The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 816–819

USE OF TRUNCATED ESTIMATOR OF VARIANCE RATIO IN RECOVERY OF INTER-BLOCK INFORMATION[1]

By K. R. Shah

University of Waterloo and Cornell University

1. Introduction. As is well known, a preliminary step in the recovery of inter-block information is to estimate the ratio of inter-to intra-block variances. Under the infinite models generally used in the literature, the true value of this ratio exceeds unity. However, when this ratio is estimated using the data, the estimate may turn out to be less than unity. In such cases it is usually recommended that the value of this estimate should be taken as unity. As pointed out by Yates (1939) this amounts to estimating the treatment effects as in a completely randomised design.

Many authors have considered combined inter-and intra-block estimators of treatment differences based on an untruncated estimator of the variance ratio. However, it is usually felt that it would be better to use a truncated estimator of variance ratio. (Stein (1966) has conjectured this.) In this paper it is shown that in *any* incomplete block design for a class of estimators of variance ratio (which includes the ones considered by the above authors) truncation at any point less than the true value leads to a smaller variance for a combined estimator of a treatment difference. (If previous experience with the experimental material indicates that unity is not a safe lower bound one might consider truncation at zero.)

A table is presented to demonstrate that much of the gain due to recovery of inter-block information could be lost by using an untruncated estimator of the variance ratio.

2. Main results. Results of this section can be readily given using the notation used in Shah (1964) and hence we shall use this notation without introducing it explicitly. In Shah (1964), the variance of $\bar{t}_s(\rho^*)$, the combined estimate of τ_s, the the sth canonical treatment contrast using ρ^* as an estimate of ρ, the variance ratio was expressed as

$$V(\bar{t}_s(\rho^*)) = V(\bar{t}_s(\rho)) + [c_s^{\,2}/a_{0s}^2(1+\rho c_s)^2]E(\omega_s^{\,2}) \tag{2.1}$$

where $\bar{t}_s(\rho)$ is the estimate of τ_s based on true value of ρ and ω_s is defined as $\omega_s = [(\rho^* - \rho)/(1+\rho^* c_s)]z_s$. [$c_s$, a_{0s} and z_s are all defined in Shah (1964). z_s is in fact proportional to the difference between intra-and inter-block estimates of τ_s, while c_s and a_{0s} are constants depending upon design parameters.] It was shown in Roy and Shah (1962) that (2.1) holds provided that

$$E\{\omega_s(\rho^*) = 0 \quad \text{and} \quad V\{\omega_s(\rho^*)\} < \infty. \tag{2.2}$$

Received March 25, 1969; revised September 23, 1970.

[1]Work supported in part by NIH grant PHS–GM–05900 and NRC grant A 7272.

Following Shah (1964) we consider a statistic P of the form

$$(2.3)\qquad P = \frac{aS_1 + \sum_{s=1}^{q} b_s z_s^2}{S_0} - d$$

where S_0 and S_1 are intra-and inter-block error sums of squares respectively, $q+1$ is the rank of the incidence matrix of the design and a, b_s, d are all nonnegative constants. Let

$$(2.4)\qquad \rho^* = \begin{cases} P & \text{if } P \geqq \rho_0, \\ \rho_0 & \text{otherwise;} \end{cases}$$

where ρ_0 is nonnegative.

It is easily seen that

$$(2.5)\qquad \omega_s^2(P) \geqq \omega_s^2(\rho^*)$$

provided that

$$(2.6)\qquad d \leqq c_s^{-1}$$

and

$$(2.7)\qquad \rho_0 \leqq \rho.$$

Equality holds in (2.4) only if $P \geqq \rho_0$. Consequently,

$$(2.8)\qquad E\{\omega_s^2(P)\} > E\{\omega_s^2(\rho^*)\}$$

and hence

$$(2.9)\qquad V\{\bar{t}_s(P)\} > V(\bar{t}_s(\rho^*)\}.$$

Using an expression for variance of combined estimator given by Roy and Shah (1962) one can easily derive a similar result for *any* treatment contrast.

Some comments on the form of P and on the conditions appear to be desirable. All estimates of ρ used in the literature are based on a statistic of form P. Conditions (2.2) appear to be natural. In fact it is not obvious if one can find an estimator $\hat{\rho}$ for which $V\{\bar{t}_s(\hat{\rho})\}$ is finite and at the same time these conditions are not satisfied.

Condition (2.6) is given in the nature of a sufficient condition and it appears that the result should be true even when this is not satisfied. In the case of BIB designs, $c_1 = c_2 = \cdots = c_q = v(k-1)/(v-k)$ and it is readily checked that (2.6) is satisfied in the case of estimators used by Graybill and Weeks (1959), Graybill and Deal (1959), Seshadri (1963) and Stein (1966).

Condition (2.7) raises the problem of proper choice of ρ_0. Under the customary infinite model $\rho \geqq 1$. This is also the truncation used by Yates (1939). However, when one has reasons to believe that ρ could take values less than unity one might use $\rho_0 = 0$.

3. Illustrations. In this section we compare efficiency factors for combined estimators of treatment differences when these estimators are based on untruncated and truncated estimators of the variance ratio. For an estimator $\hat{\rho}$ of ρ, one may define the efficiency factor $\bar{E}(\hat{\rho})$ by means of the relationship

$$\bar{E}(\hat{\rho}) = \frac{2\sigma_0^2}{r\bar{V}(\hat{\rho})} \tag{3.1}$$

where $V(\hat{\rho})$ denotes the average variance of combined estimators of all estimated paired contrasts.

TABLE 1

Efficiency factors for some selected designs

Design	ρ	P_1 Truncated	P_1 Untruncated	P_2 Truncated	P_2 Untruncated	Intra-block Efficiency Factor
BIB	1	.9333	.9091	.9691	.7143	.8889
$b = v = 4$	2	.9109	.8995	.9225	.7870	
$r = k = 3$	4	.8991	.8943	.8958	.8333	
$\lambda = 2$	8	.8934	.8916	.8857	.8598	
BIB	1	.9282	.8929	.9636	.8182	.8000
$b = 10, v = 6$	2	.8631	.8491	.8722	.8100	
$r = 5, k = 3$	4	.8292	.8252	.8249	.8053	
$\lambda = 2$	8	.8137	.8128	.8075	.8027	
BIB	1	.9379	.8972	.9641	.8571	.7500
$b = 12, v = 9$	2	.8410	.8276	.8453	.8077	
$r = 4, k = 3$	4	.7923	.7899	.7929	.7800	
$\lambda = 1$	8	.7705	.7702	.7664	.7653	
BIB	1	.9840	.9729	.9901	.9677	.8571
$b = 21, v = 15$	2	.9177	.9158	.9177	.9133	
$r = 7, k = 5$	4	.8867	.8867	.8857	.8855	
$\lambda = 2$	8	.8720	.8720	.8714	.8714	
Simple	1	.7816	.6750	.8616	.5454	.5000
Lattice	2	.6514	.6045	.6771	.5294	
with	4	.5727	.5579	.5770	.5172	
$v = 16$	8	.5340	.5306	.5231	.5094	

We shall consider two statistics of the form P, to be called P_1 and P_2 and obtain efficiency factors for both the truncated and the untruncated forms for each of these (in each case truncation will be at $\rho_0 = 1$). In accordance with (2.4), the truncated forms of P_1 and P_2 shall be denoted by $\rho_1{}^*$ and $\rho_2{}^*$ respectively.

For balanced incomplete block (BIB) designs, P_1 and P_2 are defined by

$$P_1 = \frac{(v-k)e_0}{v(k-1)(v-3)} \frac{\sum z_s^2}{S_0} - \frac{v-k}{v(k-1)} \tag{3.2}$$

$$P_2 = \frac{(v-k)e_0}{v(k-1)(v-1)} \frac{\sum z_s^2}{S_0} - \frac{v-k}{v(k-1)}. \tag{3.3}$$

Seshadri (1963) has used P_1. The author has used ρ_2*, i.e. the truncated form of P_2 in Shah (1964). We present a table giving efficiency factors for four BIB designs and the simple lattice design with sixteen treatments. For the simple lattice design we use suitably modified forms of P_1 and P_2 and the efficiency factors relate to a set of treatment contrasts on which inter-block information is available.

Table 1 indicates that the loss in efficiency due to the use of untruncated form is very considerable when ρ is small. This is especially so in the case of P_2. The loss is much smaller for designs with larger values of v and e_0, the number of treatments and the number of error degrees of freedom respectively.

REFERENCES

[1] GRAYBILL, F. A. and DEAL, R. B. (1959a). Combining unbiased estimators. *Biometrics* **15** 543–550.

[2] GRAYBILL, F. A. and WEEKS, D. L. (1959b). Combining inter-block and intra-block information in balanced incomplete blocks. *Ann. Math. Statist.* **30** 799–805.

[3] ROY, J. and SHAH, K. R. (1962). Recovery of inter-block information. *Sankhyā, Ser. A* **24** 269–280.

[4] SESHADRI, V. (1963). Combining unbiased estimators. *Biometrics* **19** 163–170.

[5] SHAH, K. R. (1964). Use of inter-block information to obtain uniformly better estimators. *Ann. Math. Statist.* **35** 1064–1078.

[6] STEIN, C. (1966). An approach to the recovery of inter-block information in balanced incomplete block designs. Research papers in Statistics ed. F. N. David. Wiley, New York. 351–366.

[7] YATES, F. (1939). The recovery of inter-block information in varietal trials arranged in three dimensional lattices. *Ann. Eugenics* **9** 136–156.