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ON THE MULTIPLE AUTOREGRESSIVE SERIES

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0. Summary. This paper deals with the finite part of multiple autoregressive series. The conditions of stationarity are derived and the inverse of the covariance matrix is evaluated (without assumption of stationarity).

1. Introduction. The models based on the autoregressive series are widely used in many fields. Their applications in economics, geophysics, hydrology etc. are well known. If we know the covariance matrix of a finite part of random series, the best linear unbiased estimates may be constructed for unknown parameters occurring in the trend. But this requires the inverse of the covariance matrix. This problem must be solved theoretically, as numerical methods are not available with respect to the order of the covariance matrix, which is commonly large.

The inverse of the covariance matrix of one-dimensional stationary autoregressive series is given in [1] in some implicit form. The further papers, namely [10], [7], ([5] Example 5B), [2] and [3] contain explicit results.

We shall keep following notation. A number $\bar{a}$ is complex conjugate to a. If $\mathbf{A} = \|a_{jk}\|$ is a matrix, then $\bar{\mathbf{A}} = \|\bar{a}_{jk}\|$. Matrix $\mathbf{A}'$ is transpose of $\mathbf{A}$. The covariance matrix of a random vector $\mathbf{Z} = (Z_1, \cdots, Z_q)'$ is denoted either by $\operatorname{Var} \mathbf{Z}$, or by $\operatorname{Var}(Z_1, \cdots, Z_q)$, whereas the covariance matrix between random vectors $\mathbf{V}$ and $\mathbf{Z}$ is $\operatorname{Cov}(\mathbf{V}, \mathbf{Z})$. It means that for $E\mathbf{V} = E\mathbf{Z} = \mathbf{0}$ we have $\operatorname{Cov}(\mathbf{V}, \mathbf{Z}) = E\mathbf{V}\mathbf{Z}'$. The unit and null matrices are denoted by $\mathbf{I}$ and $\mathbf{0}$, respectively.

Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be p-dimensional random vectors with zero mean values. Denote

$$\mathbf{B} = \operatorname{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_n')$$

the covariance matrix of np-dimensional vector $(\mathbf{X}_1', \cdots, \mathbf{X}_n')'$. Let $\mathbf{Y}_{n+1}, \cdots, \mathbf{Y}_N$ be p-dimensional random vectors such that

$$E\mathbf{Y}_t = \mathbf{0}, \qquad \operatorname{Var} \mathbf{Y}_t = \mathbf{I}, \qquad \operatorname{Cov}(\mathbf{Y}_s, \mathbf{Y}_t) = \mathbf{0}, \qquad \operatorname{Cov}(\mathbf{X}_k, \mathbf{Y}_t) = \mathbf{0}$$
$$\text{for} \quad 1 \leqq k \leqq n < s, t \leqq N.$$

Define vectors $\mathbf{X}_{n+1}, \cdots, \mathbf{X}_N$ by the recurrent formula

$$\sum_{j=0}^{n} \mathbf{A}_j \mathbf{X}_{t-j} = \mathbf{Y}_t, \qquad n < t \leqq N, \tag{1}$$

where $\mathbf{A}_0, \cdots, \mathbf{A}_n$ are given matrices of the type $p \times p$, the elements of which are real. Suppose that $|\mathbf{A}_0| \neq 0$, $\mathbf{A}_n \neq \mathbf{0}$. Then the series $\mathbf{X}_1, \cdots, \mathbf{X}_N$ is called (general) multiple autoregressive series, or, more precisely, the finite part of it. The numbers

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n and N are the order and the length of the series, respectively. The assumption $\operatorname{Var} \mathbf{Y}_t = \mathbf{I}$ may be made without any loss of generality. Another form of (1) is

$$\mathbf{X}_t = \sum_{j=1}^n \mathbf{U}_j \mathbf{X}_{t-j} + \mathbf{W}_t, \qquad n < t \leqq N, \tag{2}$$

where $\mathbf{U}_j = -\mathbf{A}_0^{-1}\mathbf{A}_j$, $\mathbf{W}_t = \mathbf{A}_0^{-1}\mathbf{Y}_t$.

2. Conditions of stationarity.

DEFINITION 1. Put $\mathbf{G} = \operatorname{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_N')$ and write $\mathbf{G}$ in terms of the $p \times p$ blocks $\mathbf{G}_{st}$, $\mathbf{G} = \|\mathbf{G}_{st}\|_{s,t=1}^N$. We say that $\mathbf{G}$ is stationary, if

$$\mathbf{G}_{st} = \mathbf{G}_{s+k,t+k} \quad \text{for} \quad 1 \leqq s, t, s+k, t+k \leqq N.$$

LEMMA 2. *The matrix* $\mathbf{G}$ *is stationary if and only if*

$$\operatorname{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_{N-1}') = \operatorname{Var}(\mathbf{X}_2', \cdots, \mathbf{X}_N').$$

PROOF. Is clear.

LEMMA 3. *If*

$$\operatorname{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_n') = \operatorname{Var}(\mathbf{X}_2', \cdots, \mathbf{X}_{n+1}'), \tag{3}$$

then under (1) $\operatorname{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_h') = \operatorname{Var}(\mathbf{X}_2', \cdots, \mathbf{X}_{h+1}')$ *holds for* $n \leqq h < N$.

PROOF. For $h = n$ the assertion holds. Further we use the induction. Let the assertion hold for some $h-1$, where $n < h < N-1$. In order to prove it for h it suffices to find out that

$$\operatorname{Cov}(\mathbf{X}_{h+1}, \mathbf{X}_{j+1}) = \operatorname{Cov}(\mathbf{X}_h, \mathbf{X}_j) \qquad \text{for} \quad 1 \leqq j \leqq h. \tag{4}$$

From (2) we get

$$\begin{aligned} \operatorname{Cov}(\mathbf{X}_{h+1}, \mathbf{X}_{j+1}) &= \sum_{i=1}^n \mathbf{U}_i \operatorname{Cov}(\mathbf{X}_{h+1-i}, \mathbf{X}_{j+1}) + \operatorname{Cov}(\mathbf{W}_{h+1}, \mathbf{X}_{j+1}), \\ \operatorname{Cov}(\mathbf{X}_h, \mathbf{X}_j) &= \sum_{i=1}^n \mathbf{U}_i \operatorname{Cov}(\mathbf{X}_{h-i}, \mathbf{X}_j) + \operatorname{Cov}(\mathbf{W}_h, \mathbf{X}_j). \end{aligned} \tag{5}$$

Let $1 \leqq j < h$. Then according to induction $\operatorname{Cov}(\mathbf{X}_{h+1-i}, \mathbf{X}_{j+1}) = \operatorname{Cov}(\mathbf{X}_{h-i}, \mathbf{X}_j)$ holds for $1 \leqq i \leqq n$; further, $\operatorname{Cov}(\mathbf{W}_h, \mathbf{X}_j) = \operatorname{Cov}(\mathbf{W}_{h+1}, \mathbf{X}_{j+1}) = \mathbf{0}$, and (4) is proved for $j < h$. Now, we repeatedly use the formulas (5) for $j = h$ and with respect to $\operatorname{Cov}(\mathbf{W}_h, \mathbf{X}_h) = \operatorname{Cov}(\mathbf{W}_{h+1}, \mathbf{X}_{h+1}) = (\mathbf{A}_0'\mathbf{A}_0)^{-1}$ we easily finish the proof.

THEOREM 4. *Define the matrices*

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ \mathbf{U}_n & \mathbf{U}_{n-1} & \mathbf{U}_{n-2} & \cdots & \mathbf{U}_1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}_0'\mathbf{A}_0)^{-1} \end{bmatrix},$$

both of them of the order $pn \times pn$. Then the relation (3) *holds if and only if the matrix* $\mathbf{B} = \mathrm{Var}\,(\mathbf{X}_1', \cdots, \mathbf{X}_n')$ *satisfies the equation*

$$\mathbf{B} = \mathbf{MBM}' + \mathbf{\Lambda}. \tag{6}$$

Equation (6) *for* $\mathbf{B}$ *has unique symmetric positive definite solution if and only if all the characteristic roots of the matrix* $\mathbf{M}$ *have their absolute values smaller than* 1. *The characteristic roots of* $\mathbf{M}$ *are the same as roots of polynomial*

$$K(z) = \left|\sum_{j=0}^{n} \mathbf{A}_j z^{n-j}\right|,$$

where $|\cdots|$ *on the right-hand side denotes the determinant* (*cf.* [4], page 192).

PROOF. It is clear that

$$\begin{bmatrix} \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_n \\ \mathbf{X}_{n+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_{n-1} \\ \mathbf{X}_n \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{A}_0^{-1}\mathbf{Y}_{n+1} \end{bmatrix}$$

and (3) is satisfied if and only if (6) holds.

We obtain by induction with respect to n that $|z\mathbf{I}-\mathbf{M}| = |\mathbf{A}_0^{-1}\sum_{j=0}^{n} z^{n-j}\mathbf{A}_j| = |\mathbf{A}_0^{-1}|K(z)$ and thus the characteristic roots of $\mathbf{M}$ are the same as roots of $K(z)$.

The rest of the proof is similar to that for one-dimensional series belonging to Professor Hájek (see [3]). Consider the series

$$\mathbf{\Lambda} + \mathbf{M\Lambda M}' + \mathbf{M}^2\mathbf{\Lambda M}'^2 + \cdots = \mathbf{B}. \tag{7}$$

If all the characteristic roots of $\mathbf{M}$ have their absolute values smaller than 1, then we obtain from the Perron formula that (7) converges in the norm. Its sum $\mathbf{B}$ satisfies (6) obviously. Using (6) repeatedly, we get that the solution must be of the form (7) and it implies unicity. Matrix $\mathbf{B} = \sum_{t=0}^{\infty} \mathbf{M}^t\mathbf{\Lambda M}'^t$ is clearly symmetric and positive semidefinite. In order to prove its definiteness, consider arbitrary p-dimensional vectors $\mathbf{y}_1, \cdots, \mathbf{y}_n$ and put $\mathbf{y} = (\mathbf{y}_1', \cdots, \mathbf{y}_n')'$. If $\mathbf{y}_n \neq \mathbf{0}$, then $\mathbf{y}'\mathbf{B}\mathbf{y} \geqq \mathbf{y}'\mathbf{\Lambda}\mathbf{y} > 0$. If $\mathbf{y}_p \neq \mathbf{0}$ and $\mathbf{y}_{p+1} = \cdots = \mathbf{y}_n = \mathbf{0}$ for some $p = 1, 2, \cdots, n-1$, then $\mathbf{M}'^{n-p}\mathbf{y} = (\mathbf{0}', \cdots, \mathbf{0}', \mathbf{y}_1', \cdots, \mathbf{y}_p')'$ and

$$\mathbf{y}'\mathbf{B}\mathbf{y} \geqq \mathbf{y}'\mathbf{M}^{n-p}\mathbf{\Lambda M}'^{n-p}\mathbf{y} > 0.$$

If z is a characteristic root of $\mathbf{M}$, then it is that of $\mathbf{M}'$, too. Suppose $|z| \geqq 1$. A vector $\mathbf{y} = (\mathbf{y}_1', \cdots, \mathbf{y}_n') \neq \mathbf{0}$ exists such that $\mathbf{M}'\mathbf{y} = z\mathbf{y}$. If it were $\mathbf{y}_n = \mathbf{0}$, then it would be

$$\mathbf{M}'\mathbf{y} = (\mathbf{0}', \mathbf{y}_1', \cdots, \mathbf{y}_{n-1}')' = z(\mathbf{y}_1', \cdots, \mathbf{y}_{n-1}, \mathbf{0}')'$$

and $\mathbf{y} = \mathbf{0}$. Consequently $\mathbf{y}_n \neq \mathbf{0}$. We obtain from (6)

$$\bar{\mathbf{y}}'\mathbf{B}\mathbf{y} = \bar{\mathbf{y}}'\mathbf{MBM}'\mathbf{y} + \bar{\mathbf{y}}'\mathbf{\Lambda}\mathbf{y} = |z|^2\bar{\mathbf{y}}'\mathbf{B}\mathbf{y} + \bar{\mathbf{y}}'\mathbf{\Lambda}\mathbf{y}.$$

This need not take place when $\mathbf{B}$ is a positive definite matrix because of $\bar{\mathbf{y}}'\mathbf{\Lambda}\mathbf{y} > 0$. The proof is finished.

COROLLARY 5. *If* $\mathbf{B}$ *is the positive definite solution of* (6) *then the matrix* $\mathbf{G}$ *is stationary.*

PROOF. Follows from Lemma 2 and Lemma 3 and Theorem 4.

3. The inverse of covariance matrix. Let us return to the general model (1) without assumption $\mathbf{G}$ to be stationary.

THEOREM 6. *Let* $\mathbf{B} = \mathrm{Var}(\mathbf{X}_1', \cdots, \mathbf{X}_n')$ *be regular and write its inverse* $\mathbf{B}^{-1} = \mathbf{E} = ||\mathbf{E}_{st}||_{s,t=1}^{n}$ *in terms of* $p \times p$ *blocks* $\mathbf{E}_{st}$. *Then the matrix* $\mathbf{G}$ *is regular and for its inverse* $\mathbf{G}^{-1} = \mathbf{H} = ||\mathbf{H}_{st}||_{s,t=1}^{N}$ *written in* $p \times p$ *blocks* $\mathbf{H}_{st}$ *the following formulas hold*:

$$\mathbf{H}_{st} = \mathbf{E}_{st} + \sum_{k=n+1}^{\min(n+s,n+t,N)} \mathbf{A}'_{k-s}\mathbf{A}_{k-t} \quad \textit{for} \quad 1 \leqq s, t \leqq n;$$

$$\mathbf{H}_{st} = \sum_{k=\max(s,t)}^{\min(n+s,n+t,N)} \mathbf{A}'_{k-s}\mathbf{A}_{k-t} \quad \textit{otherwise}.$$

Especially, $\mathbf{H}_{st} = \mathbf{0}$ *when* $|s-t| > n$.

PROOF. Define the $pN \times pN$ matrix

$$\mathbf{C} = \left[\begin{array}{cccc|cccccc} & \mathbf{E}^{\frac{1}{2}} & & & & & \mathbf{0} & & & \\ \hline \mathbf{A}_n & \mathbf{A}_{n-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_n & \cdots & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_n & \cdots & \mathbf{A}_0 \end{array}\right]$$

One sees that

$$\mathbf{C}\begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{X}_N \end{bmatrix} = \begin{bmatrix} \mathbf{E}^{\frac{1}{2}}\begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{bmatrix} \\ \mathbf{Y}_{n+1} \\ \vdots \\ \vdots \\ \mathbf{Y}_N \end{bmatrix}$$

and so $\mathbf{CGC}' = \mathbf{I}$. Thus $\mathbf{G}$ must be regular and we have $\mathbf{H} = \mathbf{C}'\mathbf{C}$. Using the expression in terms of the blocks we obtain the formulas for $\mathbf{H}_{st}$.

If the one-dimensional autoregressive series is stationary, then its covariance matrix $\mathbf{H}$ is persymmetric, as it is well known. This enables us to find an explicit formula for the matrix $\mathbf{E}$, if $N \geqq 2n$. Unfortunately, $\mathbf{H}$ is not generally persymmetric "in the blocks" in stationary case if $p > 1$. It may be shown on an example. Then the matrix $\mathbf{E}$ must be determined from $\mathbf{B}$ separately to obtain all the elements of $\mathbf{H}$. This complication is connected with the time reversing problem.

Matrix **B** may be evaluated numerically using either (6) or (7). Another method may be based on the formulas given in [6] or on the spectral theory (see [8] and [9]).

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