering Department for use of its PDP-1 computer.

REFERENCES

[1] Bell, C. B. and Doksum, K. A. (1966). Distribution-free tests of independence. *Ann. Math. Statist.* **38** 429–446.

[2] Bell, C. B., Moser, J. M. and Thompson, Rory (1966). Goodness criteria for two-sample distribution free tests. *Ann. Math. Statist.* **37** 133–142.

[3] Birnbaum, Z. W. (1953). On the power of a one-sided test of fit for continuous distribution functions. *Ann. Math. Statist.* **24** 284–289.

[4] Capon, J. (1965). On the asymptotic efficiency of the Kolmogorov–Smirnov test. *J. Amer. Statist. Assoc.* **60** 843–853.

[5] Chapman, D. G. (1958). A comparative study of several one-sided goodness-of-fit tests. *Ann. Math. Statist.* **29** 655–674.

[6] Doksum, K. A. (1966). Asymptotically minimax distribution-free procedures. *Ann. Math. Statist.* **37** 619–628.

[7] Govindarajulu, Z. (1960). Central limit theorems and asymptotic efficiency for one-sample nonparametric procedures. Technical Report No. 11, Department of Statistics, Univ. of Minnesota.

[8] Gross, S. (1966). Nonparametric tests when nuisance parameters are present. Ph.D. thesis, Statistics Department, Univ. of California, Berkeley.

[9] Gupta, M. K. (1967). An asymptotically nonparametric test of symmetry. *Ann. Math. Statist.* **38** 849–866.

[10] Hammersley, J. M. and Hanscomb, D. C. (1964). *Monte Carlo Methods.* Methuen and Co., London.

[11] Hodges, J. L. and Lehmann, E. L. (1962). Rank methods for combination of independent experiments in analysis of variance. *Ann. Math. Statist.* **33** 482–497.

[12] Hoeffding, W. (1951). "Optimum" nonparametric tests. *Proc. Second Berkeley Symp. Math. Statist. Prob.* 83–92. Univ. of California Press.

[13] Ruist, E. (1954). Comparison of tests for non-parametric hypothesis. *Ark. Mat.* **3** 133–163.

[14] Savage, I. R. (1959). Contributions to the theory of rank order tests—the one sample case. *Ann. Math. Statist.* **30** 1018–1023.

[15] Thompson, Rory, Govindarajulu, Z. and Doksum, K. A. (1967). Distribution and power of the absolute normal scores test. *J. Amer. Statist. Assoc.* **62** 966–975.