The Annals of Mathematical Statistics
1971, Vol. 42, No. 3, 1135-1137

CORRECTION NOTE

CORRECTIONS TO "STRONG CONSISTENCY OF CERTAIN SEQUENTIAL ESTIMATORS"

By Robert H. Berk
Rutgers University

In the above paper (*Ann. Math. Statist.* **40** 1492–1495), the main consistency result (Theorem 3.4) uses Theorem 2.7 to justify the assertion that $\mathscr{C}^* = \downarrow \lim_i \mathscr{C}_{t_i} \equiv \{\varnothing, \Omega\}$. However, Theorem 2.7 as stated is incorrect: $\mathscr{C}^* = \downarrow \lim \mathscr{C}_{N_i}$ can properly contain $\mathscr{C}_{N_\infty}$. Proposition 1 below illustrates an instance of this. To get around this difficulty, Proposition 2 below gives a sufficient condition that $\mathscr{C}^* \equiv \{\varnothing, \Omega\}$. The condition is seen to hold for a large variety of examples, including those considered in the paper.

Proposition 1. *Let N be a random index and let $N_n = \max\{N, n\}$, $n = 1, 2, \cdots$. Then for any decreasing sequence $\{\mathscr{C}_i\}$ with $\mathscr{C}_\infty = \downarrow \lim_i \mathscr{C}_i$, $\{N_n: 1 \leqq n \leqq \infty\}$ is C-ordered and $\mathscr{C}_{N_n} \downarrow \mathscr{B}(\mathscr{C}_\infty, (N < \infty))$, the σ-field generated by $\mathscr{C}_\infty$ and the set $(N < \infty)$.*

Proof. If $C \in \mathscr{C}_{N_n}$,

$$(1) \qquad C = \textstyle\sum_{k=1}^\infty C_{kn}(N_n = k) \cup C_{\infty n}(N_n = \infty)$$
$$= C_n(N \leqq n) \cup \textstyle\sum_{n+1}^\infty C_{kn}(N = k) \cup C_\infty(N = \infty),$$

where $C_{kn} \varepsilon \mathscr{C}_k$, $1 \leqq k \leqq \infty$ and we write $C_{nn} = C_n$ and $C_{\infty n} = C_\infty$ (note that the latter set does not depend on n). Clearly any set in $\mathscr{C}_{N_{n+1}}$ is of this form (with $C_n = C_{n,n+1} = C_{n+1}$), so $\mathscr{C}_{N_{n+1}} \subset \mathscr{C}_{N_n}$. Thus $\{N_n: 1 \leqq n \leqq \infty\}$ is C-ordered. Let $C^* = \downarrow \lim_n \mathscr{C}_{N_n}$. We note that for all n, $\mathscr{C}_\infty \subset \mathscr{C}_{N_n}$ and $(N < \infty) = (N_n < \infty) \in \mathscr{C}_{N_n}$. Hence $\mathscr{C}^* \supset \mathscr{B}(\mathscr{C}_\infty, (N < \infty))$. This already contradicts Theorem 2.7, which asserts in this case that $\mathscr{C}^* = \mathscr{C}_\infty$.

To establish the reverse inclusion for $\mathscr{C}^*$, choose $C \in \mathscr{C}^*$. Then for all n, C has a representation as in (1). Fix m. For $n > m$, it follows from (1) that $C(N \leqq m) = C_m(N \leqq m) = C_n(N \leqq m)$. Thus $\lim 1_{C_n} = 1$ on $(N \leqq m)$. Let $C_{\infty *} = \limsup C_n \in \mathscr{C}_\infty$. Then $C(N \leqq m) = C_{\infty *}(N \leqq m)$. Letting $m \to \infty$ then shows that $C = C_{\infty *}(N < \infty) \cup C_\infty(N = \infty)$. Thus $\mathscr{C}^* \subset \mathscr{B}(\mathscr{C}_\infty, (N < \infty))$. □

We note that if N is a stopping time in Proposition 1, then so are the N_n. Thus Theorem 2.7 is not even true in general for C-ordered stopping times. If one adds the hypothesis $N_\infty < \infty$ with probability one, Theorem 2.7 is true and the proof given is valid. (Whether the theorem remains true under the weaker hypothesis: for all i, $N_i < \infty$ with probability one, is not known. Note that in Proposition 1, $N_n < \infty$ with probability one if and only if $N < \infty$ with probability one and then $\mathscr{C}^* \equiv \mathscr{C}_\infty$.) Of course the case of primary interest in the paper is $N_\infty \equiv \infty$, so some suitable alternative to Theorem 2.7 seems necessary.

In the sequel we assume the structure of Section 3. Moreover, all random indices are assumed to be $\mathscr{A}_\infty$ measurable. Let Σ_n denote the permutation group on the first n positive integers and let $\Sigma = \bigcup_n \sum_n$ be all finite permutations of the positive integers. An element σ in Σ acts on $(x_1, x_2, \cdots)$ by sending it into $(x_{\sigma 1}, x_{\sigma 2}, \cdots)$. For a random index N, we let $\sigma N = N^\circ \sigma^{-1}$. Note that $\sigma(N = k) = (\sigma N = k)$ and, since the $\{x_i\}$ are i.i.d., N and σN are equidistributed.

Definition. A random index N is called tail-symmetric if for every σ in Σ there is an integer p so that N and σN coincide on $(N > p, \sigma N > p)$. That is, for all $k > p$, $(N = k)(N > p, \sigma N > p) = (\sigma N = k)(N > p, \sigma N > p)$, or

$$\forall k > p, (N = k, \sigma N > p) = (\sigma N = k, N > p). \tag{2}$$

Of course such a p, if it exists, is not unique. Then we denote by $p(\sigma, N)$ the least positive integer p for which (2) holds. A collection $\{N_i\}$ of random indices is called homogeneously tail-symmetric if each is tail-symmetric and for every σ in Σ, $\sup_i p(\sigma, N_i) < \infty$.

Proposition 2. *Suppose* $N_1 \leqq N_2 \leqq \cdots$ *are C-ordered and* $\lim N_i = +\infty$ *with probability one. If, in addition, the* $\{N_i\}$ *are homogeneously tail-symmetric, then* $\mathscr{C}_{N_i} \downarrow \mathscr{C}^* \equiv \{\varnothing, \Omega\}$.

Proof. The C-ordering implies that $\mathscr{C}_{N_i}$ decreases, to $\mathscr{C}^*$, say. Choose $C \in \mathscr{C}^*$. Since $C \in \mathscr{C}_{N_i}$, $C = \Sigma C_k(N_i = k)$, where $C_k \in \mathscr{C}_k$, $1 \leqq k \leqq \infty$. Choose an integer m, an element σ of Σ_m and let $n = \max\{m, \sup_i p(\sigma, N_i)\}$. Then $C(N_i > n) = \Sigma_{k>n} C_k(N_i = k)$, where $C_k \in \mathscr{C}_k$, for all $k > n$. Thus $\sigma\{C(N_i > n)\} = \sigma C(\sigma N_i > n) = \sum_{k>n} C_k(\sigma N_i = k)$, since for $k > n \geqq m$ the sets in $\mathscr{C}_k$ are Σ_m-invariant. Thus

$$\begin{aligned} \sigma C(\sigma N_i > n)(N_i > n) &= \textstyle\sum_{k>n} C_k(\sigma N_i = k)(N_i > n) \\ &= \textstyle\sum_{k>n} C_k(N_i = k)(\sigma N_i > n) = C(N_i > n)(\sigma N_i > n), \end{aligned} \tag{3}$$

where the second equality in (3) follows from homogeneous tail-symmetry. Since N_i and σN_i are equidistributed, it follows that $\sigma N_i \uparrow +\infty$ with probability one. Letting $i \to \infty$ in (3) then shows that $\sigma C = C$ with probability one. Since $\sigma \in \Sigma$ is arbitrary, that $\mathscr{C}^* \equiv \{\varnothing, \Omega\}$ follows from the Hewitt–Savage 0–1 law. □

Remark. Since the $\mathscr{C}_{N_i}$ decrease, it is enough for the conclusion of Proposition 2 to hold that some infinite subset of $\{N_i\}$ be homogeneously tail-symmetric.

Theorem 3.4 is then correct if one adds the requirement that the stopping times $\{t_i\}$ be homogeneously tail-symmetric. We show next that the structure assumed in Theorem 3.5 assures this, in addition to the C-ordering. Specifically, we isolate the following sufficient condition that a random index N be tail-symmetric.

Proposition 3. *Suppose there are sets* $\{D_n\}$, $D_n \in \mathscr{B}(z_n, x_{n+1}, \cdots)$, *so that for all* n, $(N > n) = \bigcap_1^n D_k$. *Then N is tail-symmetric and if* $\sigma \in \Sigma_n$, $p(\sigma, N) \leqq n$.

PROOF. Choose n, $\sigma \in \Sigma_n$ and $k > n$. We note that $(N = k) = D_k^c \bigcap_1^{k-1} D_i$ and that for $i \geqq n$, D_i is Σ_n-invariant. It follows directly that

$$(N = k, \sigma N > n) = (\sigma N = k, N > n) = D_k^c \bigcap_n^{k-1} D_i \bigcap_1^{n-1} \{D_i \cap \sigma D_i\}. \ \square$$

REMARK. In Theorem 3.5, the condition of Proposition 3 is satisfied for t_i with $D_n = (v_n \notin V_{ni})$. Thus such $\{t_i\}$ are homogeneously tail-symmetric.

Regarding the examples, it is easily seen that Proposition 3 applies to (i), (iii) and (iv). In example (ii), it is easily checked that $\{t_i\}$ is homogeneously tail-symmetric. In fact, if $\sigma \in \Sigma_n$, then for $i \geqq n$, $\sigma t_i = t_i$.