The Annals of Mathematical Statistics
1971, Vol. 42, No. 5, 1779–1781

BOOK REVIEWS

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HAYES, C. A. AND PAUC, C. Y. **Derivation and Martingales.** Springer-Verlag, Berlin, 1970. viii + 203 pp. $13.20

REVIEW BY K. KRICKEBERG
University of Heidelberg

This is a clear survey over a complex field full of technical details. The first part is devoted to a situation which, although general and abstract in its setting, still represents classical differentiation theory: the derivation of set functions by means of "bases," that is, families of generalized sequences (nets) of sets associated to the various points of the basic space like sequences of spheres converging to a point. Properties possessed by classical bases (Vitali covering properties) are assumed and derivation statements derived. Other chapters investigate implications of the converse type, and various methods of proving Vitali properties of particular bases. In part two, a fundamental change of the point of view takes place: instead of nets of sets associated to individual points, partitions of the entire space and the sigma-algebras they generate are the basic elements. This allows to consider, in addition to pointwise convergence, theorems of the global type: stochastic convergence or convergence in an Orlicz space. Finally, replacing the (atomic) sigma-algebras generated by finite or countable partitions by arbitrary sigma-algebras, martingales come in. In fact it turns out that most of the theory can be developed from the outset in this generality. On the other hand, there are many interesting particular cases and applications. The bibliography is virtually complete, and a complement collects various results connected with the field proper.

KARLIN, SAMUEL. **Total Positivity.** Stanford University Press, 1968. xi + 576 pp. $17.50.

Review by ALBERT W. MARSHALL
University of Washington

Many of us interested in total positivity have anxiously awaited the appearance of this book, which together with a promised second volume constitutes an exhaustive and definitive treatise on both the theory and its applications. The book is of major importance: it is a basic work on a subject not previously treated adequately in the literature. Moreover, there are included a considerable number of new results which have not previously appeared in print.

A special case of total positivity is particularly familiar to statisticians under the name of "monotone likelihood ratio," and its importance in statistical theory is well established and generally appreciated. Applications of total positivity have also been made, e.g., in the theories of decisions, games, mathematical economics, reliability and stochastic processes. *Annals* readers may be disappointed to find that many of these applications have been deferred to Volume II.

The uninitiated need not be frightened by the depth and breadth of Professor Karlin's book, as it can more than adequately serve anyone desiring only a basic introduction to the subject of total positivity. In fact, the first three sections of Chapter 1 present the most important definitions, a proof of the basic composition formula, and a discussion of the variation diminishing properties. Also, the most important examples of totally positive functions are listed. Very little background is required for the easy reading of these sections. Most of the remaining required definitions can be found in the first section of Chapter 2. Once these few pages have been read, nearly all of the book is accessible.

An outstanding feature of the book is its wealth of examples, often discussed in considerable detail. Partly because of this the book makes fascinating browsing.

Most chapters of the book have a "complement" section near the end. These sections contain a number of compactly stated examples and applications, and they deserve the reader's special attention. Credits and historical comments are confined to the final section of each chapter, "Notes and References." The author might have devoted a little more care to the preparation of these sections.

Many people even peripherally interested in total positivity will want a copy of Karlin's book handy for reference, although they can expect some frustration from the brevity of the index. One might hope that Volume II will contain a detailed index to both volumes.

A small warning is offered here, particularly for readers with some background in the subject. There are at least a dozen sign behavior properties introduced in the book which are related to total positivity and each is given a set of initials for abbreviations which do not always follow those used in earlier writings. The new usage seems preferable, and hopefully it will become standard. Although a symbols index would have been a welcome addition to the book, many symbols can be found defined either on pages 11, 12 or pages 46–49.

The book begins (Chapter 0) with a summary of matrix and determinant formulas. Here, results are somewhat tersely stated, and some references to fuller treatments might have been useful for the reader not already familiar with the results. E.g., difficulty could be encountered in understanding the definition of a compound matrix given on page 1, where in the same sentence "α" and "β" are used in two different senses.

The introductory Chapter 1 has already been recommended even to readers not necessarily interested in delving deeply into the subject.

Total positivity and the more general concept of sign regularity have "ordinary," "strict" and "extended" versions. The exhaustive aspect of this book becomes evident in Chapter 2, where the sometimes subtle differences between the three versions are carefully analyzed. This is one of the most involved chapters of the book, and non-specialists probably should defer a detailed reading.

Chapter 3, on composition laws and their applications, is particularly rich in detailed examples. Section 4 is devoted to examples arising in statistics; statisticians might wish for more details here and wonder why the section was singled out to appear in fine print. Section 5 is also of special interest; here one finds the Ghurye–Wallace Composition Theorem. Somewhat buried on page 137 is the fact that binomial coefficients are totally positive; this example might have been placed with the more basic examples in Chapter 1.

Chapter 4, on smoothness properties of sign-regular functions pays special attention to order two conditions of which "monotone likelihood" is an example. Section 1 contains the important observation that a function is "PF_2" if and only if its logarithm is concave. The differentiability properties investigated in Section 3 have some rather involved proofs—one is 18 pages long.

The heart of Chapter 5 on variation-diminishing transforms is given in Theorem 3.1, which takes more than a full page to state. Unfortunately, the hypotheses of this theorem include "the smoothness conditions stated above," and the reader may have difficulty in determining just what this means. Nevertheless, the theorem is an admirable summary.

Chapter 6 is a collection of miscellaneous applications of the variation diminishing property. Most readers will find something of interest here, although few will be interested in all the applications because the examples cover such a broad range of topics.

Chapter 7 is concerned primarily with probability densities that are Pólya frequency functions of order two or order infinity. Much of the material is based on early work of Professor I. J. Schoenberg, to whom the book is dedicated, but there are also a number of new results. This excellent chapter deserves the special attention of anyone interested in probability.

Pólya frequency sequences are discussed in Chapter 8; interesting connections between Pólya frequency sequences and densities are pointed out in Section 6 and elsewhere.

Chapter 9 is concerned with cyclic Pólya frequency functions. Chapter 10, the final chapter of the book, is on differential operators and total positivity. Here, the total positivity properties of Green's functions are discussed. Some applications to approximation theory are included.

Although the origins and many basic results of the theory of total positivity predate his interest in the subject, Professor Karlin, more than anyone else, is responsible for its present state of development. Even so, the writing of this book was a major undertaking which could never have been brought to fruition without a rare dedication and long sustained effort. This outstanding book should do much to promote appreciation for both the beauty and the usefulness of its subject, and to further the discovery of new results and applications.