The Annals of Mathematical Statistics
1971, Vol. 42, No. 3, 904-911

A GENERALIZED DOEBLIN RATIO LIMIT THEOREM[1]

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1. Introduction. Let us consider a discrete time parameter Markov Process $\{X_k, k \geqq 0\}$ with stationary transition probability functions defined on a general measurable state space $(X, \mathscr{B})$ where $\mathscr{B}$ is a separable (countably generated) Borel field of subsets of X containing single point sets. Furthermore, assume the Harris recurrence condition.

CONDITION (C). There exists a sigma-finite measure μ defined on X, with $\mu(X) > 0$ such that for every $S \in \mathscr{B}$ with $\mu(S) > 0$, we have that

$$P[X_k \in S \text{ infinitely often} \mid X_0 = x] = 1$$

for all $x \in X$. This then implies the existence of an invariant sigma-finite measure Π on $(X, \mathscr{B})$, unique up to a constant multiple. (Note that the Π-measure of X may be infinite.) We denote the m-step transition probability from $x \in X$ to $S \in \mathscr{B}$ by $P^{(m)}(x, S)$.

Jain [8] has considered the Doeblin Ratio of these transition probabilities, namely,

$$\frac{\sum_{k=0}^{m} P^{(k)}(x, A_1)}{\sum_{k=0}^{m} P^{(k)}(y, A_2)}$$

where $x \in X$, $y \in X$, $A_i \in \mathscr{B}$ for $i = 1, 2$, with the aforementioned conditions on $(X, \mathscr{B})$. Here Jain proved that this ratio tends to $\Pi(A_1)/\Pi(A_2)$ as $m \to \infty$ for all x, y not in a set $N(A_1, A_2)$ where $\Pi[N(A_1, A_2)] = 0$. Isaac [7] then proved that the dependence of N on A_1 and A_2 could be removed if $A_i \subset S$ for $i = 1, 2$ with $0 < \Pi(S) < \infty$.

Krengel [11] considered the special case where the space consists of one discrete ergodic class which is recurrent, but further generalized the form of the Doeblin Ratio, i.e.,

$$\frac{\sum_{k=1}^{m} \sum_{x \in X} p_x P^{(k)}(x, A_1)}{\sum_{k=1}^{m} \sum_{x \in X} q_x P^{(k)}(x, A_2)}$$

where $\{p_x\}$ and $\{q_x\}$ represent initial probability distributions on the space. He proved that this ratio converges to $\Pi(A_1)/\Pi(A_2)$ as $m \to \infty$ when $\Pi(A_1) < \infty$, $\Pi(A_2) < \infty$ and the initial distributions satisfy certain conditions.

Received February 4, 1970; revised October 30, 1970.

[1] This research was partially supported by the National Science Foundation.

Our main result (Theorem 3.3) is concerned with a generalization of the Doeblin ratio, studied by Krengel for the general measure space $(X, \mathscr{B})$, namely,

$$\frac{\sum_{k=1}^{m} \int_X P^{(k)}(x, A_1)\phi(dx)}{\sum_{k=1}^{m} \int_X P^{(k)}(x, A_2)\psi(dx)}$$

where ϕ and ψ are initial probability distributions on $(X, \mathscr{B})$. We prove that this ratio converges to $\Pi(A_1)/\Pi(A_2)$ as $m \to \infty$ for arbitrary probability distributions ϕ and ψ if the sets A_1 and A_2 satisfy certain additional conditions. (We note that this result is analogous to that of Krengel, although it is not a generalization.) From this we get a corollary which complements the above-mentioned results of Isaac [7]. The corollary yields conditions under which $N(A_1, A_2) = \varnothing$.

Recently, Metivier [13] has considered problems closely related to the above-mentioned material, and has proven theorems similar to those presented here.

A proof of a strengthened version of a theorem of Harris [6] is also presented (Theorem 3.1), the results of which are used by Jain in the proof of his results ([8], Theorem 3.4); howevei, a proof of the former does not appear in the literature.

The last theorem presented in this paper (Theorem 3.4) is that of Isaac ([7], Theorem 1). We give an alternate method of proof following the lines of the theory just developed as indicated to the author by Professor Steven Orey.

2. Definitions. The existence of the invariant sigma-finite measure Π on $(X, \mathscr{B})$, mentioned in the introduction, is a result of the following theorem of Harris [5]:

THEOREM. *Let Condition* (C) *hold. Then there exists a sigma-finite measure* Π *on* $(X, \mathscr{B})$, *unique up to a constant factor, such that*

(i) $\mu \ll \Pi$, i.e. $\Pi(S) = 0$ *implies* $\mu(S) = 0$,

(ii) $\Pi(S) = \int_X P^{(m)}(x, S)\Pi(dx)$ *for all* $S \in \mathscr{B}$, $m \geqq 0$ (*Invariance*),

(iii) $\Pi(S) > 0$ *implies that* $P[X_k \in S$ *infinitely often* $| X_0 = x] = 1$ *for all* $x \in X$.

For $m \geqq 1$ ${}_TP^{(m)}(x, S) \equiv P[X_k \notin T$ for $0 < k < m$, $X_m \in S \mid X_0 = x]$.

${}_TP^{(0)}(x, S) = P^{(0)}(x, S) =$ the indicator function of the set S.

Let ρ be an arbitrary initial probability distribution defined on $(X, \mathscr{B})$. Then we define

$$P_\rho[\cdot] = \int_X P[\cdot \mid X_0 = x]\rho(dx)$$

on the corresponding infinite cross-product space. (See the supplement in Doob [4].)

For $S \in \mathscr{B}$ with $\Pi(S) > 0$, we define the "S-process" or "Process on S" as follows: Let $\{X_{S_k}\}$ be the consecutive members of $\{X_k\}$ which fall in S. Then, $\{X_{S_k}\}$ is the Process on S. We denote the corresponding transition probability function for this process by $P_S(x, T)$, defined for all $x \in X$ and for all $T \in \mathscr{B}(S)$, where $\mathscr{B}(S) = \{T' : T' = S \cap B$ for some $B \in \mathscr{B}\}$. We then have that

$$P_S(x, T) = \sum_{m=1}^{\infty} {}_SP^{(m)}(x, T).$$

If the original Markov Process is a Harris Process, i.e., satisfies Condition (C), then so is the Process on S. We also denote the invariant probability measure for this process

$$\Pi_S(T) = \Pi(T)/\Pi(S)$$

for all $T \in \mathscr{B}(S)$.

Lastly, we introduce that which in the literature is known as

HYPOTHESIS (D). (Doob [4]) *There exists a measure θ on $\mathscr{B}$ such that* $0 < \theta(X) < \infty$ *and for some integer $\nu \geqq 1$ and some $\varepsilon > 0$*

$$P^{(\nu)}(x, S) \leqq 1-\varepsilon \quad \text{if} \quad \theta(S) \leqq \varepsilon$$

for all $x \in X$.

(For a discussion of sets satisfying Hypothesis (D), see Orey [14].)

3. Results.

THEOREM 3.1. *Let $A \in \mathscr{B}$ and let the Process on A satisfy Hypothesis (D). Then*

$$\lim_{m\to\infty} \frac{\sum_{k=0}^{m} \int_X P^{(k)}(x, A)\phi(dx)}{\sum_{k=0}^{m} \int_X P^{(k)}(x, A)\psi(dx)} = 1$$

where ϕ and ψ are arbitrary initial probability distributions on $(X, \mathscr{B})$.

We need the following lemma.

LEMMA 3.1. *Under the hypothesis of Theorem 3.1, if $B \in \mathscr{B}(A)$ with $\Pi_A(B) > \delta$ for some $\delta > 0$, then for some constant K_δ*

$$\sum_{k=0}^{\infty} {}_BP_A{}^{(k)}(x, A) \leqq K_\delta$$

for all $x \in X$.

PROOF. Hypothesis (D) implies that $\lim_{m\to\infty} P_A{}^{(m)}(x, A-B) = \Pi_A(A-B)$ uniformly in x, $x \in A$. Thus for some fixed n and for all $x \in A$

$$P_A{}^{(n)}(x, A-B) < \Pi_A(A-B)+\delta/2 < 1-\delta/2.$$

For $m = 1, 2, \cdots, n$, $k = 0, 1, 2, \cdots$ and any $x \in X$

$$
\begin{aligned}
&{}_BP_A{}^{(kn+m+1)}(x, A) = \\
&\int_{A-B} \overset{(k+1)}{\cdots} \int_{A-B} {}_BP_A{}^{(1)}(x, dy_1)\, {}_BP_A{}^{(n)}(y_1, dy_2) \cdots {}_BP_A{}^{(n)} \cdot (y_k, dy_{k+1})\, {}_BP_A{}^{(m)}(y_{k+1}, A), \\
&\leqq \int_{A-B} \overset{(k)}{\cdots} \int_{A-B} {}_BP_A{}^{(1)}(x, dy_1)\, {}_BP_A{}^{(n)}(y_1, dy_2) \cdots {}_BP_A{}^{(n)} \cdot (y_{k-1}, dy_k)\, {}_BP_A{}^{(n)}(y_k, A-B), \\
&\leqq (1-\delta/2)^k.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\infty} {}_BP_A{}^{(k)}(x, A) &= {}_BP_A{}^{(0)}(x, A) + {}_BP_A{}^{(1)}(x, A) + \sum_{k=0}^{\infty} \sum_{m=1}^{n} {}_BP_A{}^{(kn+m+1)}(x, A) \\
&\leqq 2 + n \sum_{k=0}^{\infty} (1-\delta/2)^k = 2+2n.
\end{aligned}
$$

PROOF OF THEOREM 3.1. Let us define

$$H_N(x) = \sum_{k=0}^{N-1} P^{(k)}(x, A) \quad \text{and} \quad H_N(\rho) = \int_X H_N(x)\rho(dx)$$

for an arbitrary initial probability distribution ρ on $(X, \mathscr{B})$.

$$\lim_{m\to\infty} \frac{\sum_{k=0}^{m} \int_X P^{(k)}(x, A)\phi(dx)}{\sum_{k=0}^{m} \int_X P^{(k)}(x, A)\psi(dx)} = \lim_{N\to\infty} \frac{H_N(\phi)}{H_N(\psi)} = \lim_{N\to\infty} \frac{H_N(\phi)/H_N(\Pi_A)}{H_N(\psi)/H_N(\Pi_A)}.$$

Hence it is sufficient to show that

$$\lim_{N\to\infty} \frac{H_N(\rho)}{H_N(\Pi_A)} = 1.$$

For $N \geqq 1$ and arbitrary $\delta > 0$, let

$$B_{N,\delta} = \{x : x \in A,\ H_N(x) \leqq (1+\delta)H_N(\Pi_A)\}.$$

$$H_N(\Pi_A) = \int_A H_N(x)\Pi_A(dx) \geqq \int_{A-B_{N,\delta}} H_N(x)\Pi_A(dx)$$
$$> (1+\delta)H_N(\Pi_A)\Pi_A(A-B_{N,\delta}).$$

For all N sufficiently large so that $H_N(\Pi_A) > 0$

$$1 > (1+\delta)[1-\Pi_A(B_{N,\delta})] = 1+\delta-(1+\delta)\Pi_A(B_{N,\delta}).$$

Therefore $\Pi_A(B_{N,\delta}) > \delta/(1+\delta)$.

$$H_N(\rho) \leqq \int_X \sum_{k=0}^{N} P^{(k)}(x, A)\rho(dx)$$
$$\leqq \int_X \sum_{k=0}^{N} [\sum_{v=1}^{k-1} \int_{B_{N,\delta}}\, {}_{B_{N,\delta}}P^{(v)}(x, dy)P^{(k-v)}(y, A) + {}_{B_{N,\delta}}P^{(k)}(x, A)]\rho(dx)$$
$$\leqq \int_X [\sum_{v=1}^{N-1} \int_{B_{N,\delta}}\, {}_{B_{N,\delta}}P^{(v)}(x, dy) \sum_{s=1}^{N-v} P^{(s)}(y, A) + \sum_{k=0}^{\infty} {}_{B_{N,\delta}}P^{(k)}(x, A)]\rho(dx)$$

$$H_N(\rho) \leqq \int_X [\sum_{k=0}^{\infty} {}_{B_{N,\delta}}P^{(v)}(x, A)]\rho(dx)$$
$$+\sup_{y\in B_{N,\delta}} H_N(y) \int_X [\sum_{v=1}^{\infty} {}_{B_{N,\delta}}P^{(v)}(x, B_{N,\delta})]\rho(dx)$$

$$H_N(\rho) \leqq \int_X [\sum_{k=0}^{\infty} {}_{B_{N,\delta}}P_A{}^{(k)}(x, A)]\rho(dx) + (1+\delta)H_N(\Pi_A).$$

By Lemma 3.1

$$\sum_{k=0}^{\infty} {}_{B_{N,\delta}}P_A{}^{(k)}(x, A) \leqq K_\delta.$$

Choose N_δ so large that for all $N \geqq N_\delta$

$$K_\delta \leqq \delta H_N(\Pi_A).$$

Therefore for all $N \geqq N_\delta$

$$H_N(\rho) \leqq (1+2\delta)H_N(\Pi_A)$$

implying that

$$\limsup_{N\to\infty} H_N(\rho)/H_N(\Pi_A) \leqq 1.$$

For $N \geqq 1$ and arbitrary $\delta > 0$, let $C_{N,\delta} = \{x : x \in A,\ H_N(x) \geqq (1-\delta)H_N(\Pi_A)\}$.

$$H_N(\Pi_A) = \int_{C_{N,\delta}} H_N(x)\Pi_A(dx) + \int_{A-C_{N,\delta}} H_N(x)\Pi_A(dx).$$

For all $N \geqq N_\delta$, with ρ chosen so that x has initial probability measure one,

$$H_N(\Pi_A) < (1+2\delta)H_N(\Pi_A)\Pi_A(\ _{N,\delta}) + (1-\delta)H_N(\Pi_A)\Pi_A(A-C_{N,\delta}),$$

$$1 < (1+2\delta)\Pi_A(C_{N,\delta}) + (1-\delta)[1-\Pi_A(C_{N,\delta})] = 1-\delta+3\delta\Pi_A(C_{N,\delta}).$$

Therefore

$$\Pi_A(C_{N,\delta}) > \tfrac{1}{3}$$

for all $N \geqq N_\delta$. For $N \geqq 1$, $m \geqq 1$

$$H_{N+m}(\rho) \geqq \int_X [\textstyle\sum_{k=0}^{N+m-1} \sum_{v=1}^{k} \int_{C_{N,\delta}C_{N,\delta}} P^{(v)}(x, dy)P^{(k-v)}(y, A)]\rho(dx),$$

$$H_{N+m}(\rho) \geqq \int_X [\textstyle\sum_{v=1}^{m} \int_{C_{N,\delta}\,C_{N,\delta}} P^{(v)}(x, dy) \sum_{s=0}^{N+m-v-1} P^{(s)}(y, A)]\rho(dx)$$

$$\geqq \inf_{y \in C_{N,\delta}} H_N(y) \int_X [\textstyle\sum_{v=1}^{m}{}_{C_{N,\delta}} P^{(v)}(x, C_{N,\delta})]\rho(dx)$$

$$H_{N+m}(\rho) \geqq (1-\delta)H_N(\Pi_A)\textstyle\sum_{v=1}^{m} \int_{X\,C_{N,\delta}} P^{(v)}(x, C_{N,\delta})\rho(dx).$$

Since the A-process satisfies Hypothesis (D) and $\Pi(C_{N,\delta}) > \frac{1}{3}$ for all $N \geqq N_\delta$, for arbitrary $\varepsilon > 0$ there exists $M_0 = M_0(\delta, \varepsilon) > 0$ such that $P_\rho[E] > 1-\varepsilon$ where $E = \{$Event of visiting $C_{N,\delta}$ during the first M_0 visits to $A\}$. As this is a Harris Process, we can find $m_0 = m_0(M_0, \varepsilon, \rho)$ such that $P_\rho[F] > 1-\varepsilon$ where $F = \{$Event of visiting A at least M_0 times in the first m_0 steps$\}$. Thus m_0 is a function of δ, ε, ρ and is independent of N if $N \geqq N_\delta$.

For $N \geqq N_\delta$

$$P_\rho[\text{Event of visiting } C_{N,\delta} \text{ in the first } m_0 \text{ steps}] = \textstyle\sum_{v=1}^{m_0} \int_{X\,C_{N,\delta}} P^{(v)}(x, C_{N,\delta})\rho(dx)$$

$$\geqq P_\rho[E \cap F] \geqq 1-2\varepsilon.$$

Therefore

$$\frac{H_{N+m_0}(\rho)}{H_N(\Pi_A)} \geqq (1-\delta)(1-2\varepsilon).$$

Since $\lim_{N\to\infty} H_N(\Pi_A) = \infty$ and $0 \leqq H_{N+m}(\rho) - H_N(\rho) \leqq m$

$$\liminf_{N\to\infty} \frac{H_N(\rho)}{H_N(\Pi_A)} = \liminf_{N\to\infty} \frac{H_{N+m_0}(\rho)}{H_N(\Pi_A)}.$$

Due to the fact that δ and ε are arbitrary,

$$\liminf_{N\to\infty} H_N(\rho)/H_N(\Pi_A) \geqq 1.$$

Therefore, for any initial probability measure ρ on $(X, \mathscr{B})$

$$\lim_{N\to\infty} H_N(\rho)/H_N(\Pi_A) = 1.$$

THEOREM 3.2. (Jain) *If* $A_i \in \mathscr{B}$ *for* $i = 1, 2$ *with* $0 < \Pi(A_2) < \infty$ *then*

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{m} P^{(k)}(x, A_1)}{\sum_{k=1}^{m} P^{(k)}(y, A_2)} = \frac{\Pi(A_1)}{\Pi(A_2)}$$

for all x and for all y not in a Π*-null set* $N(A_1, A_2)$.
(Here $\Pi(A_1)/\Pi(A_2) = \infty$ if $\Pi(A_1) = \infty$.)

PROOF. This is Theorem 3.4 of Jain [8], the proof of which utilizes Theorem 3.1. (An alternate proof using Theorem 3.1 may be obtained via theorems of Chacon [1] and Chacon and Ornstein [2].)

THEOREM 3.3. *Let* $A_i \in \mathscr{B}$ *and let the Process on* A_i *satisfy Hypothesis* (D) *for* $i = 1, 2$. *Then*

$$\lim_{m\to\infty} \frac{\sum_{v=1}^{m} \int_X P^{(v)}(x, A_1)\phi(dx)}{\sum_{v=1}^{m} \int_X P^{(v)}(x, A_2)\psi(dx)} = \frac{\Pi(A_1)}{\Pi(A_2)}$$

where ϕ *and* ψ *are arbitrary probability measures on* X.

PROOF. Let

$$R_m(\phi, A_1; \psi, A_2) = \frac{\sum_{v=1}^{m} \int_X P^{(v)}(x, A_1)\phi(dx)}{\sum_{v=1}^{m} \int_X P^{(v)}(x, A_2)\psi(dx)}.$$

By Theorem 3.2, we may choose $z \in X - N(A_1, A_2)$, thus

$$\lim_{m\to\infty} R_m(z, A_1; z, A_2) = \Pi(A_1)/\Pi(A_2).$$

(In our notation, when a point is used in place of a measure, we mean a measure the mass of which is all concentrated at that point.)

$\lim_{m\to\infty} R_m(\phi, A_1; z, A_1) = 1$ and $\lim_{m\to\infty} R_m(z, A_2; \psi, A_2) = 1$ by Theorem 3.1.

Since

$$\lim_{m\to\infty} R_m(\phi, A_1; \psi, A_2) = \lim_{m\to\infty} R_m(\phi, A_1; z, A_1)R_m(z, A_1; z, A_2) \cdot R_m(z, A_2; \psi, A_2)$$

we have the desired result.

COROLLARY 3.1. *Let* $A_i \in \mathscr{B}$ *and let the Process on* A_i *satisfy Hypothesis* (D) *for* $i = 1, 2$. *Then for all* $x \in X$, *for all* $y \in X$,

$$\lim_{m\to\infty} \frac{\sum_{v=1}^{m} P^{(v)}(x, A_1)}{\sum_{v=1}^{m} P^{(v)}(y, A_2)} = \frac{\Pi(A_1)}{\Pi(A_2)}.$$

PROOF. Choose ϕ and ψ in Theorem 3.3 so that $x \in A_1$ and $y \in A_2$ have initial probability measures one, respectively.

REMARK. The null set of Theorem 3.2 apparently depends upon the sets A_1 and A_2 involved in the limit. Isaac [7] has shown that this dependence is justified in general and, furthermore, has proven a theorem ([7], Theorem 1) showing when the null set is actually independent of these sets. We give here an alternative proof of that theorem. Note also that under the hypothesis of Theorem 3.3 (as demonstrated in Corollary 3.1) we find that this null set is empty.

THEOREM 3.4. (Isaac) *For any fixed set* $S \in \mathscr{B}$ *with* $0 < \Pi(S) < \infty$, *if* $A_i \in \mathscr{B}(S)$, $i = 1, 2$ *with* $\Pi(A_2) > 0$ *then*

$$\lim_{m\to\infty} \frac{\sum_{k=1}^{m} P^{(k)}(x, A_1)}{\sum_{k=1}^{m} P^{(k)}(y, A_2)} = \frac{\Pi(A_1)}{\Pi(A_2)}$$

for all x *and for all* y *outside of a fixed* Π-*null set* $N(S)$ *which is independent of* A_1 *and* A_2.

PROOF. Let $R_m(x, A_1; y, A_2) = \sum_{k=1}^{m} P^{(k)}(x, A_1)/\sum_{k=1}^{m} P^{(k)}(y, A_2)$. From Theorem 1 of [13], there exists $S_i \in \mathscr{B}$, $i = 1, 2, \cdots$ such that $\Pi(S_i) > 0$, the Process on S_i satisfies Hypothesis (D) and $S_i \uparrow S$. Then for any fixed $q \in S_1$, Theorem 3.2 and Corollary 3.1 imply that

$$\lim_{m\to\infty} R_m(x, S; q, S_1) = \Pi(S)/\Pi(S_1)$$

for all $x \in S - N(S)$ where $\Pi(N(S)) = 0$. (Note that the null set $N(S)$ is indeed a function of S alone. If not, there exists $y \in S_1$ such that $\lim_{m\to\infty} R_m(x, S; y, S_1) \neq \Pi(S)/\Pi(S_1)$ for some $x \in S - N(S)$. However, there is a contradiction since $\lim_{m\to\infty} R_m(x, S; y, S_1) = \lim_{m\to\infty} R_m(x, S; q, S_1) R_m(q, S_1; y, S_1)$ and

$$\lim_{m\to\infty} R_m(q, S_1; y, S_1) = 1$$

by Corollary 3.1.)

By Corollary 3.1, for each $i \geqq 1$

$$\lim_{m\to\infty} R_m(x, S_i; q, S_1) = \Pi(S_i)/\Pi(S_1)$$

for all $x \in X$. Therefore, for all $x \in S - N(S)$,

$$\begin{aligned}
&\lim_{i\to\infty} \lim_{m\to\infty} R_m(x, S - S_i; q, S_1) \\
&\quad = \lim_{i\to\infty} \lim_{m\to\infty} [R_m(x, S; q, S_1) - R_m(x, S_i; q, S_1)] \\
&\quad = \lim_{i\to\infty} \left[\frac{\Pi(S)}{\Pi(S_1)} - \frac{\Pi(S_i)}{\Pi(S_1)}\right] = 0.
\end{aligned}$$

Thus for all $x \in S - N(S)$

$$\begin{aligned}
&\lim_{i\to\infty} \lim_{m\to\infty} R_m(x, A_1 \cap S_i^c; q, S_1) \\
&\qquad \leqq \lim_{i\to\infty} \lim_{m\to\infty} R_m(x, S - S_i; q, S_1) = 0.
\end{aligned}$$

By Corollary 3.1 $\lim_{m\to\infty} R_m(x, A_1 \cap S_i; q, S_1) = \Pi(A_1 \cap S_i)/\Pi(S_1)$ for all $x \in X$ and for all $i \geqq 1$.

Therefore, for all $x \in S - N(S)$

$$\begin{aligned}
&\lim_{m\to\infty} R_m(x, A_1; q, S_1) \\
&\quad = \lim_{i\to\infty} \lim_{m\to\infty} [R_m(x, A_1 \cap S_i; q, S_1) + R_m(x, A_1 \cap S_i^c; q, S_1)] \\
&\quad = \lim_{i\to\infty} \frac{\Pi(A_1 \cap S_i)}{\Pi(S_1)} = \frac{\Pi(A_1)}{\Pi(S_1)}.
\end{aligned}$$

Since

$$\lim_{m\to\infty} R_m(x, A_1; y, A_2) = \lim_{m\to\infty} R_m(x, A_1; q, S_1)R_m(q, S_1; y, A_2)$$

we have the desired result.

Acknowledgment. The author wishes to express his deep gratitude to Professor Naresh Jain for his correspondence and to Professor Steven Orey whose suggestions and encouragement have been invaluable.

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