The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 784–788

LOCAL THEOREMS IN STRENGTHENED FORM FOR LATTICE RANDOM VARIABLES

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1. Introduction. Let $\{X_n\}$ be a sequence of independent integral-valued lattice random variables such that the distribution of X_n is one of the distinct nondegenerate distributions $H_1, \cdots, H_r$ $(r \geqq 2)$. With the assumption that there are sequences $\{A_n\}$ and $\{B_n\}$ $(0 < B_n \to \infty)$ such that $Z_n \equiv B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a nondegenerate distribution G, this paper investigates some conditions which are sufficient for $\{X_n\}$ to satisfy a local theorem in strengthened form.

2. Discussions and results. V. M. Kruglov [2] noted that a result of A. A. Zinger [5] implies that G has an absolutely integrable characteristic function ϕ and, therefore, a bounded density g.

We say that a local limit theorem holds for $\{X_n\}$ if

$$\lim_{n\to\infty} \sup_{N\in Z} |B_n P(\sum_{i=1}^n X_i = N) - g((N/B_n) - A_n)| = 0,$$

where Z denotes the set of all integers.

We say that $\{X_n\}$ satisfies a local theorem in strengthened form (L.T.S.) if a local limit theorem (in the usual form) holds for any sequence $\{X_n'\}$ which differs from $\{X_n\}$ only by a finite number of terms.

Since it is possible that for some i among $1, \cdots, r$ the number of times that H_i appears among the distributions of $X_1, \cdots, X_n$ eventually does not depend on n, we let $F_1, \cdots, F_k$ be those among $H_1, \cdots, H_r$ for which this does not occur. Let $n_i(n)$ denote the number of times that F_i appears among the distributions of $X_1, \cdots, X_n$. Then $n_i(n) \to \infty$ as $n \to \infty$ for $i = 1, \cdots, k$. Let h_i denote the maximum span of F_i for $i = 1, \cdots, k$.

LEMMA (Petrov). *A necessary condition that* $\{X_n\}$ *satisfies* L.T.S. *is that*

$$\text{g.c.d.}(h_1, \cdots, h_k) = 1. \tag{1}$$

(Here, as usual, g.c.d. means the greatest common divisor.)

The proof of this Lemma is contained in the proof of Theorem 1 of [4].

Because of this lemma, we will always assume that (1) holds.

Received February 9, 1970.

[1] Reproduction in whole or in part is permitted for any purpose of the United States Government. This research was supported, in part, by the Office of Naval Research, Contract No. N 0001 4–69–A–0423, Nr. 042–261.

THEOREM 1. *If* $h_1 = \cdots = h_k = 1$, *then* $\{X_n\}$ *satisfies* L.T.S.

In [5], it is shown that the Lévy spectral function N of G is of the form, for $x > 0$,

$$N(\pm x) = \textstyle\sum_{i=1}^{l} a_i^{\pm} x^{-\lambda_i}\{1+\sum_{j=1}^{k_i}(A_{ij}^{\pm}\cos v_{ij}\log x + B_{ij}^{\pm}\sin v_{ij}\log x)\},$$

where $l \leqq k$, $0 < \lambda_1 < \cdots < \lambda_l < 2$, $v_{ij} > 0$, $1+2k_i \leqq l_i$, $\sum_{i=1}^{l} l_i \leqq l$.

THEOREM 2. *If there is an i such that*

$$\lim_{n\to\infty}[cn_i(n)-(1/\lambda_l)\log n] = \infty, \quad \textit{for any constant } c > 0, \tag{2}$$

and the maximum span of F_i *is equal to unity, then* $\{X_n\}$ *satisfies* L.T.S.

We remark that the limit condition on $n_i(n)$ is satisfied if there is $\delta > 0$ such that $\liminf n_i(n)/n^{\delta} > 0$.

THEOREM 3. *If* F_i *is in the domain of attraction of a stable law with characteristic exponent* λ_i *with* $0 < \lambda_1 \leqq \cdots \leqq \lambda_k = 2$, *and if G has a Gaussian component, then* $\{X_n\}$ *satisfies* L.T.S.

Note that in order for G in Theorem 3 to have a Gaussian component it is necessary that $\lambda_k = 2$.

THEOREM 4. *If G is the composition of k stable laws with characteristic exponents* λ_i *such that* $0 < \lambda_1 < \cdots < \lambda_k = 2$, *then* $\{X_n\}$ *satisfies* L.T.S.

In Theorem 1, the case where $k = 1$ is proved in [1] page 235 (with a slight modification). Theorem 3 generalizes a Theorem in [4] where $\lambda_1 = \cdots = \lambda_k = 2$.

3. Proofs and lemmas. Let v_i, f_n denote the characteristic functions of X_i, Z_n, respectively.

LEMMA A (Kruglov). *There is a neighborhood of the origin not depending on n such that for t in this neighborhood*

$$|f_n(t)| \leqq \exp\{-c|t|^{\beta}\},$$

where c, β are positive constants.

LEMMA B (Kruglov). *There is* $A > 0$ *such that* $B_n \leqq An^{1/\lambda_l}$ *for all n.*

In a slightly different setting, these lemmas are proved in [2].

PROOF OF THEOREM 1. Letting $x = (N/B_n)-A_n$, we have

$$\begin{aligned} &B_nP(\textstyle\sum_{i=1}^{n} X_i = N)-g(x) \\ &= \frac{1}{2\pi}\left\{B_n\int_{-\pi}^{\pi}\exp\{-itN\}\prod_{j=1}^{n} v_j(t)\,dt - \int_{-\infty}^{\infty}\exp\{-itx\}\phi(t)\,dt\right\} \\ &= \frac{1}{2\pi}\left\{\int_{-B_n\pi}^{B_n\pi}\exp\{-itx\}f_n(t)\,dt - \int_{-\infty}^{\infty}\exp\{-itx\}\phi(t)\,dt\right\}. \end{aligned}$$

Hence,

$$\begin{aligned}\sup_N &\left|B_n P\left(\textstyle\sum_{i=1}^n X_i = N\right) - g(x)\right| \\ &= \frac{1}{2\pi}\sup_N \Big|\int_{|t|<L} \exp\{-itx\}(f_n(t)-\phi(t))\,dt \\ &\quad + \int_{L<|t|<\varepsilon B_n} \exp\{-itx\} f_n(t)\,dt \\ &\quad + \int_{\varepsilon B_n<|t|<\pi B_n} \exp\{-itx\} f_n(t)\,dt \\ &\quad - \int_{|t|>L} \exp\{-itx\}\phi(t)\,dt\Big| \\ &\leqq \frac{1}{2\pi}(I_1+I_2+I_3+I_4),\end{aligned}$$

where

$$I_1 = \int_{|t|<L} |f_n(t)-\phi(t)|\,dt, \qquad I_2 = \int_{L<|t|<\varepsilon B_n} |f_n(t)|\,dt,$$
$$I_3 = \int_{\varepsilon B_n<|t|<\pi B_n} |f_n(t)|\,dt, \qquad I_4 = \int_{|t|>L} |\phi(t)|\,dt,$$

and $0 < \varepsilon < \pi$, $L > 0$.

Since f_n converges to ϕ uniformly on bounded intervals, I_1 can be made arbitrarily small for fixed L by choosing n sufficiently large.

Since ϕ is absolutely integrable, I_4 can be made arbitrarily small by choosing L sufficiently large.

By letting ε be sufficiently small to apply Lemma A,

$$I_2 \leqq \int_{L<|t|} \exp\{-c|t|^\beta\}\,dt.$$

Hence, by choosing L sufficiently large, I_2 can be made arbitrarily small.

By a corollary of ([1] page 60), for each j there is a constant $a_j > 0$ such that for $\varepsilon B_n < |t| < \pi B_n$, $|v_j(t/B_n)| \leqq \exp(-a_j)$.

Letting $a = \min\{a_1, a_2, \cdots\}$ and using Lemma B, we have

$$\begin{aligned}I_3 &= \int_{\varepsilon B_n<|t|<\pi B_n} \textstyle\prod_{j=1}^n |v_j(t/B_n)|\,dt \\ &\leqq 2\pi B_n \exp\{-an\} \leqq 2\pi A n^{1/\lambda_l} e^{-an}.\end{aligned}$$

Hence, for n sufficiently large, I_3 is arbitrarily small.

PROOF OF THEOREM 2. Letting I_1, I_2, I_3 and I_4 be as before, it suffices to show that I_3 can be made arbitrarily small by choosing n sufficiently large. By Lemma B, we have

$$\begin{aligned}I_3 &\leqq 2\pi B_n \exp\{-a_i n_i(n)\} \\ &\leqq 2\pi A \exp -\{a_i n_i(n) - (1/\lambda_l)\log n\}.\end{aligned}$$

By (2), this last bound may be made as small as desired.

PROOF OF THEOREM 3. Without loss of generality, we assume

$$P(X_n = 0) \geqq P(X_n = a) \qquad \text{for all } n \text{ and all } a. \tag{3}$$

All that is required is to show that I_3 can be made arbitrarily small. To this end we utilize a method suggested in [4].

We designate by d the g.c.d. of all integers m which correspond to a positive probability under at least one of the distributions $F_1, \cdots, F_k$. By (3), each of the numbers $h_1, \cdots, h_k$ is divisible by d. Hence, from (1) $d = 1$. Thus, the g.c.d. of all integers m such that

$$\textstyle\sum_{i=1}^{\infty} P(X_i = m) = \infty \tag{4}$$

is equal to unity.

Therefore, it follows that there is a positive integer M_0 such that the g.c.d. of all integers m, with $|m| < M_0$, for which (4) holds is equal to unity. Clearly, M_0 may be chosen so that there is an integer $m_1 \neq 0$ such that $|m_1| < M_0$ and $P(X_n = m_1) > 0$ for all n. Set $M = \max(M_0, \frac{1}{2}\varepsilon)$, where $\varepsilon > 0$ is the same as in I_2. Then $I_3 \leqq B_n(R_1 + R_2)$, where,

$$R_1 = \textstyle\int_{\frac{1}{2}M}^{\pi} \prod_{j=1}^{n} |v_j(t)|\, dt, \qquad R_2 = \int_{-\pi}^{-\frac{1}{2}M} \prod_{j=1}^{n} |v_j(t)|\, dt.$$

Let

$$p_{jm} = P(X_j = m), \qquad \tilde{p}_{jm} = \textstyle\sum_s p_{j,m+s} p_{js}.$$

Then

$$\begin{aligned} \textstyle\prod_{j=1}^{n} |v_j(t)| &\leqq \exp\{(\tfrac{1}{2}) \textstyle\sum_{j=1}^{n} (|v_j(t)|^2 - 1)\} \\ &= \exp\{(\tfrac{1}{2}) \textstyle\sum_{j=1}^{n} \sum_m \tilde{p}_{jm} (\cos mt - 1)\}. \end{aligned}$$

Denote the points in the segment $[\frac{1}{2}M, \pi]$ of the form $2\pi r/m$ (r and m relatively prime, $1 \leqq r \leqq [m/2]$, $2 \leqq m \leqq M$) taken in increasing order by $t_1, \cdots, t_\nu$. Evidently, $t_\nu = \pi$, $t_1 > \frac{1}{2}M$. If we set

$$\begin{aligned} \Delta_1 &= [\tfrac{1}{2}M, (t_1 + t_2)/2], \\ \Delta_\mu &= [(t_{\mu-1} + t_\mu)/2, (t_\mu + t_{\mu+1})/2], \qquad \text{for } \mu = 2, \cdots, \nu - 1, \\ \Delta_\nu &= [(t_{\nu-1} + t_\nu)/2, \pi], \end{aligned}$$

we can represent R_1 as a sum of integrals over Δ_μ, $\mu = 1, \cdots, \nu$.

We will examine a fixed segment Δ_μ containing the point t_μ. Let $t_\mu = 2\pi r_0/m_0$. Clearly,

$$\begin{aligned} \textstyle\sum_m \tilde{p}_{jm} (\cos mt - 1) \leqq &-2\textstyle\sum' \tilde{p}_{jm} \sin^2(mt/2) \\ &-2\textstyle\sum'' \tilde{p}_{jm} \sin^2(mt/2), \end{aligned}$$

where Σ' is the sum over all integers m with $|m| < M$ and $m \not\equiv 0 \pmod{m_0}$, and Σ'' is the sum over all integers m with $|m| < M$, $m \equiv 0 \pmod{m_0}$, $m \neq 0$. The minimum distance between points t_μ is not smaller than $2\pi/M^2$. Hence, for any

$t \in \Delta_\mu$, any $m \not\equiv 0 \pmod{m_0}$, $|m| < M$, and any integer s, we have $|mt - 2\pi s| > \varepsilon_1$; therefore, $\sin^2(mt/2) > \varepsilon_2$ (the constants $\varepsilon_1, \varepsilon_2, \cdots$ are positive and depend only on M). For $t \in \Delta_\mu$, for $m \equiv 0 \pmod{m_0}$, $|m| < M$, $m \neq 0$, we have $\sin^2(mt/2) \geqq \varepsilon_3(t-t_\mu)^2$.

Hence, for $t \in \Delta_\mu$

$$\prod_{j=1}^n |v_j(t)| \leqq \exp\{-g_n - k_n(t-t_\mu)^2\},$$

where

$$g_n = \varepsilon_2 \sum_{j=1}^n \sum' \tilde{p}_{jm}, \qquad k_n = \varepsilon_3 \sum_{j=1}^n \sum'' \tilde{p}_{jm}.$$

By (3) and (4), $\lim_{n\to\infty} g_n = \infty$. By (4), by the existence of m_1, and by the fact that there is a constant $b > 0$ such that $B_n \sim b(n_k(n))^{\frac{1}{2}}$ (see [3]), we have $B_n^2 = O(g_n + k_n)$.

For n such that $g_n \geqq k_n$,

$$B_n \int_{\Delta_\mu} \prod_{j=1}^n |v_j(t)|\, dt \leqq B_n \exp\{-g_n\}.$$

For n such that $g_n < k_n$,

$$B_n \int_{\Delta_\mu} \prod_{j=1}^n |v_j(t)|\, dt \leqq B_n(\pi/k_n)^{\frac{1}{2}} \exp\{-g_n\}.$$

Since both of these bounds go to zero as $n \to \infty$, R_1 can be made arbitrarily small. Clearly, R_2 permits the same treatment. Thus, $\lim I_3 = 0$.

Proof of Theorem 4. This follows from Theorem 3 due to the fact that if G is the composition of k stable laws with distinct characteristic exponents, then the distributions $F_1, \cdots, F_k$ can be re-indexed so that F_i is in the domain of attraction of a stable law with exponent λ_i, $i = 1, \cdots, k$ ([5], Theorem 3).

REFERENCES

[1] Gnedenko, B. V. and Kolmogorov, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*, tr. K. L. Chung. Addison-Wesley, Reading.

[2] Kruglov, V. M. (1968). A local limit theorem for non-identically distributed random variables. *Theor. Probability Appl.* **13** 332–334.

[3] Mason, J. D. (1970). Convolutions of stable laws as limit distributions of partial sums. *Ann. Math. Statist.* **41** 101–114.

[4] Petrov, V. V. (1965). Limit theorems for k-sequences of independent random variables. *Litovsk. Mat. Sb.* **5** 443–455. (In Russian.)

[5] Zinger, A. A. (1965). A class of limit distributions for normalized sums of independent random variables. *Theor. Probability Appl.* **10** 607–626.