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ABSTRACTS OF PAPERS

(*Abstract of a paper presented at the Eastern Regional meeting, University Park, Pennsylvania, April* 21–23, 1971.)

129-44. Methods for assessing multivariate normality (Invited). D. F. ANDREWS, R. GNANADESIKAN AND J. L. WARNER, Bell Telephone Laboratories.

The paper reviews a variety of techniques for evaluating the normality of the distribution of a body of multivariate data. Two broad categories of approaches are considered: (i) methods associated with data-based transformations for improving normality, and (ii) other techniques. Under each category, both simple though not sufficient methods (e.g. assessment of marginal normality) as well as more complete techniques (i.e. evaluating joint normality) are discussed. More specifically, the essential idea in the approaches associated with transformations is to assess the deviations of the estimated values of the transformation parameters involved from "null" (i.e. no transformation is required) values. Among the methods not related to transformations, firstly there are the univariate techniques useful for evaluating marginal normality of the individual variables. Secondly, linearity of the regressions among the variables and the behaviour of the generalized distances of the observations from their centroid may be investigated. Thirdly, and most generally, methods for assessing joint normality can be developed. The χ^2 goodness-of-fit test is a classical example, but additional techniques are needed and a few are discussed in this paper.

The detailed developments are for the bivariate case, but extensions to higher dimensions are implicit and, in principle, direct. Several illustrative examples are included. (Received April 21, 1971.)

(*Abstracts of papers presented at the Annual meeting, Fort Collins, Colorado, August* 23–26, 1971. *Additional abstracts will appear in future issues.*)

131-5. Arbitrary event initial conditions for branching Poisson processes. A. J. LAWRENCE, Thomas J. Watson Research Center.

Branching Poisson processes are now a well-known class of stationary point processes, being introduced by Bartlett, *J. Roy. Statist. Soc. Ser. B* **25** 264–296, and Lewis, *J. Roy. Statist. Soc. Ser. B* **26** 398–456; they are built up by the superposition of main Poisson events and events from finite renewal subsidiary processes which are initiated by the Poisson events. Arbitrary time (equilibrium) initial conditions were given by Lewis, *J. Appl. Probability* **6** 355–371 and here the corresponding arbitrary event initial conditions are obtained from an extension of Khintchine's analytic notion of an arbitrary event in a stationary point process. These conditions are shown to jointly specify the distribution of the number of subsidiary processes running (which is found to be the mixture of a Poisson variate and a Poisson variate plus one) together with the distributions of the numbers of events remaining in each and their future durations. In such an initiated branching Poisson process it is also shown that these conditions repeat themselves at every subsequent event. (Received April 7, 1971.)

131-6. A note on asymptotic joint normality. C. L. MALLOWS, Bell Telephone Laboratories, Inc.

The concept of asymptotic normality takes on some new aspects when the dimensionality of the vector random variable under consideration is allowed to increase indefinitely. A necessary and sufficient condition for joint asymptotic normality in a new (strong) sense, in the case of independence, is given. (Received April 8, 1971.)

131-7. On the possibility of a multivariate extension of the variance stabilizing transformations. PAUL W. HOLLAND, Harvard University.

After reviewing the asymptotic variance stabilizing transformations in one dimension, a generalization of these to multivariate cases is discussed. Results are given for the uniqueness of solutions when they exist, but unlike the one-dimensional case, covariance stabilizing transformations need not exist. In the two-dimensional case, a necessary and sufficient condition is given for the existence of solutions. It takes the form of a second order partial differential equation that the elements of any square root of the inverse of the limiting covariance matrix must satisfy. This condition is applied to three examples with the conclusion that no covariance stabilizing transformation exists for the trinomial distribution. It is conjectured that this non-existence of solutions is true for the general multinomial. Finally, some alternative formulations of the problem are mentioned. (Received April 20, 1971.)

131-8. Comparison of some two-sample nonparametric tests for scale. BENJAMIN S. DURAN, Texas Institute for Rehabilitation and Research.

Suppose $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ are independent random samples from two populations having unknown absolutely continuous cumulative distribution functions $F(x)$ and $G(x)$, respectively. Consider $W_1 = \sum_{i=1}^n R_i$ and $W_2 = \sum_{i=1}^n R_i^2$ where R_i denotes the rank of the ith smallest $|Y|$ in the combined ordered sample of $|X|$'s and $|Y|$'s. This paper discusses the statistics W_1 and W_2 as test statistics for testing H_0: $F(x) = G(x)$ against $H_1: G(x) = F(x\theta)$, $\theta \neq 1$. The statistic W_1 is asymptotically equivalent to others previously proposed, among them that of Sukhatme (*Ann. Math. Statist.* **24** 188–194). The asymptotic relative efficiency of the W_1-test relative to the W_2-test is 0.80 when normal alternatives are assumed. Other comparisons are also discussed. (Received April 20, 1971.)

131-9. On the distribution of a linear combination of correlated quadratic forms. P. R. KRISHNAIAH AND V. B. WAIKAR, Aerospace Research Laboratory.

Let $\mathbf{X}_j' = (\mathbf{X}_{1j}', \cdots, \mathbf{X}_{qj}')$, $j = 1, \cdots, n$ be independently distributed as m-variate normal with mean vector $\boldsymbol{\mu}_j$ and covariance matrix Σ. Let $y_i = \sum_{j=1}^{n} \mathbf{X}_{1j}' A_i \mathbf{X}_{ij}$, $i = 1, \cdots, q$ where $A_1, \cdots, A_q$ are positive definite. Further, let $y = \sum_{i=1}^{q} c_i y_i$ where $c_1, \cdots, c_q$ are arbitrary positive constants. In this paper, the authors proved the infinite divisibility of $\phi(t)$ where $\phi(t)$ is the characteristic function of y. Further, using the property of infinite divisibility of $\phi(t)$, two representations have been given for the distribution of y^* where the characteristic function of y^* is $[\phi(t)]^{\beta}$, β being a positive real number, and assuming $\boldsymbol{\mu}_j = \boldsymbol{\mu}$, $j = 1, \cdots, n$. Finally, the authors have derived the density of the random vector $\mathbf{T} = y^{-\frac{1}{2}}\mathbf{Z}$ where $\mathbf{Z}$ is distributed independently of y as a p-variate normal with mean vector $\boldsymbol{v}$ and covariance matrix V and y is as stated above with $\boldsymbol{\mu}_j = \mathbf{0}$, $j = 1, \cdots, n$. Thus the authors derived a generalization of the noncentral multivariate t-distribution. (Received April 21, 1971.)

131-10. Distribution-free interval estimation of the largest α-quantile. M. HASEEB RIZVI AND K. M. LAL SAXENA, Stanford University and University of Nebraska.

Consider $k(\geqq 1)$ distributions with unknown continuous cdfs F_i, $i = 1, \cdots, k$. Let $x_\alpha(F_i)$ be the unique α-quantile $(0 < \alpha < 1)$ of F_i and let $\theta = \max_{1 \leqq i \leqq k} x_\alpha(F_i)$. For specified γ, we want a random interval I such that $\inf P\{\theta \in I\} \geqq \gamma$, where infinum is taken over the set Ω of all possible k-tuples $(F_1, F_2, \cdots, F_k)$. Let $Y_{r,i}$ denote the rth order statistic from F_i obtained from independent random samples of common size n from each F_i and let $Y_r = \max_{1 \leqq i \leqq k} Y_{r,i}$ for $r = 1, \cdots, n$. Define $Y_0 = -\infty$ and $Y_{n+1} = +\infty$. For $s < t$ consider the random interval $I_0 = (Y_s, Y_t)$ and assert that $\theta \in I_0$, where s and t are chosen to satisfy the above probability requirement. The minimization of the probability of coverage is given by the following. THEOREM. *With* $G_r(\alpha)$ *denoting the incomplete beta function,* $\inf_\Omega P\{\theta \in I_0\}$ *is* $G_s^{\,k}(\alpha)$, $1 - G_t(\alpha)$ *or* $\min\{G_s(\alpha) - G_t(\alpha), G_s^{\,k}(\alpha) - G_t^{\,k}(\alpha)\}$ *according as* $I_0 = (Y_s, \infty)$, $I_0 = (-\infty, Y_t)$ *or* $I_0 = (Y_s, Y_t)$ *with* $0 < s < t < n+1$. An optimality criterion is proposed and an algorithm is given for two-sided intervals. Large sample approximations are also considered. (Received April 23, 1971.)

131-11. Multivariate tests for nonadditivity: a general procedure. LYMAN L. MCDONALD AND GEORGE A. MILLIKEN, University of Wyoming and Kansas State University.

In a conventional univariate linear model it is often possible to examine a more complete model which includes nonlinear terms. This leads to tests for non-additivity in some of the commonly used experimental designs. Consider, for example, Tukey's one degree of freedom for nonadditivity in the two-way crossclassification model. This paper extends this theory to the multivariate case (i.e., the case when more than one variate or response is measured on each experimental unit) and tests are developed for the hypothesis that the multivariate model is additive with respect to each variate against the alternative that at least one variate requires a nonadditive model. (Received April 27, 1971.)

131-12. Two examples of invariant Bayes procedures. RICHARD E. SCHWARTZ, Department of Defense.

Kiefer and Schwartz (*Ann. Math. Statist.* **36** (1965) 747–770) describe techniques for the construction of proper Bayes tests which are invariant under non-compact groups. Their work, as well as subsequent papers on the subject, is limited to *testing* problems concerning *exponential* families of p.d.f's. The present note shows, by means of two simple examples, that the same methods can sometimes be applied more generally. One example considers estimation of the ratio of the mean to the variance for the normal distribution. The second example considers testing problems in which the p.d.f's have an exponential factor as well as a non-exponential factor. (Received May 14, 1971.)

131-13. A test for normality based on a characterization of the Univariate Normal. WILLIAM B. OWEN, Central Washington State College.

A characterization of the Univariate Normal is used as a basis for testing any continuous density for normality. Empirical power comparisons of this test are made with the Kolmogorov–Smirnov test and the chi-square goodness-of-fit test for several densities, including the exponential, uniform, and Cauchy. (Received May 14, 1971.)

131-14. A test for weak bandwidth stationarity. EDWARD L. MELNICK, New York University.

Observations generated by time series models contain interrelationships which are extremely difficult to analyze. The mathematics become more tractable if the generating process is assumed to be weakly stationary, i.e., the first two moments exist and are independent of time. In this paper a procedure is developed for testing the null hypothesis that a harmonizable process is weakly stationary. The procedure assumes only one observation to be available at each time point. The derived tests investigate predetermined frequency bands, under the assumption of a smoothness property by an application of complex demodulation. Basically, the random variables $du(\omega)$ and $dv(\omega)$ in the representation $X(t) = \int_0^\infty \cos \omega t du(\omega) + \int_0^\infty \sin \omega t dv(\omega)$ are tested using the discrimination information statistics. The tests will be illustrated by applying them to artificial data whose properties are known. (Received May 14, 1971.)

131-15. WITHDRAWN.

131-16. Improved predictive mean square error. S. C. NARULA AND JOHN S. RAMBERG, University of Iowa.

A method is proposed to improve the predictive mean square error (variance + squared bias) of a least squares prediction equation. The equation is multiplied by a constant (λ) which is a function of the model parameters and the values of the predictor variables. The distribution of λ is derived and a simple illustrative example is included. (Received May 14, 1971.)

131-17. The estimation of heteroscedasticity parameters from a marginal likelihood function. HANS LEVENBACH, Bell Telephone Laboratories, Inc.

The linear regression model with normally distributed heteroscedastic errors is analyzed in the conditional framework of the theory of structural inference. The inference procedure produces a marginal likelihood function for the parameters that describe the heteroscedasticity (unequal variances) in the model. This marginal likelihood function is based on the distribution of a vector of standardized residuals that is constant on orbits in the partition generated by the regression-scale group of transformations on a Euclidean space. Given a plausible value or set of values for the heteroscedasticity parameters, the theory also provides structural distributions of the regression coefficients and the scale parameter. Comparisons are made in two simple regression models using two sets of data drawn from the literature. (Received May 14, 1971.)

131-19. Asymptotic joint distribution of $\binom{p}{t}$ multiple correlation coefficients between a certain variable and t variables among p other variables ($t < p$). MINORU SIOTANI, Kansas State University.

Let $(x_0, x_1, \cdots, x_p)$ be subject to a $(p+1)$-variate normal distribution. Then an asymptotic joint distribution of $\binom{p}{t}$ squared sample multiple correlation coefficients (m.c.c.) $r^2_{0(\alpha)}$ between x_0 and $(x_{\alpha_1}, \cdots, x_{\alpha_t})$ was obtained. Since $r^2_{0(\alpha)}$ are functions of the sample covariance matrix S with n degrees of freedom, $r^2_{0(\alpha)}$'s are asymptotically normal with mean $\rho^2_{0(\alpha)}$'s, corresponding squared population m.c.c. Elements of the covariance matrix are evaluated by using the delta method and the following formula: $\operatorname{Cov}\{f(S), g(S)\} \simeq (2/n)f(\Sigma)g(\Sigma) \operatorname{tr}(\Phi\Sigma\Psi\Sigma)$ where Σ is the population covariance matrix, $\Phi = (\partial_{\alpha\beta} \ln f(\Sigma))$, $\Psi = (\partial_{\alpha\beta} \ln g(\Sigma))$, $\partial_{\alpha\beta} = (\frac{1}{2})(1+\delta_{\alpha\beta})\partial/\partial\sigma_{\alpha\beta}$, and $\delta_{\alpha\beta}$ = the Kronecker symbol (Mimeo Series No. 595, University of North Carolina at Chapel Hill). Let us use the symbols $(\alpha) \equiv (i, k) = (i_1, i_{t-r}, k_1, \cdots, k_r)$ and $(\beta) \equiv (k, j) = (k_1, \cdots, k_r, j_1, \cdots, j_{t-r})$ with the emphasis of common variables. Then the obtained covariances are:

$$\begin{aligned}
\operatorname{Cov}\left(r^2_{0(i,k)}, r^2_{0(k,j)}\right) &\simeq (2/n)\cdot \\
&[(1-\rho^2_{0(i,k)})(1-\rho^2_{0(k,j)})(\rho^2_{0(i,k)}+\rho^2_{0(k,j)}-1) \\
&+\{1-\rho^2_{0(k,j)}-\mathbf{q}_i{}'A^{-1}_{(i)k}\mathbf{q}_i/\sigma_{00}+\mathbf{q}_i{}'A^{-1}_{(i)k}(\Sigma^{(i)}_{(j)}-\Sigma^{(i)}_{(k)}\Sigma^{(k)}_{(j)})A^{-1}_{(j,k)}\mathbf{q}_j/\sigma_{00}\}\cdot \\
&\{1-\rho^2_{0(i,k)}-\mathbf{q}_j{}'A^{-1}_{(j)k}\mathbf{q}_j/\sigma_{00}+\mathbf{q}_i{}'A^{-1}_{(i)k}(\Sigma^{(i)}_{(j)}-\Sigma^{(i)}_{(k)}\Sigma^{-1}_{(k)}\Sigma^{(k)}_{(j)})\cdot \\
&A^{-1}_{(j)k}\mathbf{q}_j/\sigma_{00}\}], \text{ where } \mathbf{q}_u = \boldsymbol{\sigma}_{0(u)}-\Sigma^{(u)}_{(k)}\Sigma^{-1}_{(k)}\boldsymbol{\sigma}_{0(k)},
\end{aligned}$$

$A_{(u)k} = \Sigma_{(u)}-\Sigma^{(u)}_{(k)}\Sigma^{-1}_{(k)}\Sigma^{(k)}_{(u)}$, $(u = i, j)$, $\Sigma^{(u)}_{(v)}$ is the covariance matrix of $(x_{u_1}, \cdots, x_{u_s})$ and $(x_{v_1}, \cdots, x_{v_s})$, $\Sigma^{(u)}_{(u)} \equiv \Sigma_{(u)}$, σ_{00} the variance of x_0, and $\boldsymbol{\sigma}_{0(u)}$ the column vector of covariances of x_0 and $x_{u_1}, \cdots, x_{u_s}$. (Received May 18, 1971.)

131-20. Estimates of the renewal function when the second moment is infinite.

MORTON R. DUBMAN, Bell Telephone Laboratories, Inc.

Let $F(x)$, $-\infty < x < \infty$, be a nonlattice probability distribution function with finite positive first moment μ and with $\int_{-\infty}^{0} x^2\, dF(x) < \infty$. Let $F^{(k)}$ denote the k-fold convolution of F with itself and H the renewal function, given by $H = \sum_{k=0}^{\infty} F^{(k)}$. Set $Q(x) = 1-F(x)$ for $x \geqq 0$, and $Q(x) = -F(x)$ for $x < 0$. Set $T_1(x) = x$ for $x \geqq 0$, and $T_1(x) = 0$ for $x < 0$. For $m \geqq 2$ define the functions T_m by $T_m = \mu T_{m-1} - T_{m-1} {*} Q$, where $*$ denotes the convolution operation. THEOREM. *Assume that*: (i) $F(x+1)-F(x) = O(x^{-1-\alpha})$ *as* $x \to \infty$, *where* $1 < \alpha < 2$, *and* (ii) $1-F(x) \sim x^{-\alpha}L(x)$ as $x \to \infty$, *where L is a slowly varying function. Suppose that* $n \geqq 2$. *If* $(n+1)/n < \alpha \leqq n/(n-1)$, *then* $H(x) = x/\mu + \sum_{m=2}^{n} \mu^{-m} T_m(x) + o_x(1)$ *as* $x \to \infty$, *where* $T_m(x) \sim L_m(x)x^{(m-1)(1-\alpha)+1}$ *as* $x \to \infty$, *with* L_m *being a slowly varying function simply determined by m and L.* In the proof Fourier analysis methods are used to estimate the remainder term in the expansion of H. Similar results, valid for $2 \leqq n \leqq 5$, are proved when assumption (i) is dropped. Assumptions of the form $1-F(x) = O(x^{-\alpha})$, $x \to \infty$, are also considered. The results of the paper provide refinements of earlier estimates of the renewal function under condition (ii) due to Feller (*Trans. Amer. Math. Soc.* **67** (1949) 98–119) and Teugels (*J. London Math. Soc.* **2** (1970) 179–190). (Received May 19, 1971.)

131-21. Invariance and randomization in fractional factorial designs. J. N. SRIVASTAVA AND B. L. RAKTOE, Colorado State University and University of Guelph.

In this paper, we first prove: THEOREM. *Consider an* $(s_1 \times s_2 \times \cdots \times s_m)$ *factorial. Let* $p_0(v \times 1)$ *denote the set of (possibly) nonzero unknown parameters, the remaining parameters being assumed zero. Let* T *be any (arbitrary) design, i.e. various level-combinations may occur zero, one or more times in* T. *Let* M *denote the "information matrix" for* p_0 *using the design* T. *Let* p_0 *be such that if* $A_1^{k_1} A_2^{k_2} \cdots A_m^{k_m} \in p_0$ *with* $k_1 \neq 0$, *then* $A_1^{k'_1} A_2^{k_2} \cdots A_m^{k_m} \in p_0$ *for every* $k_1' \neq 0$. *Let* $(\omega_0, \omega_1, \cdots, \omega_{s_1-1})$ *denote any permutation of* $(0, 1, 2, \cdots, s_1 - 1)$. *Let* T^* *be the design obtained from* T *by changing level* i $(i = 0, 1, \cdots, s_1 - 1)$ *of the first factor to level* ω_i. *Let* M^* *be the information matrix for* T^*. *Then* $M^* = F'MF$, *where* F *is an orthogonal matrix.* (Obviously, the theorem also holds if the levels of several factors are transformed, instead of just one factor). This generalizes the results of Paik and Federer (*Ann. Math. Statist.* **41** (1970) 369) in several directions. (i) They considered main effects plans, for the symmetrical prime power case. Our p_0 is very general, T is arbitrary, and the s_i are not necessarily equal and may be *any* positive integers. (ii) Their method of obtaining T^* from T is a very special case of ours. (iii) They proved only that $|M| = |M^*|$, etc. Finally, we extend their results on randomization to our general setup. (Received May 21, 1971.)

131-22. Reliability estimation in presence of an outlier observation. S. K. SINHA, University of Manitoba.

Consider a situation in which the random variables $(X_1, X_2, \cdots, X_n)$ are such that $(n-1)$ of them are distributed as $f(x, \sigma) = (1/\sigma) \exp(-x/\sigma)$, $x \geqq 0$ and one of them is distributed as $f(x, \sigma/\alpha)$, $0 < \alpha \leqq 1$, while each X_i has a priori probability $1/n$ of being distributed as $f(x, \sigma/\alpha)$. The behavior of $\hat{R}_t$, the uniformly minimum variance unbiased estimator of the reliability function for the family $f(x, \sigma)$ has been studied in presence of an outlier observation $X \sim f(x, \sigma/\alpha)$. Bounds of $E(\hat{R}_t \mid \alpha)$ and the mean-squared-error of $(\hat{R}_t \mid \alpha)$ have been obtained and a new estimator R_t^* has been proposed. Similar problems have been considered when the pdf's of X are $g(x, \mu) = \exp\{-(x-\mu)\}$, $x \geqq \mu$ and $g(x, \mu+\delta)$, $\delta \geqq 0$. A semi-Bayesian approach, where α and δ are treated as random variables with Beta and Gamma type priors, will also be discussed. (Received May 24, 1971.)

131-23. Un test des hypothèses: $f(x)$ spécifié représente ou ne represente pas la densité de la population. EUGÈNE H. LEHMAN, Université du Québec.

La statistique Cramér–Rao, L_n, est définie comme: $L_n = \Sigma(F_i - 1/m)^2$, $i = 1, \cdots, n$; $m = n+1$; F_i = la distribution cumulative de l'observation ordonnée, x_i; x_1 est la plus petite observation, x_n la plus grande. On veut vérifier si F est en réalité la distribution de la population. On dérive une formule tres élaborée pour calculer l'espérance $E(L_n^p)$, $p = 1, \cdots, 6$. Avec ces 6 moments-ci on peut approximer la densité h de L_n pour chaque $n = 1, \cdots, 100$. Prochainement, on peut calculer les valeurs de $L_n(a)$ où $1-a =$ intégrale (0 à $L_n(a))h(L_n)dL_n$; $a = 0.0001, 0.0002, \cdots, 0.5, \cdots, 0.9998, 0.9999$. Armé de ces chiffres, on peut rejeter H_1: F est en effet la densité de la population; si L_n excède $L_n(a)$ où a est la taille du test; semblablement, on peut rejeter H_2: F n'est point la densité, si L_n n'excède pas $L_n(1-a)$. (Received May 25, 1971.)

131-24. The asymptotic behavior of the joint distribution of maxima from bivariate samples. JANET W. CAMPBELL AND CHRIS P. TSOKOS, NASA Langley Research and Virginia Polytechnic Institute and State University.

A workable method is presented for obtaining the asymptotic joint distribution of the maximum X and the maximum Y in samples drawn from a continuous bivariate population. It is assumed that the marginals of this bivariate distribution possess asymptotic extreme-value distributions and that the probability density function associated with the original bivariate distribution has a canonical series expansion, i.e., is "ϕ^2-bounded" as defined by Lancaster, [The structure of bivariate distributions, *Ann. Math. Statist.* **29** (1958) 719–736]. Using the canonical expansion of the density function and the univariate extreme-value distributions of the marginals, the authors derive a general form for a bivariate extreme-value distribution. Their technique is then applied to the bivariate gamma distribution, and the resulting asymptotic extreme-value distribution is shown to be the product of two double-exponential distributions, the univariate extreme-value distributions associated with gamma marginals. Thus, it is proved that the maxima from bivariate gamma samples, like those from bivariate normal samples, are asymptotically independent. (Received May 25, 1971.)

131-25. An optimal test of fit based on order statistics. JAMES E. NORMAN, University of Georgia.

Let $\mathbf{Y} = \{Y_{i_1}, Y_{i_2}, \cdots, Y_{i_k}\}$ be a vector of K order statistics from a random sample of size $n \geqq k$. Denote the mean vector and covariance matrix of $\mathbf{Y}$ by μ and Σ respectively. The parent distribution function is to be tested as H_0: $F(x)$ (completely specified) vs. H_A: $G(x)$ (completely specified). The result that $(\mathbf{Y}-\boldsymbol{\mu})' \times \Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})$ has an asymptotic chi-square (k degrees of freedom) distribution can be used to make this test and the choice of $i_1, i_2, \cdots, i_k$ can be made in such a way that the power is a maximum. (Received May 25, 1971.)

131-26. Improved predictive mean square error. S. C. NARULA AND JOHN S. RAMBERG, University of Iowa.

A method is proposed to improve the predictive mean square error (variance + squared bias) of a least squares prediction equation. The equation is multiplied by a constant (λ) which is a function of the model parameters and the values of the predictor variables. The distribution of λ is derived and a simple illustrative example is included. (Received May 28, 1971.)

131-27. On three level symmetrical factorial designs and ternary group codes. BODH RAJ GULATI AND E. G. KOUNIAS. Southern Connecticut State College and McGill University.

Consider a finite $(t+r-1)$-dimensional projective space PG $(t+r-1, 3)$ based on three elements 0, 1 and 2. A set of k distinct points in PG $(t+r-1, 3)$, no t of which are linearly dependent, is said to be a (k, t)-set; such a set is maximal if it is not contained in any (k^*, t)-set with $k^* > k$. The number of points in a maximal set is denoted by $m_t(t+r, 3)$. In the language of experimental designs, $m_t(t+r, 3)$ represents the maximum number of variables in a symmetrical factorial design, in which factors operate at 3 levels, blocks are of size 3^{t+r} and no main effect or t-factor or lower order interactions is confounded with block effects. The problem of constructing fractional replicates of 2^m designs has been approached in the literature from several closely related points of view, but work on 3^m designs has scarcely begun. In this paper, we have explored a possibility of adjoining $n \leqq t$ points to the basic set of E_i, $i = 1, 2, 3, \cdots, t+r$, where E_1 is a point with a one in the ith position and zeros elsewhere. In general, $m_t(t+r, 3) = t+r+n$ for $\sum_{j=1}^{n-1} [t \cdot 3^{-j}] \leqq r < \sum_{j=1}^{n} [t \cdot 3^{-j}]$ where $[x]$ is the largest integer not exceeding x. (Received June 1, 1971.)

131-27. On detecting a spurious observation. K. S. MOUNT and B. K. KALE, University of Manitoba.

Suppose we take a sample of size $n(X_1, \cdots, X_n)$ with $n-1$ of the observations having df F and one (spurious) observation having df G where $F < G$. A priori, each X_i has probability $1/n$ of being the spurious observation. Let u_r be the probability that the rth order statistic $X_{(r)}$ corresponds to the spurious observation. Kale and Sinha [*Ann. of Math.* **46**, 752] have proved that u_r is a monotone increasing function of r when F and G are exponential with unknown scale parameters. We generalize this result when: (i) F and G are members of a monotone likelihood ratio (in X) family; (ii) $G \in \mathscr{A}(F) = \{G: G(x) = \sum_{k=1}^{\infty} C_k[F(x)]^k, C_k \geqq 0, \sum_{k=1}^{\infty} C_k = 1\}$. Some properties of the class of df's $\mathscr{A}(F)$ are studied. We show that $\mathscr{A}(F)$ is closed under two types of convergence: (a) weak convergence; (b) convergence in L^2 norm of the vector of constants determining the member of $\mathscr{A}(F)$. The above results are then applied to study the slippage problem, i.e., to test $H_0: X_j \sim F_0, j = 1, \cdots, n$, vs. H_i: $X_j F_0$, $j \neq i$, $X_i \sim G$ where $F_0 < G$. The optimal test such that $P\{\text{rej. } H_0 \mid H_0\} = \alpha$ and $P\{\text{acc. } H_i \mid H_i\}$ is maximized and $P\{\text{acc. } H_i \mid H_i\}$ is independent of i, has critical region $X_{(n)} > C_{n,\alpha}$. (Received June 3, 1971.)

131-28. A family of binary balanced incomplete block designs with two unequal block sizes and two unequal replications. A. HEDAYAT AND W. T. FEDERER, Cornell University.

The following theorem is proved. THEOREM. If there exists a balanced incomplete block design with parameters $v, b_1, r_1, k_1, \lambda_1$ and a balanced incomplete block design with parameters $v, b_2, r_2, k_2, \lambda_2$, then there exists a binary balanced incomplete block design consisting of $v+1$ treatments (the new treatment will be called the augmented treatment), $c_1 b_1$ blocks of size k_1+1 and $c_2 b_2$ blocks of size k_2 with $c_2 = k_2(r_1-\lambda_1)/d$ and $c_1 = \lambda_2(k_1+1)/d$, where d is the greatest common divisor of $k_2(r_1-\lambda_1)$ and $\lambda_2(k_1+1)$. The augmented treatment is replicated $c_1 r_1$ times while the other v treatments are each replicated $c_1 r_1 + c_2 r_2$ times. The corresponding matrix of the adjusted normal equations is of the form $r_1\lambda_2(v+1)\times I + r_1\lambda_2 J$, where I is the identity matrix or order $v+1$ and J is the $(v+1)\times(v+1)$ matrix with unit entries everywhere. Thus every elementary contrast is estimable with the same variance, and hence the design is balanced. (Received June 4, 1971.)

131-29. An optimal sequential decision procedure for comparing two binomial probabilities. THOMAS L. BRATCHER, University of Southwestern Louisiana.

A straightforward Bayes solution is found for the two-decision problem of comparing two binomial probability parameters where the data are taken in pairs. The Bayes sequential procedure is shown to be truncated and an upper bound on the point of truncation is derived. A linear loss function is utilized and the beta distribution is taken to represent the prior information. (Received June 4, 1971.)

131-30. Percentage points of the statistic for testing hypotheses on mean vectors of multivariate normal populations with missing observations. D. S. BHOJ, Rutgers—the State University, Camden.

We have considered the problem of testing of hypotheses on the mean vector of a multivariate normal distribution with unknown and positive definite covariance matrix when a sample with missing observations from that population is available. We have assumed that the missing observations follow a special, though not unusual, pattern. The null distributions are not in a suitable form to determine the critical region. We have used the cumulants of the test statistic to get the approximate percentage points. The accuracy of the percentage points has been checked by comparing them with some exact percentage points which are calculated for complete samples and some special incomplete samples. The comparison showed that our percentage points are in good agreement with exact percentage points. We extended the above work to the problem of testing the hypothesis of equality of two mean vectors of two multivariate normal populations with the same, though unknown, covariance matrix. (Received June 4, 1971.)

(Abstracts contributed by title)

71T-44. Testing equality of means of two normal populations under multiplicative model. J. SINGH AND B. N. PANDEY, Banaas Hindu University.

Consider two Normal populations $N(\mu i, \sigma_i^2)$ $(i = 1, 2)$ where $\mu_2 = a\mu_1$, $\sigma_2{}^2 = a^2\sigma_1{}^2$. For testing the hypothesis $H_0: a = 1$ against the alternative hypothesis $H_1 = a > 1$, the following test statistics have been used (i) $U = S_2{}^2/S_1{}^2$ where $s_i{}^2 = \sum_{j=1}^{ni} (\chi ij - \bar{\chi} i)^2/(ni-1)(i = 1,\ 2)$ (ii) Unilateral statistics $Y(\gamma_1\gamma_2) = (\bar{X}_1 - \bar{X}_2)^2/[(n_1-1)\gamma_1 S_1{}^2 + (n_2-1)\gamma_2 S_2{}^2]$ where γ_1 and γ_2 are suitable constants (McCullough and L. Rogen-burg (1960) (iii) $p^2 = (\bar{X}_1 - \bar{X}_2)^2 n_1 n_2 (n_1 + n_2 - 2)/[(n_1+n_2)\{(n_i-1)S_1{}^2 + (n_2-1)S_2{}^2\}]$. We have investigated the power function of these statistics by taking $n_1, n_2 = 3, 5, 7, 9$. We have also considered the case when the two populations have the same coefficient of variation i.e. we have considered $N(\mu i, \beta^2\mu i^2)$ $(i = 1, 2)$, $\mu_2 = a\mu_1$. For testing the hypothesis $H_0: a = 1$ against the alternative hypothesis $H_1: a > 1$ the likelihood ratio criterion is used and the test statistic used is $\lambda = [2[\{(\bar{X}_1{}^2 + 2s_1{}^2)(\bar{X}_2{}^2 + 2s_2{}^2)\}^{\frac{1}{2}} - \bar{X}_1\bar{X}_2]]^n/[(\bar{X}_1 - \bar{X}_2)^2 + 2(s_1{}^2 + s_2{}^2)]$. (Received April 5, 1971.)

71T-45. On multitype Galton–Watson processes with ρ near 1 (preliminary report). M. P. QUINE, Australian National University.

Let $(\mathbf{Z}_n) = (\{Z_{n,1}, \cdots, Z_{n,k}\})$, $n = 0, 1, 2, \cdots$, be a k-type Galton–Watson process which has probability generating function (p.g.f.)$\mathbf{F} \equiv \mathbf{F}(\mathbf{s}) = \{F_1(\mathbf{s}), \cdots, F_k(s)\}$, $\mathbf{s} \in [0, 1]^k$, and expectation matrix $\mathbf{M}(\mathbf{F}) = \|\partial F_\alpha(\mathbf{1})/\partial s_\beta\|$. Given a positive integer U, and constants $a > 0$, $b > 0$, $c < \infty$, let K denote the class of all p.g.f.'s $\mathbf{F}$ of the above kind which are proper, and satisfy (i) $\{\mathbf{M}^U(\mathbf{F})\}_{\alpha\beta} \geqq a$, $1 \leqq \alpha, \beta \leqq k$; (ii) $\sum_{\alpha,\beta,\gamma} \partial^2 F_\alpha(\mathbf{1})/\partial s_\beta \partial s_\gamma \geqq b$; (iii) $\sum_{\alpha,\beta,\gamma,\delta} \partial^3 F_\alpha(\mathbf{1})/\partial s_\beta \partial s_\gamma \partial s_\delta \leqq c$. Denote by ρ_F the spectral radius of $\mathbf{M}(\mathbf{F})$. Let $\mathbf{i} = \{i_1, \cdots, i_k\}$ be a fixed vector where $i_1, \cdots, i_k$ are nonnegative integers and $\sum_\alpha i_\alpha > 0$. Let $\boldsymbol{\mu}_n \equiv \{\mu_{n,1}, \cdots, \mu_{n,k}\}$ denote the conditional expectation $E[\mathbf{Z}_n \mid \mathbf{Z}_0 = \mathbf{i}, \mathbf{Z}_n \neq \mathbf{0}]$. For $n = 1, 2, \cdots$, define $\mathbf{Z}_n{}^*$ as the vector of the normed random variables $Z_{n,\alpha}^* = Z_{n,\alpha}/\mu_{n,\alpha}$, $1 \leqq \alpha \leqq k$. For two vectors $\mathbf{a}$, $\mathbf{b}$, we write $\mathbf{a} \geqq \mathbf{b}$ if each element of $\mathbf{a}$ is not less than the corresponding element of $\mathbf{b}$. Given any $\varepsilon > 0$, there exist $\delta > 0$ and $N < \infty$ such that for $n > \mathrm{N}$, $\sup_{x \in R^k} | P[\mathbf{Z}_n{}^* \leqq \mathbf{x} | \mathbf{Z}_0 = \mathbf{i}, \mathbf{Z}_n \neq \mathbf{0}] - S(\mathbf{x}) | < \varepsilon$, uniformly for all processes with p.g.f. $\mathbf{F} \in k$ satisfying $|1 - \rho_F| < \delta$, where $S(\mathbf{x}) = 1 - \exp(-\min_\alpha x_\alpha)$ when $\mathbf{x} \geqq \mathbf{0}$, and $S(\mathbf{x}) = 0$ otherwise. A similar theorem has been proved for the process with immigration, in which case conditioning on non-extinction is unnecessary, and the limit distribution relates to the gamma rather than the exponential. (Received April 5, 1971.)

71T-46. On bounds for the frequency of misleading Bayes inferences. RAY E. SCHAFER, Hughes Aircraft Company.

If the posterior probability of a true hypothesis is small, say $\leqq p$, then one could be led to the misleading inference that the true hypothesis is false. Suppose given a finite number k of hypotheses one true, say T, and $k-1$ false such that (i) the k hypotheses have equal prior probability and (ii) all k are simple hypotheses. Then Kerridge [*Ann. Math. Statist.* **34** (1963) 1109–1110] has given, for general stopping rule, the upper bound $(k-1)p/(1-p)$ on the frequency of misleading Bayes inferences. That is, $\lim_{n\to\infty} \sum^* P(X_n \mid T) \leqq (k-1)p/(1-p)$ where $\sum^*$ denotes summation over all vectors X_n for which sampling has terminated and $P(T \mid X_n) \leqq p$. In this paper we remove the two mentioned restrictions. The k hypotheses need not have equal prior probabilities and they need not be simple. Denoting the prior probabilities of the true hypothesis T and of the union of the $k-1$ false hypotheses by $P(T)$ and $P(F)$ respectively, we obtain for an upper bound on the frequency of misleading Bayes inferences: $cp/(1-p)$ where $c = P(F)/P(T)$. That is, $\lim_{n\to\infty}\sum^* P(X_n \mid T) \leqq cp/(1-p)$. (Received April 27, 1971.)

71T-47. Markov decision processes with a new optimality criterion. STRATTON C. JAQUETTE, Stanford University.

Standard finite state and action Markov decision processes with discounting are studied using a new optimality criterion called moment optimality. A policy is moment optimal if it lexicographically maximizes the sequence of signed moments of return with a positive (negative) sign if the moment is odd (even). For discrete time processes it is shown that a stationary policy is moment optimal by examining the negative of the Laplace transform of the total return random variable. An algorithm is developed to construct all stationary moment optimal policies. The algorithm iteratively examines the moment of return beginning with the first moment. The algorithm is shown to be finite. The corresponding results for continuous time processes are proved when general piecewise constant policies are allowed. A convergent Laurent expansion for all moments of return is derived for stationary policies, and this expansion is used to show that there are stationary policies that are moment optimal for all sufficiently small interest rates. The criterion of maximizing the expected utility of total return for exponential utility functions is also considered as related to the rest of the paper, and it is shown that for two related Markov decision processes stationary policies maximize utility. (Received May 3, 1971.)

71T-48. Doubly non-central F distribution-tables and applications. M. L. TIKU, McMaster University.

The distribution of $F'' = (f_2\chi_1'^2/f_1\chi_2'^2)$, where $\chi_1'^2$ and $\chi_2'^2$ are two independent non-central chi-square variates with degrees of freedom f_1 and f_2 and noncentrality parameters λ_1 and λ_2, respectively, is called the doubly non-central F-distribution. This distribution has applications in analysis-of-variance in which interaction or bias effects occur (Scheffé, H. *The Analysis of Variance*, Wiley, New York; Madow, W. G. *Ann. Math. Statist.* **19** (1948) 351–9, and in information theory and engineering (Price, R. *IRE Trans. Inf. Theory* **IT–8** 305–16). In this paper an expression for the probability integral of F'' is obtained from the Laguerre series expansion of the distribution of noncentral chi-square (Tiku, M. L. *Biometrika* **52** (1965) 415–27). The probability integral is tabulated for various values of the parameters f_1, f_2, λ_1 and λ_2. An application in analysis-of-variance is discussed. (Received May 4, 1971.)

71T-49. The law of the iterated logarithm for U-statistics and related von Mises statistics. R. J. SERFLING, Florida State University.

Let $X_1, X_2, \cdots$ be i.i.d. random vectors with df F. Consider estimation of $\theta(F) = E_F h(X_1, \cdots, X_m)$, where h is symmetric and $E_F h^2 < \infty$. For samples of size $n \geqq m$, Hoeffding (*Ann. Math. Statist.* **19** (1948)) introduced the unbiased estimators $U_n = \binom{n}{m}^{-1}\sum_c h(X_{i_1}, \cdots, X_{i_m})$, summation being over all $\binom{n}{m}$ choices of distinct $\{i_1, \cdots, i_m\}$ from $\{1, \cdots, n\}$. Define $h^*(x) = E_F h(X_1, X_2, \cdots, X_{m-1}, x) - \theta(F)$ and set $\zeta_1 = \mathrm{Var}_F h^*(X_1)$. THEOREM. *Let* $\zeta_1 > 0$. *Then, with probability* 1, $\limsup_{n\to\infty} n^{\frac{1}{2}}(U_n - \theta(F))/(2m^2\zeta_1 \ln\ln n)^{\frac{1}{2}} = 1$. PROOF. By the Hartman–Wintner (*Amer. J. Math.* **63** (1941)) law of iter. log. for i.i.d. sums, the statement of the theorem holds if U_n is replaced by $\hat{U}_n = \theta(F) + mn^{-1}\sum_1^n h^*(X_i)$. By Corollary 4.2 of Geertsema (*Ann. Math. Statist.* **41** (1970)), which is attributed to a referee (who, incidentally, is not the present writer), we have, with probability 1, $U_n - \hat{U}_n = o(n^{-1}\ln n)$ and thus $n^{\frac{1}{2}}(U_n - \hat{U}_n)/(2m^2\zeta_1 \ln\ln n) \to 0$, completing the proof. The related von Mises (*Ann. Math. Statist.* **18** (1947)) statistic for estimation of $\theta(F)$ is $V_n = \theta(F_n)$, the result of putting the sample df, F_n, in place of F in the definition of $\theta(F)$. Ghosh and Sen (*Calcutta Statist. Assoc. Bull.* **23** (1970)) show that for any $\alpha < 1$, with probability 1, $n^\alpha(U_n - V_n) \to 0$, provided that $E_F V_m{}^2 < \infty$. Accordingly, under this added proviso, the above theorem applies also with U_n replaced by V_n. (Received May 6, 1971.)

71T-50. On the variance of the number of zeros of a stationary Gaussian process. DONALD GEMAN, University of Massachusetts.

For a real, stationary Gaussian process $X(t)$, it is well known that the mean number of zeros of $X(t)$ in a bounded interval is finite exactly when the covariance function $r(t)$ is twice differentiable. Cramér and Leadbetter have shown that the variance of the number of zeros of $X(t)$ in a bounded interval is finite if $(r''(t) - r''(0))/t$ is integrable around the origin. We show that this condition is also necessary. Applying this result, we then answer the question raised by several authors regarding the connection, if any, between the existence of the variance and the existence of continuously differentiable sample paths. We exhibit counterexamples in both directions. (Received May 10, 1971.)

71T-51. On optimal estimation methods using stochastic approximation procedures. DAN ANBAR, University of California, Berkeley.

Robbins Monro type of stochastic approximation procedures for estimating the zero of a regression function $M(x)$, are considered. The procedures are defined recursively by, $X_{n+1} = X_n - An^{-1}Y_n$. We are interested in minimizing the asymptotic variance of X_n, by transforming the observations Y_n. The problem is investigated in the translation parameter case. A class of transformations $\mathscr{E}$ is defined. It is shown that if the underlying distribution F is absolutely continuous with density f, then if $g_0(x) = c(d/dx)\,[\log f(x)]$ is in $\mathscr{E}$, g_0 is the optimal transformation. The optimization is done with respect to the function g and the constant A. As it turns out the optimal solution depends on α_1, the slope of $M(x)$ at the zero. Under conditions essentially due to Sacks (*Ann. Math. Statist.* **29** (1958) 373–405), we estimate α_1 by $\hat{\alpha}_1 = \sum_{k=N_1}^{N}(X_k - X_N)Y(X_k)/\sum_{k=N_1}^{N}(X_k - X_N)^2$ with $N_1 = N_1(N)$ is such that $N_1/N \to 0$ and $N_1(N) \to \infty$ as $N \to \infty$, and show that $\hat{\alpha}_1 \to \alpha_1$ a.s. A two stage optimal estimation procedure which is free of unknown parameters is suggested. A number of applications are discussed with special reference to the bio assay problem. (Received May 11, 1971.)

71T-52. A solution to an open problem of Bechofer–Kiefer–Sobel. SALLY SIEVERS, Cornell University.

The inequality below is used to prove that the sequential ranking procedure $P_B{}^*$ [Bechhofer–Kiefer–Sobel, Sequential Identification and Ranking Procedures] ensures probability P^* of choosing all populations having τ's δ^* greater than the smallest $k-t$ and rejecting all those with τ's δ^* smaller than the largest t τ's. ($P_B{}^*$ chooses from k K–D populations the t with the largest τ-parameters). This extends the desired protection of $P_B{}^*$ beyond their case $\tau_{k-t+1} - \tau_{k-t} \geqq \delta^*$. Given two ordered k-vectors $\boldsymbol{\tau}$, $\tau_1 \leqq \cdots \leqq \tau_k$, and $\mathbf{y}$, $y_1 \leqq \cdots \leqq y_k$, we can form $k!$ permuted products $\boldsymbol{\tau}\cdot\alpha\mathbf{y}$, $\alpha \in S_k$, the permutation group, where $\alpha\mathbf{y} = (y_{\alpha(1)}, \cdots, y_{\alpha(k)})$. Given $\delta^* > 0$ and t, $1 \leqq t \leqq k$, define $r = r(\boldsymbol{\tau})$ and $s = s(\boldsymbol{\tau})$ so that exactly r components of $\boldsymbol{\tau}$ are $\leqq \tau_{k-t+1} - \delta^*$ and s components $\geqq \tau_{k-t} + \delta^*$. Define $\alpha \in \Omega(\boldsymbol{\tau}) \subset S_k$ if $\{\alpha(1), \cdots, \alpha(r)\} \subseteq \{1, \cdots, t\}$ and $\{\alpha(k-s+1), \cdots, \alpha(k)\} \subseteq \{k-t+1, \cdots, k\}$. Let $Q(\boldsymbol{\tau}) = (\sum_{\alpha \in \Omega(\boldsymbol{\tau})} \exp \boldsymbol{\tau}\cdot\alpha\mathbf{y})/(\sum_{\gamma \in S_k} \exp \boldsymbol{\tau}\cdot\gamma\mathbf{y})$. Then $Q(\boldsymbol{\tau}) \geqq Q(\boldsymbol{\tau}^*)$, for $\boldsymbol{\tau}^*$ such that $\tau_1{}^* = \cdots = \tau_{k-t}^*, \tau_{k-t+1}^* = \cdots = \tau_k{}^*$, and $\tau_{k-t+1}^* - \tau_{k-t}^* = \delta^*$. Here $Q(\boldsymbol{\tau}^*) = (\exp \boldsymbol{\tau}^*\cdot\mathbf{y})/(\sum_\beta \exp \boldsymbol{\tau}^*\cdot\beta\mathbf{y})$ where β enumerates the $\binom{k}{t}$ partitions of $\{1, \cdots, k\}$ into $k-t$- and t-cells. After crossmultiplying, each term $\exp(\boldsymbol{\tau}\cdot\gamma\mathbf{y} + \boldsymbol{\tau}^*\cdot\mathbf{y})$ is majorized by a term $\exp(\boldsymbol{\tau}\cdot\alpha\mathbf{y} + \boldsymbol{\tau}^*\cdot\beta\mathbf{y})$ such that $\boldsymbol{\tau}\cdot(\gamma\mathbf{y} - \alpha\mathbf{y}) \geqq \boldsymbol{\tau}^*\cdot(\mathbf{y} - \beta\mathbf{y})$, by using techniques related to Sobel [*Proc. Amer. Math. Soc.* **5** (1954) 596–602]. (Received May 14, 1971.)

71T-53. On the stochastic approximation of the root of a linear function with independent standard normal errors. R. A. AGNEW AND R. M. EMMERICHS, Air Force Institute of Technology.

We consider the relation $Y_n = \beta(X_n - \theta) + Z_n$ where $\beta > 0$, θ is a real number, Z_n is a sequence of independent $N(0, 1)$ random variables, Y_n is a sequence of observations, and X_n is a sequence of estimates for θ. When β is known and θ is unconstrained, various natural stage-wise parametric estimates correspond to Robbins–Monro procedures. The situation is somewhat different when θ is constrained or β is unknown. (Received May 18, 1971.)

71T-54. A model for studying stability of variance and informativeness of labels in sampling with varying probabilities. CLAES MAGNUS CASSEL, AND CARL ERIK SÄRNDAL, University of Umeå and University of British Columbia.

This paper presents a model for investigating how information contained in the labels of a finite population may be utilized in order to reduce variance when sampling from a finite population with varying probabilities. The model assumes a population of N values of a characteristic X given as $X_I = Y_I + Z_I (I = 1, \cdots, N)$, where the mean of X is to be estimated. Here Y_I is a "systematic" component in the sense that its relation to the label I is a priori known, or intelligently guessed at, while Z_1 is a "random noise" component. In the with replacement case, a sample of labels is selected with possibly unequal probabilities $p_I (I = 1, \cdots, N)$ attached to the various labels in each of the n draws. Assuming that N is reasonably large, the variance of the usual estimator is studied under various assumptions about the shape of the continuous distribution approximating the Y-distribution (e.g., gamma, Pareto, Weibull shapes), various assumptions about the influence of the random noise component Z, and different designs for distributing the probabilities p_I among the N labels. (Received May 21, 1971.)

71T-55. Skorokhod embedding of multivariate rv's, and the sample df. J. KIEFER, Cornell University.

Let $A_1 = (A_{11}, A_{12}, \cdots, A_{1k})$ be a zero-expectation martingale with finite fourth moments. Let $\{A_i\}$ be i.i.d. copies of A_1 and $\sum_1^n A_i = S_n = (S_{n1}, \cdots, S_{nk})$. (More generally, A_i can be a martingale in i, too.) Let $\{\xi(s, t);\ s, t \geqq 0\}$ be the 0-expectation Gaussian process with two-dimensional time and $E\xi(s_1, t_1)\xi(s_2, t_2) = \min(s_1, s_2) \min(t_1, t_2)$. We define rv's $T(k, n)$ such that $\{\xi(k, T(k, n));\ k, n \in Z^+\}$ has the same law as $\{S_n, n \in Z^+\}$ and $n^{-\frac{1}{2}}[\xi(k, T(k, n)) - \xi(k, ET(k, n))]$ is of order $n^{-\frac{1}{4}}(\log n)^\lambda$wp 1. This generalization of the Skorokhod embedding yields vector analogues of weak convergence error estimates of Skorokhod, Rosenkrantz, Sawyer for $k = 1$; Strassen's strong invariance principles and Rosenkrantz's von Mises statistic results are extended. For $k \to \infty$, one obtains results for certain martingale-related processes like the sample df, F_n, for uniform rv's. Some of our results on the latter (where $n^{-\frac{1}{4}}$ becomes $n^{-\frac{1}{5}}$ or $n^{-\frac{1}{6}}$) are only minor improvements over recent ones of D. W. Müller, but our embedding differs from his and that of Breiman by exhibiting the martingale $n[(s+1)F_n(s/(s+1))-s]$ explicitly in $\xi(s, n)$ for all s, n. It is explained why the Skorokhod approach cannot achieve the $n^{-\frac{1}{2}}$ of its scalar counterpart or of the multivariate Berry–Esseen theorem. (Received May 24, 1971).

71T-56. On a measure of efficiency of an estimating equation. V. P. BHAPKAR, University of Kentucky.

Godambe (*Ann. Math. Statist.* **31** (1960)) has shown that for estimating a real-valued parameter the maximum likelihood estimating equation enjoys an optimal property in a certain sense under some regularity assumptions. That optimality criterion can be used to define a measure of information, $J_g(\theta) = [\xi_\theta g'(X, \theta)]^2/\xi_\theta g^2(X, \theta)$ contained in an estimating equation $g(x, \theta) = 0$ and, thus, its efficiency. We prove the following THEOREM. *If T is sufficient for $\{P_\theta, \theta \in \Omega\}$ and $g^*(t, \theta) = \xi_\theta[g(x, \theta) \mid t]$, then $J_g(\theta) \leqq J_{g^*}(\theta)$ with equality* iff $g(x, \theta) = g^*(t(x), \theta)$ a.e. (P_θ). This theorem and the earlier results are then extended to the case of a vector-valued parameter. The results here are seen to be generalizations of the Rao–Blackwell Theorem and of the multi-parametric versions of the Cramér–Rao inequality and the R–B Theorem so as to include estimating functions involving the parameter. (Received May 25, 1971.)

71T-57. The log likelihood ratio in segmented regression. PAUL I. FEDER, Yale University and Princeton University.

This paper deals with the asymptotic distribution of the log likelihood ratio statistic in regression models which have different analytical forms in different regions of the domain of the independent variable. It is shown that under suitable identifiability conditions, the asymptotic chi square results of Wilks and Chernoff are applicable. It is shown by example that if there are actually fewer segments than the number assumed in the model, then the least squares estimates are not asymptotically normal and the log likelihood ratio statistic is not asymptotically χ^2. The asymptotic behavior is then more complicated, and depends on the configuration of the independent variable observation points. (Received May 25, 1971.)

71T-58. Maximum likelihood estimation for stationary *m*-dependent sequences. M. SIVA PRASAD AND B. L. S. PRAKASA RAO, Indian Institute of Technology, Kanpur.

Let $\{X_n, n \geqq 1\}$ be a stationary m-dependent sequence of random variables. Let $p(x_1, \cdots, x_n; \theta)$ denote the n-dimensional joint density function, which depends on a single unknown parameter θ. In this paper, we proved the weak consistency asymptotic normality and first-order efficiency of a maximum likelihood estimator (MLE) $\hat{\theta}_n$ of θ. The proof runs on the same lines as that of Sarma, Y. R. K. [Premiere These 1968; Sur les tests et sur L'estimation de parame'tres pour certains Processus stochastiques Stationnaires] for Markov chains. The strong consistency of the MLE is proved under a different set of conditions. (Received May 26, 1971.)

71T-59. Optimal resource allocation. E. LASKA, M. MEISNER AND C. SIEGEL, Rockland State Hospital.

To analyze the optimal allocation of a resource among n jobs, a mathematical definition of a procedure, λ, is given. The service times $\{X_i\}$, of the jobs are assumed to be independent, identically distributed random variables with common distribution function F having a monotone hazard function h: $(h(x) = -d/dx \log (1-F(x))$. A procedure λ (actually $\lambda(t)$ since it is time dependent) induces completion time random variables $S_j(\lambda)$, $j = 1, \cdots, n$ representing the times at which j of the n jobs are completed. The partial sums, $W_j(\lambda)$, of the completion times are introduced and may be interpreted as the total cumulative waiting times required for the first j completions. If the hazard function is increasing, it is shown that the procedure of allocating the full resource individually to each job until its completion simultaneously minimizes the quantities $E[W_j(\lambda)]$ for all j. The procedure which at any instant of time equally allocates the resource among all of the remaining jobs, (absolute time sharing) is shown to minimize $E[W_n(\lambda)]$ if the hazard is decreasing. Non-identically distributed service times having hazard functions that are monotone and are of the form $h(x+A_i)$ $i = 1, 2, \cdots, n$, are introduced and optimal procedures in the sense of minimizing $E[W_n(\lambda)]$ are determined. (Received May 27, 1971.)

71T-60. A new matrix product and its applications. D. S. TRACY AND R. P. SINGH, University of Windsor.

Matrix differential calculus, as discussed by Neudecker [*J. Amer. Statist. Assoc.* **64** (1969) 953–963] and by Tracy and Dwyer [*J. Amer. Statist. Assoc.* **64** (1969) 1576–1594], involves Kronecker product of matrices. In this paper, a new matrix product is introduced. For arbitrarily partitioned matrices $A = [A^{ij}]$ and $B = [B^{uv}]$, the product $A \otimes B$ is the partitioned matrix $[(A^{ij} \otimes B^{uv})]$, where $A^{ij} \otimes B^{uv}$ is the Kronecker product $(a_{\alpha\beta}^{ij} \otimes B^{uv})$. Some properties of this product are stated. Two elementary matrices, related to partitioned identity matrices, are defined. These, together with the new matrix product, are useful in differentiating partitioned matrix functions. Formulae for partitioned matrix derivatives of some general matrix functions are presented. The results extend those of Tracy and Dwyer to the partitioned situation. An econometric application of the above partitioned matrix differential calculus is provided. (Received June 1, 1971.)

71T-61. Some families of designs for multistage experiments: mutually balanced Youden designs when the number of treatments is prime power or twin primes. A. HEDAYAT, E. SEIDEN AND W. T. FEDERER, Cornell University, Michigan State University and Cornell University.

In this paper the concepts of "balance for ordered and for unordered pairs of treatments" are introduced. Methods for constructing multistage experiment designs which are Youden designs at each stage are given. In the construction of these designs we have tried to accommodate as much orthogonality and balance, both in our sense and the classical sense, as is possible in these multistage experiments. These constructions are given via several theorems of which the following results highlight the content of the paper. In one theorem we give a uniform method of converting a set of t mutually orthogonal Latin squares of order n into a t-stage balanced (for ordered pairs and also in the classical sense) $(n-1)\times n$ Youden designs. If one wants to apply this theorem he should first construct t mutually orthogonal Latin squares of order n. Unfortunately if $n = 6$ then there are no orthogonal Latin squares of order 6 and, besides, the known methods of construction of orthogonal Latin squares of order $n = 4t+2$ is not uniform. We have partially overcome these difficulties by giving a uniform method for constructing 2-stage $(n-1)\times n$ Youden designs for all even n. In another theorem we give a method of constructing $(2\lambda+1)$-stage balanced (for unordered pairs and also in the classical sense) $(2\lambda+1)\times(4\lambda+3)$ Youden designs whenever $4\lambda+3$ is a prime power. A method of construction of $(p^\alpha-1)$-stage balanced (in the classical sense) $(v-1)/2\times v$ Youden designs is given in another theorem, whenever $v = 4\lambda+3 = p^\alpha q^\beta, q^\beta = p^\alpha+2, p$ and q primes and α a positive integer. These constructions mainly depend on difference sets based on the elements of Galois fields. (Received June 4, 1971.)

71T-62. Properties of Brownian motion first passage time process. M. T. WASAN, Queen's University.

Let $\{x(t),\ t \geqq 0\}$ be the first passage time process of Brownian motion with positive drift with density function $f(x, t) = t(2\pi x^3)^{-\frac{1}{2}} \exp[-(x-t)^2/(2x)]$, $x > 0$, $t > 0$; $f(x, t) = 0$, otherwise. It is shown that the process is of bounded variation and a characterization of the process is given. It is proved that $x(t)/t \to 1$ as $t \to \infty$ with probability one. The stochastic integral of the process is defined and its properties are investigated. (Received June 4, 1971.)