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ON THE NUMBER OF GENERATORS OF SATURATED MAIN EFFECT FRACTIONAL REPLICATES[1]

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1. Introduction. Recently Paik and Federer (1970) have presented a method to generate from a given saturated main effect plan of the s^m factorial a class of plans such that the determinant of the information matrix remains invariant. They did not however enumerate the total number of generators, nor did they enumerate the number of plans generated by each generator. This paper solves this problem using a group-theoretic approach.

2. Some known results. Consider the well-known Abelian group G of order s^m, which consists of row vectors of the form $u' = (u_1, u_2, \cdots, u_m)$, where prime indicates transpose and each u_i is an element of $GF(s)$, s being a prime or a power of a prime. The following lemma is given in most elementary text books on group theory:

LEMMA 2.1. (a) *Every subgroup of G is of order s^k, k being an integer.*

(b) *The number of distinct subgroups of order s^k of G is equal to*:

$$\alpha = \alpha(k, m, s) = \frac{(s^m-1)(s^m-s)(s^m-s^2)\cdots(s^m-s^{k-1})}{(s^k-1)(s^k-s)\,(s^k-s^2)\cdots(s^k-s^{k-1})}. \tag{2.1}$$

(c) *The number of cosets into which G can be partitioned relative to a subgroup of order s^k is equal to*:

$$\beta = \beta(m, s, k) = s^{m-k}. \tag{2.2}$$

Now, let F denote the set of $[m(s-1)+1] \times m$-matrices such that the rows of each matrix form an $[m(s-1)+1]$-subset of G. Note that the order of F is equal to:

$$\gamma = \gamma(m, s) = \binom{s^m}{m(s-1)+1}. \tag{2.3}$$

Consider linear functionals f from G into R(= real line) of the form:

$$f(u') = c'\phi \tag{2.4}$$

where c' is an $[m(s-1)+1]$-row vector of known real coefficients and ϕ is an $[m(s-1)+1]$-column vector of unknown real parameters. These functionals lead to the fact that to each $A \in F$ there corresponds a square matrix B of order $[m(s-1)+1]$, the rows of which are vectors of the form c'. Denote the set of B's by H, which incidentally has the same order as F. The following theorem has been proved recently by Paik and Federer (1970):

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THEOREM 2.1. (Paik–Federer). *If two matrices A_1 and A_2 in F are such that the rows of A_2 are obtained from A_1 by adding a particular vector of G to all the rows of A_1 then the corresponding matrices B_1 and B_2 in H are such that* $|B_1'B_1| = |B_2'B_2|$. The matrix A_1 is in this instance called the *generator* of A_2.

In the next section we establish the number of distinct *generators* admitted by G and the number of matrices which each *generator* generates in F such that each class has the invariant property of the above theorem.

3. The number of generators and the number of generated plans. There are two distinct cases to be reckoned with, because the results will depend on whether $[m(s-1)+1]$ is equal to s^k or is not equal to s^k for some k. If $[m(s-1)+1]$ is equal to s^k (i.e. the condition $m = (s^k-1)/(s-1)$ is satisfied for some k), then the possibility exists that the rows of $A \in F$ form a subgroup. If this is the case, then by Lemma 2.1 we know that G has α distinct subgroups of order s^k each giving rise to β cosets. If on the other hand the rows of $A \in F$ do not form a subgroup, then from A we may obtain s^m matrices by adding to each row of A an element of G. If we denote by $\delta(m, s)$ the number of *generators* in F such that the rows of each do not form a subgroup, then we must have:

$$\alpha \cdot \beta + \delta \cdot s^m = \gamma. \tag{3.1}$$

Hence

$$\delta = (\gamma - \alpha \cdot \beta)/s^m \tag{3.2}$$

so that the total number of *generators* in the case $[m(s-1)+1] = s^k$ is equal to $\alpha + \delta$.

Now consider the case where $m(s-1)+1 \neq s^k$, i.e. the rows of $A \in F$ cannot form a subgroup. In this instance every $A \in F$ leads us to precisely s^m other A's by adding an element of G to every row of A. Hence the number of generators must equal γ/s^m.

We have thus proved the following theorem:

THEOREM 3.1. *If $[m(s-1)+1]$ is equal to s^k for some k, then the number of generators in F is equal to $\alpha+\delta$ and when $[m(s-1)+1]$ is not equal to s^k then the number of generators is equal to γ/s^m.*

4. Discussion. The result obtained above and the result of Paik and Federer (1970) lead us to interesting aspects of the enumeration problem in saturated main effect fractional replicates. First of all one needs to study only the *generators* in order to assess the determinant of $B'B$ for every member in the generated class. Knowing the number of *generators* and the number of plans each *generator* generates, a knowledge of the possible values of $B'B$ for the *generators* and their frequencies will enable us to resolve the distribution of $|B'B|$ in the case of main effect fractional replicates. For the 2^m series attempts have been made by Metropolis and Stein (1967) and by Raktoe and Federer (1970a), (1970b) for the singular class. The problems are far from resolved at present.

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