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A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY ORDER STATISTICS[1]

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0. Introduction. Several results characterizing the exponential distribution have appeared in the literature in recent years (Basu [1], Crawford [2], Ferguson [3], Govindarajulu [4] and Tanis [6]). Many of these results are based on the independence of suitable functions of order statistics. Here a different type of theorem, which characterizes the exponential distribution, is given and the key idea is to present a function of the order statistics having the same distribution as the one sampled. As a consequence a result on the characterization of a power distribution is obtained.

1. The results. Let X be a random variable with the distribution function $F(\cdot)$. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from F and let $W = \min(X_1, X_2, \cdots, X_n)$.

THEOREM. *If $F(\cdot)$ is a nondegenerate distribution function, then for each positive integer n, nW and X are identically distributed if and only if $F(x) = 1-\exp(-\lambda x)$, for $x \geqq 0$, where λ is a positive constant.*

PROOF. The distribution function of nW is given by

$$F_{nW}(w) = \Pr\{W \leqq w/n\} = 1-[1-F(w/n)]^n. \tag{1}$$

It is easy to verify that $F_{nW}(w) = F_X(w) \equiv F(w)$ when

$$\begin{aligned} F(w) &= 1-\exp(-\lambda w), & w &\geqq 0 \\ &= 0 & w &< 0. \end{aligned} \tag{2}$$

The main task is to show that for real w

$$F_{nw}(w) = F(w) \Rightarrow F(w) \qquad \text{is given by (2).} \tag{3}$$

Let $G(w) = 1-F(w)$. Now $F_{nW}(w) = F(w)$ can be expressed as

$$G(nw) = G^n(w) \quad \text{for real} \quad w \quad \text{and integers} \quad n \geqq 1. \tag{4}$$

From (4), it follows that $G(0) = G^n(0)$, and this implies that $G(0) = 0$ or $G(0) = 1$, since $0 \leqq G(w) \leqq 1$. We shall now show that $G(0) = 1$.

To prove this, let us assume that there is a negative number w_0 such that $G(w_0) < 1$. Then, as $n \to \infty$, $G^n(w_0) \to 0$ and $G(nw_0) \to 1$, contradicting (4). Thus, for all $w < 0$, $G(w) \geqq 1$ and hence $G(w) = 1$, for $w < 0$. Since F is nondegenerate we cannot have $G(w) = 0$ for all $w \geqq 0$, which implies that $G(w_1) > 0$

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for some $w_1 \geqq 0$. If $w_1 = 0$, then $G(0) > 0$, and hence $G(0) = 1$. On the other hand if $G(w_1) > 0$ for $w_1 > 0$, then $G(w) > 0$ for $0 \leqq w \leqq w_1$, since G is non-increasing. From this observation and the fact that G is a right continuous function, it follows that $G(0) = 1$. Thus

$$G(w) = 1 \quad \text{for} \quad w \leqq 0,$$
$$G(w) > 0 \quad \text{for} \quad 0 < w \leqq w_1. \tag{5}$$

Suppose there is a constant $w_2 > w_1$ such that $G(w_2) = 0$. Then, from (4), $G(w_2/n) = 0$, for integers $n \geqq 1$. This implies $G(w) = 0$ for $w > w_2/n$; so for sufficiently large n, $G(w) = 0$, when $w_2/n \leqq w \leqq w_1$. This contradicts (5). Hence $G(w) > 0$ for $w > 0$.

We proceed as in, say, Karlin ([6], cf. Theorem 2.2, page 182) to determine explicitly the function $G(w)$ for $w > 0$. From (4), we have that

$$G(n/m) = [G(1)]^{n/m} \quad \text{for integers} \quad m, n > 0. \tag{6}$$

Since $G(w)$ and $[G(1)]^w$ are both non-increasing and coincide for positive rational w, and $[G(1)]^w$ is continuous, it follows that $G(w) = [G(1)]^w = \exp\{w \ln G(1)\}$ for all $w > 0$. But F is a nondegenerate distribution function, and so $\lim_{w\to\infty} G(w) = 1 - \lim_{w\to\infty} F(w) = 0$; this implies that $G(1) < 1$. Hence

$$G(w) = \exp(-\lambda w), \qquad \text{for} \quad w > 0$$
$$= 1 \qquad \text{for} \quad w \leqq 0$$

where $\lambda = -\ln G(1)$. This completes the proof.

Taking $X = -\ln Y$ and $X_i = -\ln Y_i (1 \leqq i \leqq n)$, we obtain the following result from the above theorem.

COROLLARY. *Let Y be a nondegenerate random variable with the distribution function $H(y)$ where $H(0) = 0$. Let $(Y_1, Y_2, \cdots, Y_n)$ $(n \geqq 2)$ be a random sample from $H(\cdot)$. Then Y and $Z = \max(Y_1^n, Y_2^n, \cdots, Y_n^n)$ are identically distributed (for each integer $n \geqq 2$) if and only if $H(y) = y^\lambda$ for some $\lambda > 0$.*

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