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OPTIMAL DESIGNS WITH A POLYNOMIAL SPLINE REGRESSION WITH A SINGLE MULTIPLE KNOT AT THE CENTER

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0. Summary. In this paper it is shown that the optimal design for estimating any specific parameter in a polynomial spline regression with a single multiple knot at the center is supported by one of two sets of points. Familiarity with the notation and terminology used in the paper by Studden (1968) is assumed.

1. Introduction. As regression functions we consider the $(2n-k+2)$ linearly independent and continuous functions $\{f_i\}_0^n\ U\{g_j\}_{j=k}^n$ where

$$f_i = x^i; \qquad i = 0, \quad 1, 2, \cdots, n$$

$$g_j = x_+{}^j; \qquad j = k, k+1, \cdots, n$$

defined on $[-1, 1]$ and

$$x_+{}^i = x^i \quad \text{if } x \geqq 0;$$
$$= 0 \quad \text{if } x < 0.$$

The regression coefficient associated with f_i is denoted by θ_i; $i = 0, 1, \cdots, n$ and that associated with g_j is denoted by θ_j'; $j = k, k+1, \cdots, n$.

A "polynomial" is a linear combination of these $(2n-k+2)$ functions. We now show the existence and uniqueness of two polynomials $W(x)$, and $W_1(x)$ and state some of their properties.

2. Polynomials W and W_1.

THEOREM 1. *There exists a unique polynomial $W(x)$ unique up to $\pm$ satisfying*

(i) *odd if k is even and even if k is odd.*

(ii) $|W(x)| \leqq 1$.

(iii) *The set $E = \{x \colon |W(x)| = 1\}$ contains precisely $(2n-k+2)$ points. These points include -1, 0, and 1 and the remaining $(2n-k-1)$ points are symmetrically located about 0 if k is odd. If k is even these points include -1 and 1 and the remaining $2n-k$ points are symmetrically located about 0.*

(iv) *$W(x)$ attains its supremum at each of the points of the set E with alternating signs and is of the form*

$$\textstyle\sum_{j=0}^{l} a_{2j+\delta} x^{2j+\delta} + \sum_{j=0}^{[(n-k)/2]} a_{k+2j}[x^{k+2j} - 2x_+{}^{k+2j}]$$

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where

$$l = \left[\frac{n}{2}\right] \qquad \text{if } \delta = 0,$$
$$= \left[\frac{n-2}{2}\right] \qquad \text{if } \delta = 1;$$
$$\delta = 0 \qquad \text{if } k \text{ is odd},$$
$$= 1 \qquad \text{if } k \text{ is even};$$

and the coefficients a_j *are* $\neq 0$.

PROOF. We prove the theorem for the case where both n and k are odd. The proof for other cases is virtually the same. Consider the function $f(x) = 2x_+{}^k - x^k$; clearly $f(x)$ is an even function. Let V be the linear space spanned by $\{f_i\}_0{}^n U\{g_j\}_{k+1}^n$. Then $g(x) \in V$ implies $g(-x) \in V$. Hence there exists a best approximation of $f(x)$, say $h(x)$ with respect to V which is also even (see Meinardus (1967) pages 26–27). Thus $h(x)$ has the form

$$h(x) = \sum_{j=0}^{\frac{1}{2}(n-1)} \alpha_{2j} x^{2j} + \sum_{j=1}^{\frac{1}{2}(n-k)} \alpha_{k+2j}(2x_+{}^{k+2j} - x^{k+2j}).$$

We may thus consider only the space V_1 spanned by

$$\{x^{2j}\}_0{}^{\frac{1}{2}(n-1)} U\{2x_+{}^{k+2j} - x^{k+2j}\}_1{}^{\frac{1}{2}(n-k)}.$$

Each function in V_1 is even and f is even. Therefore a best approximation of f with respect to V_1 is also a best approximation of x^k with respect to the space V_2 spanned by $\{x^{2j}\}_0{}^{1/2(n-1)}\ U\{x^{k+2j}\}_1{}^{1/2(n-k)}$ on the interval $[0, 1]$ and the dimension of V_2 is $n - \frac{1}{2}(k-1)$. But on $[0, 1]$ the spanning set of functions of the space V_2 is a T-system with a unit element and hence best approximation of $f(x)$ with respect to V_2 is unique; i.e. $h(x)$ is unique and $f-h$ possesses precisely $n-\frac{1}{2}(k-3)$ extremal points including the end points 0 and 1 and $f-h$ attains its norm at these points with alternating signs (see Meinardus (1967) page 29). Thus best approximation of f with respect to V_1 on $[-1, 1]$ is unique and has precisely $(2n-k+2)$ extremal points including -1, 0, and 1 at each of which $f(x)-h(x)$ attains its norm with alternating signs.

Set $W(x) = [f(x)-h(x)]/\|f-h\|$

where

$$\|f-h\| = \sup_{-1 \leq x \leq 1} |f(x)-h(x)|.$$

Now it is easily seen that $W(x)$ satisfies all the conditions of Theorem 1. Note that the $(2n-k+2)$ extreme points of $W(x)$ are symmetric about 0 and $n-\frac{1}{2}(k-1)$ are in $[-1, 0)$ and $n-\frac{1}{2}(k-1)$ are in $(0, 1]$. To show that the $W(x)$ constructed above is the only polynomial satisfying (i) to (iv) we assume that there exists another polynomial $R(x)$ having all the properties (i) to (iv). Consider the case where k is odd, and n is odd. Then $R(x)$ must be an even function and hence is of the form

$$R(x) = \sum_{j=0}^{\frac{1}{2}(n-1)} b_{2j} x^{2j} + \sum_{j=0}^{\frac{1}{2}(n-k)} b_{k+2j}(x^{k+2j} - 2x_+{}^{k+2j}).$$

Since $R(x)$ satisfies (iii) it has $n-\frac{1}{2}(k+1)$ extreme points in (0, 1) and an equal number in $(-1, 0)$, together with -1, 0 and 1. Hence it has $n-\frac{1}{2}(k-1)$ distinct zeros in (0 1]. From Descartes' rule it is seen that none of the b's can be zero. Since $R(x)$ satisfies (iv)

$$\frac{1}{b_k}[\sum_{j=0}^{\frac{1}{2}(n-1)} b_{2j}x^{2j} - \sum_{j=1}^{\frac{1}{2}(n-k)} b_{k+2j}x^{k+2j}]$$

is a best approximation of x^k with respect to the linear space spanned by $\{x^{2j}\}_{j=0}^{\frac{1}{2}(n-1)} U\{x^{k+2j}\}_{j=1}^{\frac{1}{2}(n-k)}$ on the interval [0, 1]. This is a contradiction since there is one and only one best approximation of x^k with respect to this space as the spanning set of functions is a T- system. This establishes the uniqueness of W.

THEOREM 2. *There exists a unique polynomial $W_1(x)$ unique up to $\pm$ satisfying*

(i) $W_1(x)$ *is odd or even according as k is odd or even.*

(ii) $|W_1(x)| \leqq 1$.

(iii) *The set* $E_1 = \{x: |W_1(x)| = 1\}$ *contains precisely* $(2n-k+1)$ *points. These points include* -1 *and* 1 *and the remaining* $(2n-k-1)$ *points are symmetrically located about* 0 *if k is odd. If k is even these points include* -1, 0 *and* 1 *and the remaining* $(2n-k-2)$ *points are symmetrically located about* 0.

(iv) $W_1(x)$ *attains its supremum with alternating signs at each of the points of the set* E_1 *and is of the form*

$$\sum_{j=0}^{m} b_{2j+\delta_1}x^{2j+\delta_1} + \sum_{j=0}^{[n-k-2]/2} b_{k+2j+1}(x^{k+2j+1} - 2x_+{}^{k+2j+1})$$

where

$$m = \left[\frac{n-2}{2}\right] \quad \text{if} \quad \delta_1 = 1,$$
$$= \left[\frac{n}{2}\right] \quad \text{if} \quad \delta_1 = 0;$$
$$\delta_1 = 1 \quad \text{if} \quad k \text{ is odd},$$
$$= 0 \quad \text{if} \quad k \text{ is even};$$

and the coefficients b_j *are* $\neq 0$.

If $k = n$, *the terms* $(x^{k+2j+1} - 2x_+{}^{k+2j+1})$ *are omitted. The polynomial* W_1 *in this case is* $T_n(x)$, *the Tchebycheff polynomial of the first kind and of degree n.*

PROOF. The construction of W_1 for $k \leqq n-1$ is exactly similar to that of $W(x)$, except we start with $f_1(x) = 2x_+{}^{k+1} - x^{k+1}$ and consider its best approximation with respect to V_1, spanned by $\{x^i\}_0^n U\{x_+{}^i\}_{k,\, i\neq k+1}^n$ and set $W_1(x) = [f_1(x) - h_1(x)]/\|f_1 - h_1\|$ where h_1 is the unique best approximation of f_1 with respect to V_1. If $k = n$, then clearly $T_n(x)$ satisfies all the conditions of the Theorem 2.

3. Zeros of a polynomial.

THEOREM 3. *Let* $S(n, k; x) = \sum_{i=0}^{n} d_i x^i + \sum_{i=k}^{n} d_i' x_+{}^i$ *with at least one of the* d_i*'s* $= 0$ *for some* $i \geqq k-1$; *then* $S(n, k; x)$ *cannot have more than* $(2n-k)$ *distinct zeros unless it vanishes identically between some two of them.*

PROOF. We first prove the theorem for $k = 1$. Since $d_i = 0$ for some $i > 0$ we consider two cases. (i) $d_0 = 0$. Then S can have at most $(n-1)$ distinct zeros in $[-1, 0)$, and at most $(n-1)$ distinct zeros in $(0, 1]$. Thus it can have at most $(2n-1)$ distinct zeros, including 0. Hence if it has $2n$ distinct zeros it is clearly $\equiv 0$. (ii) If $d_0 \neq 0$ and $d_i = 0$ for some $i \geqq 1$ then S can have at most $(n-1)$ distinct zeros in $[-1, 0)$ and at most n in $(0, 1]$ and thus can have at most $(2n-1)$. Hence $S \equiv 0$ if it has $2n$ distinct zeros. Thus the theorem is true for $k = 1$. Let $k > 1$. If S has $(2n-k+1)$ distinct zeros and does not vanish identically in any interval containing two of these zeros, then we claim that its derivative S', by Rolle's theorem has $(2n-k)$ distinct zeros which are separated by the zeros of S and S' and cannot vanish identically in between any two of these zeros. For suppose S' vanishes identically between two such zeros z_1 and z_2. Then S is a constant on $[z_1, z_2]$ and has one of its distinct zeros in its interior and as such is $\equiv 0$, on $[z_1, z_2]$, a contradiction. Differentiating S, $(k-1)$ times we have

$$S^{(k-1)} = d_{k-1}^* + d_k{}^* x + \cdots + d_n{}^* x^{n-k+1} + \sum_{i=1}^{n-k+1} d_i{}^* x_+^i$$

and since $d_i = 0$ for some $i \geqq k-1$, we have $d_j{}^* = 0$ for some $j \geqq k-1$ and $S^{k-1} = S(N, 1; x)$ where $N = n-k+1$ and has by Rolle's theorem $(2n-k+1)-(k-1) = 2N$ distinct zeros and does not vanish identically in any interval containing two of these zeros. This contradicts the theorem already proved for $k = 1$. Therefore if $S(n, k; x)$ has $2n-k+1$ zeros it must vanish identically between two of them.

4. Minimizing property of the polynomials W and W_1. Let $\mathscr{P}_1$ denote the class of all "Polynomials" $u(x)$ with coefficients of x^p equal to unity where $1 \leqq p \leqq k-1$; $k \geqq 1$, $\mathscr{P}_2$ denote the class of all "Polynomials" $u(x)$ with coefficient of x^p equal to unity $p \geqq k$; $k \geqq 1$ and $\mathscr{P}_3$ be the class of all "Polynomials" $u(x)$ with coefficient of $x_+{}^p$ equal to unity $p \geqq k$.

THEOREM 4.

$$\inf \sup_{u \in \mathscr{P}_1 - 1 \leqq x \leqq 1} |u(x)| = \sup_{-1 \leqq x \leqq 1} |W(x)/a_p| \quad \textit{if } k-p \textit{ is odd,}$$
$$= \sup_{-1 \leqq x \leqq 1} |W_1(x)/b_p| \quad \textit{if } k-p \textit{ is even;}$$
$$\inf \sup_{u \in \mathscr{P}_2 - 1 \leqq x \leqq 1} |u(x)| = \sup_{-1 \leqq x \leqq 1} |W(x)/a_p|;$$
$$\inf \sup_{u \in \mathscr{P}_3 - 1 \leqq x \leqq 1} |u(x)| = \sup_{-1 \leqq x \leqq 1} |W(x)/a_p| \quad \textit{if } p-k \textit{ is even,}$$
$$= \sup_{-1 \leqq x \leqq 1} |W_1(x)/b_p| \quad \textit{if } p-k \textit{ is odd.}$$

PROOF. Let p be even; $p \neq 0$, $p \leqq k-1$ and k be odd so that $k-p$ is odd. Consider the space V, spanned by $\{x^i\}_{i=0,\, i\neq p}^{n} U\{x_+{}^i\}_k{}^n$. If $g(x) \in V$, then so does $g(-x)$. Let $f(x) = x^p$; $x \in [-1, 1]$. $f(x)$ is even, and hence there exists a best approximation $P(x)$ of f with respect to V which is also even.

$$\sup_{-1 \leqq x \leqq 1} |x^p - P(x)| \leqq \sup_{-1 \leqq x \leqq 1} |x^p - Q(x)|$$

where $Q(x)$ is any linear combination of $\{x^i\}_{i=0,\, i\neq p}^{n} U\{x_+{}^i\}_{i=k}^{n}$. If we consider $W(x)/a_p$ ($a_p \neq 0$ from (iv) of Theorem 1) it is readily seen that it is of the form $x^p - Q(x)$

$$\therefore \ \sup_{-1 \leqq x \leqq 1} |x^p - P(x)| \leqq \sup_{-1 \leqq x \leqq 1} |W(x)/a_p| = \|W(x)/a_p\|.$$

Hence the difference $W(x)/a_p - [x^p - P(x)]$ must either vanish at one of the extreme points of $W(x)$ or it has $(2n-k+1)$ distinct zeros in $[-1, 1]$ since the graph of $x^p - P(x)$ must stay within $\|W(x)/a_p\|$, and does not vanish identically in any interval containing two of these zeros. In the first case

$$\|W/a_p\| = \sup_{-1 \leqq x \leqq 1} |W(x)/a_p| = |x^p - P(x)| \leqq \|x^p - P(x)\| \leqq \|W/a_p\|.$$

Hence $\|W/a_p\| = \|x^p - P(x)\|$.

This implies

$$\inf \ \sup_{u \in \mathscr{P}_1 - 1 \leqq x \leqq 1} |u(x)| = \sup_{-1 \leqq x \leqq 1} |W(x)/a_p|.$$

In the second case the difference is easily seen to be of the form

$$\beta_0 + \beta_2 x^2 + \cdots + \beta_{p-2} x^{p-2} + \beta_{p+2} x^{p+2} + \cdots + \beta_{n-1} x^{n-1}$$
$$+ \beta_k (x^k - 2x_+{}^k) + \cdots + \beta_n (x^n - 2x_+{}^n)$$

and vanishes at $n - \frac{1}{2}(k-1)$ points in $[-1, 0)$ and $n - \frac{1}{2}(k-1)$ points in $(0\ 1]$. But from Descartes' rule of signs, it can have at most $n - \frac{1}{2}(k+1)$ zeros in $(0, 1]$. Hence the difference vanishes identically. This completes the proof for this case. The proof of the other cases is treated similarly. In the case where $p \geqq k$ the difference $W(x)/a_p - [x^p - P(x)]$ is of the form $\sum_{0,\, i\neq p}^{n} d_i x^i + \sum_{i=k}^{n} d_i' x_+^i$ and will either vanish at one of the extreme points of W or has $(2n-k+1)$ distinct zeros and does not vanish identically between any two of these. Hence from Theorem 3 it must be identically equal to zero. This completes the proof.

We now recall some of the definitions given in Studden's paper (1968), so as to facilitate ready reference. Let $f_0, f_1, \cdots, f_n$ be $(n+1)$ regression functions and $V(x)$ be the unique polynomial $\sum_{i=0}^{n} a_i^* f_i$ which attains its supremum 1 with alternating signs at $(n+1)$ distinct points $s_0, s_1, \cdots, s_n$ where $s_0 = -1$ and $s_n = 1$, and $s_0 < s_1 \cdots < s_n$. For any vector c with $(n+1)$ components different from the null vector

$$D_\nu(c) = \begin{vmatrix} f_0(s_0) & f_0(s_1) \cdots f_0(s_{\nu-1}) & f_0(s_{\nu+1}) \cdots f_0]s_n] & c_0 \\ f_1(s_0) & f_1(s_1) \cdots f_1(s_{\nu-1}) & f_1(s_{\nu+1}) \quad f_1(s_n) & c_1 \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\ f_n(s_0) & f_n(s_1) \quad f_n(s_{\nu-1}) & f_n(s_{\nu+1}) \quad f_n(s_n) & c_n \end{vmatrix}.$$

If $D_\nu(c) = 0$, the sign of $D_\nu(c)$ may be defined as -1 or $+1$.
c_p is an $(n+1)$ component vector with a one only in the $(p+1)$ *st* component and the rest being zeros. $p = 0, 1, 2, \cdots n$.
R denotes the class of vectors $c = (c_0, c_1, \cdots, c_n)$ such that $\varepsilon D_\nu(c) \geqq 0$ for $\nu = 0, 1, \cdots, n$ where ε is fixed to be $+1$ or -1 for a given vector c. i.e. the $D_\nu(c)$, $\nu = 0, 1, \cdots, n$ all have the same sign in a weak sense.

In our case we have $(2n-k+2)$ regression functions and any vector c will be a $(2n-k+2)$ component vector. c_p will have a one in the $(p+1)$ *st* component and zeros elsewhere for $p = 0, 1, \cdots, n$ c_p' will have a one in the $(p+1)$ *st* component and zeros elsewhere for $p = n+1, n+2, \cdots, (2n-k+1)$. So that if θ denotes the $(2n-k+2)$ component vector $(\theta_0, \theta_1, \cdots, \theta_n, \theta_k', \theta'_{k+1}, \cdots, \theta_n')$ then $(c_p, \theta) = \theta_p$ for $p = 0, 1, \cdots, n$ and $(c_p', \theta) = \theta'_{p+k-(n+1)}$ for $p = n+1, n+2, \cdots, (2n-k+1)$.

5. Optimal designs of individual regression coefficients.

THEOREM 5. *For $p = 0$ the unique c_p-optimum design concentrates mass one at $x = 0$. For $p \neq 0; p < k, k \geqq 1$.*

(a) *If $k-p$ is odd then $c_p \in R$, i.e. the unique optimal design is supported by the full set of the points of the set E.*

(b) *If $k-p$ is even the unique c_p-optimum design is supported by the full set of points of the set E_1.*

(c) *If $p \geqq k$; $k \geqq 2$, then the unique c_p-optimum design is supported by the full set of points of the set E. When $k = 1$, the support is the set $E \cap [-1, 0]$.*

(d) *If $p \geqq k$, $p-k$ is even, the unique c_p'-optimum design is supported by the full set E, and if $p-k$ is odd the c_p'-optimum design is supported by the full set E_1.*

Theorem 4 is analogous to Studden's Theorem 4.2 ((1968) page 1443) and now using the same arguments as given in Lemma 4.1 (Studden (1968) page 1443) we obtain the supports of the optimal designs for estimating $\theta_p(p = 0, 1, \cdots, n)$, and $\theta_p'(p = k, k+1, \cdots, n)$, i.e. the c_p and c_p' optimal designs stated in Theorem 5. We also note that any linear combination of the $(2n-k+2)$ regression functions $\{x^i\}_{i=0}^n \, U\{x_+{}^j\}_{j=k}^n$ can have at most $(2n-k+1)$ distinct zeros; such that the linear combination does not vanish identically on any interval containing two of these zeros. (Lemma 2.2, Schumaker (1967)). Hence these regression functions satisfy Assumption (v) stated on page 1442 of Studden's paper (1968). Hence if we show that the supports are full, then the Kiefer and Wolfowitz (1965) characterization of optimal designs described in Section 3 of Studden's paper (1968) guarantees that $c_p \in R$. Thus we show in the next section that the supports are full.

6. Supports of the optimal designs. Let n and k be odd. Consider θ_j, $j \leqq k-1$; $j \neq 0$ and j even. From Theorem 5 we know that the optimal design for θ_j in this case has for its support the set E consisting of $(2n-k+2)$ points. Let $\{x_i\}_0^{2n-k+1}$ be these points with $x_0 = -1$, $x_{2n-k+1} = 1$, $x_{n-\frac{1}{2}(k-1)} = 0$, and the remaining are symmetric about zero. Moreover $x_0 < x_1 \cdots < x_{2n-k+1}$. Let $\{p_i\}_0^{2n-k+1}$ be the probabilities associated with these points by the optimal design. Then there exists a solution $\{\varepsilon_\nu p_\nu\}$, by Elfving's Theorem (see Studden (1968) page 1437), to the system of equations

$$\beta c_j = \sum_{\nu=0}^{2n-k+1} \varepsilon_\nu p_\nu f(x_\nu); \qquad \text{where}$$

$$\beta^{-1} = \text{coefficient of } x^j \text{ in } W(x), \text{ in absolute value.}$$

Suppose $p_i = 0$ where $i > n-\frac{1}{2}(k-1)$. Then there exists a polynomial

$$P(x) = \sum_{i=0}^{n} d_i x^i + \sum_{i=k}^{n} d_i' x_+^i$$

such that (i) $\Sigma\, d_i^2 + \Sigma\, d_1'^2 > 0$, (ii) $d_j = 0$, and (iii) $P(x_\nu) = 0$ for $\nu \neq i$. Consider $Q(x) = P(x)+P(-x)$, and note that, $x_+^i - (-x)_+^i \equiv x^i$ if i is odd and $x_+^i + (-x)_+^i \equiv x^i$ if i is even. Then

$$Q(x) = P(x)+P(-x) = \sum_{\nu=0,\,\nu\neq j/2}^{\frac{1}{2}(n-1)} 2d_{2\nu}x^{2\nu} + \sum_{\nu=1}^{\frac{1}{2}(n-k)} d'_{k+2\nu-1}x^{k+2\nu-1} + \sum_{\nu=0}^{\frac{1}{2}(n-k)} d'_{k+2\nu}[2x_+^{k+2\nu} - x^{k+2\nu}]$$

and for $x < 0$

$$Q(x) = \sum_{\nu=0,\,\nu\neq j/2}^{\frac{1}{2}(k-1)} 2d_{2\nu}x^{2\nu} + \sum_{\nu=1}^{\frac{1}{2}(n-k)} 2d_{k+2\nu-1}x^{2\nu+k-1} + \sum_{\nu=1}^{\frac{1}{2}(n-k)} d'_{k+2\nu-1}x^{k+2\nu-1} - \sum_{\nu=0}^{\frac{1}{2}(n-k)} d'_{k+2\nu}x^{k+2\nu}.$$

Therefore $Q(x)$ can at most have $n-\frac{1}{2}(k+3)$ zeros in $[-1, 0)$ unless $Q(x)$ is $\equiv 0$ in $[-1, 0)$. (Use Descartes' rule of signs, and note also that $d_0 = 0$; as $P(0) = 0$.) But actually it has $(n-1)-\frac{1}{2}(k-1)$ zeros in $[-1, 0)$, and since $(n-1)-\frac{1}{2}(k-1) > n-\frac{1}{2}(k+3)$ we have

$$d_{2\nu} = 0; \qquad \nu = 0, 1, 2, \cdots, \tfrac{1}{2}(k-1)$$

$$2d_{k+2\nu-1} + d'_{k+2\nu-1} = 0; \qquad \nu = 1, 2, \cdots, \tfrac{1}{2}(n-k)$$

$$d'_{k+2\nu} = 0; \qquad \nu = 0, 1, \cdots, \tfrac{1}{2}(n-k).$$

But this implies that for $x < 0$

$$P(x) = \sum_{\nu=1}^{\frac{1}{2}(n-k)} d_{k+2\nu-1}x^{k+2\nu-1} + \sum_{\nu=0}^{\frac{1}{2}(n-k)} d_{k+2\nu}x^{k+2\nu}$$

and hence can have at most $n-k$ zeros in $[-1, 0)$, but actually has $n-\frac{1}{2}(k-1)$, which implies that

$$d_{k+2\nu} = 0; \qquad \nu = 0, 1, \cdots, \tfrac{1}{2}(n-k)$$

$$d_{k+2\nu-1} = 0; \qquad \nu = 1, 2, \cdots, \tfrac{1}{2}(n-k)$$

i.e. $P(x) \equiv 0$, a contradiction.

The proof that $p_j \equiv 0$ when $j < n-\frac{1}{2}(k-1)$ is exactly similar. When $j = n-\frac{1}{2}(k-1)$, we proceed with $P(x)-P(-x)$ and follow the same arguments. When k is even these roles are reversed.

It may be noted that the above method is exactly similar to the one used by Studden (1968). [See page 1444]. For those θ's whose support is on the set E_1, the proof of $p_i \neq 0$ is reduced to the earlier situation, by dropping the component corresponding to x_+^k, in the system of equations

$$\beta c_j = \textstyle\sum_{\nu=0}^{2n-k} \varepsilon_\nu p_\nu f(t_\nu);$$

β^{-1} is the coefficient of x^j in W_1 in absolute value and $\{t_\nu\}_0{}^{2n-k}$ are the points of the set E_1.

7. Illustrations.

EXAMPLE 1. Let $n = 2$, $k = 1$ and the regression equation be denoted by $\theta_0+\theta_1 x+\theta_2 x^2+\theta_1' x_+ +\theta_2' x_+^2$; $x \in [-1, 1]$. Then the polynomials $W(x)$ and $W_1(x)$ are

$$W(x) = 1+8x+8x^2-16x_+$$

$$W_1(x) = -\frac{2}{c}x-\frac{x^2}{c^2}+\frac{2}{c^2}x_+^2; \qquad c = 2^{\frac{1}{2}}-1.$$

The sets E and E_1 are

$$E = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\}$$

$$E_1 = \{-1, -c, c, 1\}.$$

The optimal designs for θ_1, θ_2, and θ_1' are supported on the set E, with respective weights

$$(\tfrac{1}{8}, \tfrac{4}{8}, \tfrac{3}{8}, 0, 0);$$

$$(\tfrac{1}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0, 0); \qquad \text{and}$$

$$(\tfrac{1}{16}, \tfrac{4}{16}, \tfrac{6}{16}, \tfrac{4}{16}, \tfrac{1}{16}).$$

The optimal design for θ_2' is supported on the set E_1 with weights

$$\left\{\frac{c}{2(1+c)}, \frac{1}{2(1+c)}, \frac{1}{2(1+c)}, \frac{c}{2(1+c)}\right\}.$$

EXAMPLE 2. Let $n = 2$, $k = 2$ and the regression equation be denoted by $\theta_0+\theta_1 x+\theta_2 x^2+\theta_2' x_+^2$; $x \in [-1, 1]$.

The polynomial $W(x)$ is the same as $W_1(x)$ of the previous example, and the set E is the same as E_1 of the previous example. The optimal designs for θ_1, θ_2 and θ_2' are supported on the full set $E_1 = \{-1, -c, c, 1\}$ with respective weights

$$\left\{\frac{c^2}{2(1+c^2)}, \frac{1}{2(1+c^2)}, \frac{1}{2(1+c^2)}, \frac{c^2}{2(1+c^2)}\right\},$$

$$\left\{\frac{c(2+c)}{2(1+c)^2}, \frac{1+2c}{2(1+c)^2}, \frac{1}{2(1+c)^2}, \frac{c^2}{2(1+c)^2}\right\}, \quad \text{and}$$

$$\left\{\frac{c}{2(1+c)}, \frac{1}{2(1+c)}, \frac{1}{2(1+c)}, \frac{c}{2(1+c)}\right\}.$$

EXAMPLE 3. Let $n = 3$, $k = 3$ and the regression equation be denoted by $\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_3' x_+^3$; $x \in [-1, 1]$.

The polynomials $W(x)$ and $W_1(x)$ are

$$W(x) = -1 + 27/2x^2 + 27/2x^3 - 27x_+^3$$

$$W_1(x) = -3x + 4x^3.$$

The sets E and E_1 are

$$E = \{-1, -\tfrac{2}{3}, 0, \tfrac{2}{3}, 1\}$$

$$E_1 = \{-1, -\tfrac{1}{2}, \tfrac{1}{2}, 1\}.$$

The optimal designs for θ_2, θ_3, and θ_3' are supported on the set E with respective weights

$$\{8/108, \quad 27/108, \quad 38/108, \quad 27/108, \quad 8/108\}$$

$$\{32/180, \quad 63/180, \quad 50/180, \quad 27/180, \quad 8/180\} \quad \text{and}$$

$$\{4/36, \quad 9/36, \quad 10/36, \quad 9/36, \quad 4/36\}.$$

The optimal design for θ_1 is supported on E_1 with weights $\{1/18, 8/18, 8/18, 1/18\}$.

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