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THE DISTRIBUTIONS OF SOME TEST CRITERIA IN MULTIVARIATE ANALYSIS[1]

C. G. TROSKIE
University of Cape Town

1. Introduction. Let the random vector $\mathbf{X}: p \times 1$ be distributed according to a multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let $\mathbf{X}_{(1)}, \cdots, \mathbf{X}_{(N)} (N > p)$ be a random sample of N observations on $\mathbf{X}$ and let

$$\mathbf{A} = \sum_{\alpha=1}^{N} (\mathbf{X}_{(\alpha)} - \overline{\mathbf{X}})(\mathbf{X}_{(\alpha)} - \overline{\mathbf{X}})'$$

be the Wishart matrix. $\overline{\mathbf{X}} = N^{-1} \sum_{\alpha=1}^{N} \mathbf{X}_{(\alpha)}$ and $\mathbf{A}/N$ are the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, respectively.

Let $\mathbf{X}$ be partitioned into two sets of components

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \end{bmatrix} \tag{1.1}$$

where $\mathbf{X}^{(1)}$ is $q \times 1$ and $\mathbf{X}^{(2)}$ is $r \times 1$ with $q \leqq r$, $q + r = p$. Partition $\boldsymbol{\Sigma}$ and $\mathbf{A}$ accordingly, that is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{1.2}$$

where $\boldsymbol{\Sigma}_{11}$ and $\mathbf{A}_{11}$ are $q \times q$, $\boldsymbol{\Sigma}_{12}$ and $\mathbf{A}_{12}$ are $q \times r$ and $\boldsymbol{\Sigma}_{22}$ and $\mathbf{A}_{22}$ are $r \times r$ matrices.

Several criteria have already been proposed in the literature to test the independence of the sets $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, that is to test the hypothesis $H_0 : \boldsymbol{\Sigma}_{12} = 0$. Nearly all these criteria are based on the generalised multiple correlation matrix

$$\mathbf{R} = \mathbf{A}_{11}^{-\frac{1}{2}} \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \mathbf{A}_{21} \mathbf{A}_{11}^{-\frac{1}{2}} \tag{1.3}$$

which was defined by Khatri (1964) as a measure of the correlation between the sets $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$. $\mathbf{A}_{11}^{\frac{1}{2}}$ is the positive definite square root of $\mathbf{A}_{11}$ (positive definite) and can either be the symmetric or lower-triangular square root. We also adopt the convention that in an expression like $\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}$ the post-multiplier is $(\mathbf{A}^{\frac{1}{2}})'$. Thus $\mathbf{R}$ defined by (1.2) is always a symmetric matrix.

Some of the criteria proposed are

(i) The likelihood ratio criterion (Wilks (1932)), Anderson (1958))

$$\lambda^{2/N} = W = |\mathbf{I} - \mathbf{R}|. \tag{1.4}$$

(ii) Roy's largest root criterion based on the largest canonical correlation coefficient, that is the largest characteristic root of $\mathbf{R}$.

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(iii) Hotelling's generalised $T_0^{\,2}$ (or Pillai's $U^{(q)}$) defined by (see Pillai (1955))

$$(1.5)\qquad U^{(q)} = T_0^{\,2}/c = \operatorname{tr}\mathbf{R}(\mathbf{I}-\mathbf{R})^{-1} = \textstyle\sum_{i=1}^{q} r_i^{\,2}/(1-r_i^{\,2})$$

where c is a constant and $r_1^{\,2} < r_2^{\,2} < \cdots < r_q^{\,2}$ are the characteristic roots of $\mathbf{R}$.

(iv) The criteria proposed by Pillai (1955)

$$(1.6)\qquad V^{(q)} = \operatorname{tr}\mathbf{R} = \textstyle\sum_{i=1}^{q} r_i^{\,2},$$

$$(1.7)\qquad H^{(q)} = q(\operatorname{tr}(\mathbf{I}-\mathbf{R})^{-1})^{-1} = q(\textstyle\sum(1-r_i^{\,2})^{-1})^{-1},$$

$$(1.8)\qquad R^{(q)} = q(\operatorname{tr}\mathbf{R}^{-1})^{-1} = q(\textstyle\sum 1/r_i^{\,2})^{-1},$$

$$(1.9)\qquad T^{(q)} = q(\operatorname{tr}(\mathbf{I}-\mathbf{R})\mathbf{R}^{-1})^{-1} = q(\textstyle\sum(1-r_i^{\,2})/r_i^{\,2}\}^{-1}.$$

Wilks criterion is the qth power of a geometric mean, while $U^{(q)}$ and $V^{(q)}$ are q times the arithmetic means. $H^{(q)}$, $R^{(q)}$ and $T^{(q)}$ are based on the harmonic mean.

For simplification let

$$(1.10)\qquad U = U^{(q)} = \operatorname{tr}\mathbf{R}(\mathbf{I}-\mathbf{R})^{-1},$$

$$(1.11)\qquad V = V^{(q)} = \operatorname{tr}\mathbf{R}$$

and

$$(1.12)\qquad Q = \operatorname{tr}(\mathbf{I}-\mathbf{R})\mathbf{R}^{-1}.$$

Then $H^{(q)} = (1+U/q)^{-1}$, $R^{(q)} = (1+Q/q)^{-1}$ and $T^{(q)} = q/Q$; and thus we need only derive the distributions of U, V and Q which is the main purpose of this paper.

2. The central and noncentral distributions of U, V and Q.

The statistics U, V and Q are functions of the generalised multiple correlation matrix $\mathbf{R}$ or of its characteristic roots. For the central case, that is when the two sets $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent, $\mathbf{R}$ has a multivariate Beta Type 1 distribution but for the noncentral case the distribution of $\mathbf{R}$ becomes untraceable (Troskie). However the joint distribution of the characteristic roots, $r_1^{\,2}, \cdots, r_q^{\,2}$ of $\mathbf{R}$, that is the squares of the canonical correlation coefficients, is known and is given by (Constantine (1963));

$$(2.1)\qquad (\textstyle\prod^{\frac{1}{2}q^2}/\Gamma_q(\frac{1}{2}q))\beta_1(\mathbf{R},\frac{1}{2}r,\frac{1}{2}n-\frac{1}{2}r)|\mathbf{I}-\mathbf{P}|^{\frac{1}{2}n}\alpha_q(\mathbf{R})\,{}_2F_1(\frac{1}{2}n,\frac{1}{2}n;\frac{1}{2}r;\mathbf{P},\mathbf{R});\qquad 0 \leqq r_1^{\,2} \leqq \cdots \leqq r_q^{\,2} < 1$$

where

$$(2.2)\qquad \beta_1(\mathbf{R},\tfrac{1}{2}r,\tfrac{1}{2}n-\tfrac{1}{2}r) = (\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}r)\Gamma_q(\tfrac{1}{2}n-\tfrac{1}{2}r)) \cdot |\mathbf{R}|^{\frac{1}{2}(r-q-1}|\mathbf{I}-\mathbf{R}|^{\frac{1}{2}(n-r-q-1)}$$

$$(2.3)\qquad \alpha_q(\mathbf{R}) = \textstyle\prod_{i>j}(r_i^{\,2}-r_j^{\,2}),\qquad \mathbf{P} = \operatorname{diag}(\rho_1^{\,2},\cdots,\rho_q^{\,2}),$$

the Gamma coefficient $\Gamma_q(a)$ and the hypergeometric function are defined by James (1964).

LEMMA 2.1. (Khatri and Pillai (1968)). *If A_κ denotes the integral*

$$\int_D |\mathbf{Z}|^{\frac{1}{2}(f-q-1)}\alpha_q(\mathbf{Z})C_\kappa(\mathbf{Z})\,dz_1,\cdots,dz_{q-1} \tag{2.4}$$

where $Z = \operatorname{diag}(z_1, \cdots, z_q)$, $z_q = 1-z_1-\cdots-z_{q-1}$ *and D is given by*

$$D:(0<z_1<z_2<\cdots<z_{q-2}<z_{q-1}<z_q=1-z_1-z_2-\cdots-z_{q-1}), \tag{2.5}$$

then

$$A_\kappa = ((\tfrac{1}{2}f)_\kappa\Gamma_q(\tfrac{1}{2}f)\Gamma_q(\tfrac{1}{2}q)C_\kappa(\mathbf{I}_q)/\textstyle\prod^{\frac{1}{2}q^2}(\tfrac{1}{2}fq)_k\Gamma(\tfrac{1}{2}fq)). \tag{2.6}$$

LEMMA 2.2. *For symmetric matrix* $\mathbf{S}$

$$|\mathbf{I}+\mathbf{S}|^{-a}(a)_\kappa C_\kappa((\mathbf{I}+\mathbf{S})^{-1}) = \textstyle\sum_{m=0}^{\infty}\sum_\eta\sum_\delta((-1)^m(a)_\delta g_{\kappa,\eta}^{\delta}C_\eta(\mathbf{S})C_\delta(\mathbf{I})/C_\eta(\mathbf{I})m!) \tag{2.7}$$

where $\delta = (\delta_1, \cdots, \delta_q)$, $\delta_1 \geqq \delta_2 \geqq \cdots \geqq \delta_q \geqq 0$,

$\sum\delta_i = m+k = d$; $\kappa = (k_1, \cdots, k_q)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_q \geqq 0$, $\sum k_i = k$, *is a partition of the integer k into not more than q parts*; $\eta = (m_1, \cdots, m_q)$, $m_1 \geqq m_2 \geqq \cdots \geqq m_q$, $\sum m_i = m$, *is a partition of the integer m into not more than q parts. The coefficient* $g_{\kappa,\eta}^{\delta}$ *has been tabulated by Khatri and Pillai* (1968) *for various values of the arguments.*

PROOF. The lemma follows immediately from the following relationship which is valid for symmetric $\boldsymbol{\theta}$,

$$\begin{aligned}
&\Gamma_q(a)|\mathbf{I}+\mathbf{S}|^{-a}\textstyle\sum_{k=0}^{\infty}\sum_\kappa((-1)^k(a)_\kappa C_\kappa((\mathbf{I}+\mathbf{S})^{-1})\cdot C_\kappa(\theta)/k!C_\kappa(\mathbf{I}))\\
&\quad=\textstyle\int_{B>0}\int_{O(q)}\int_{O(q)}\operatorname{etr}(-(\mathbf{I}+\mathbf{H}_1\mathbf{S}\mathbf{H}_1'+\mathbf{H}_2\boldsymbol{\theta}\mathbf{H}_2')\mathbf{B})|\mathbf{B}|^{a-\frac{1}{2}(q+1)}\,d\mathbf{B}\,d\mathbf{H}_1\,d\mathbf{H}_2\\
&\quad=\Gamma_q(a)\textstyle\sum_{k=0}^{\infty}\sum_\kappa\sum_{m=0}^{\infty}\sum_\eta\sum_\delta((-1)^{m+k}C_\eta(\mathbf{S})C_\kappa(\theta)C_\delta(\mathbf{I})(a)_\delta g_{\kappa,\eta}^{\delta}\\
&\qquad\div C_\eta(\mathbf{I})C_\kappa(\mathbf{I})m!k!).
\end{aligned} \tag{2.8}$$

We will now derive the densities of U, V and Q.

THEOREM 2.1. *The density function of* $U = \operatorname{tr}\mathbf{R}(\mathbf{I}-\mathbf{R})^{-1} = \sum r_i^2/(1-r_i^2)$ *is given by*

$$\begin{aligned}
&(\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}n-\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}rq))|\mathbf{I}-\mathbf{P}|^{\frac{1}{2}n}\\
&\quad\cdot U^{\frac{1}{2}rq-1}\textstyle\sum_{k=0}^{\infty}\sum_\kappa\sum_{d=0}^{k}\sum_\delta((-1)^{k+d}U^k(\tfrac{1}{2}n)_\kappa\\
&\quad\cdot(\tfrac{1}{2}n)_\delta(\tfrac{1}{2}r)_\kappa a_{\kappa,\delta}\ {}_\kappa(\mathbf{I})C_\delta(\mathbf{P})/k!(\tfrac{1}{2}rq)_k(\tfrac{1}{2}r)_\delta C_\delta(\mathbf{I}))
\end{aligned} \tag{2.9}$$

for $|U| < 1$ *and where* $a_{\kappa,\delta}$ *has been defined by Constantine* (1966) *as*

$$C_\kappa(\mathbf{I}+\mathbf{A})/C_\kappa(\mathbf{I}) = \textstyle\sum_{d=0}^{k}\sum_\delta a_{\kappa,\delta}C_\delta(\mathbf{A})/C_\delta(\mathbf{I}).$$

PROOF. The proof of (2.9) is straightforward. It is easy to show that $U = \operatorname{tr}\mathbf{E}^{-1}\mathbf{B}$ where $\mathbf{E} = \mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ and $\mathbf{B} = \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$. Now the conditional distribution of U for given $\mathbf{A}_{22}$ has already been derived by Constantine (1966). Multiplying Constantine's result with the marginal density of $\mathbf{A}_{22}$ (which is $W(\mathbf{\Sigma}_{22}, n)$) and integrating over $\mathbf{A}_{22}$ yields the result.

COROLLARY 2.1. *If the sets* $\mathbf{X}^{(1)}$ *and* $\mathbf{X}^{(2)}$ *are independent, that is* $\mathbf{\Sigma}_{12} = 0$ *and* $\mathbf{P} = \mathbf{0}$, *then from* (2.9) *follows the central density of* U *as*

$$(2.10)\quad (\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}n-\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}rq))U^{\frac{1}{2}rq-1}\textstyle\sum_{m=0}^{\infty}\sum_{\eta}((\tfrac{1}{2}n)_{\eta}(\tfrac{1}{2}r)_{\eta}(-1)^m U^m \cdot C_{\eta}(\mathbf{I})/(\tfrac{1}{2}rq)_m m!) \qquad \text{for } |U|<1.$$

The density (2.10) was first derived by Constantine (1966). The density of U, for given $\mathbf{A}_{22}$, given by Constantine (1966) is the density of U for the multiple regression problem, that is when the set of variables $\mathbf{X}^{(1)}$ depends on the fixed set of variables $\mathbf{x}^{(2)}$. When there is "no regression," that is all the regression coefficients of $\mathbf{X}^{(1)}$ on $\mathbf{x}^{(2)}$ are zero, then the density of Constantine of course also reduces to (2.10). It is well known that the central densities of the test criteria for the "regression" problem and the "correlation" problem are identical, but that the noncentral densities are different.

THEOREM 2.2. *The density function of the test criterion* $V = \operatorname{tr}\mathbf{R} = \sum_{i=1}^{q} r_i^2$ *is given by*

$$(2.11)\quad \begin{aligned}&(\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}n-\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}rq))|\mathbf{I}-\mathbf{P}|^{\frac{1}{2}n}\\ &\cdot V^{\frac{1}{2}rq-1}\textstyle\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\kappa}\sum_{\eta}\sum_{\delta}((\tfrac{1}{2}n)_{\kappa}(\tfrac{1}{2}n)_{\kappa}\\ &\cdot(\tfrac{1}{2}(q+1+r-n))_{\eta}g_{\kappa,\eta}^{\delta}V^{d}(\tfrac{1}{2}r)_{\delta}C_{\kappa}(\mathbf{P})\\ &\cdot C_{\delta}(\mathbf{I})/(\tfrac{1}{2}r)_{\kappa}(\tfrac{1}{2}rq)_{d}m!k!C_{\kappa}(\mathbf{I})) \qquad \text{for } 0<V<1.\end{aligned}$$

PROOF. Now $\operatorname{tr}\mathbf{R} = \operatorname{tr}(\mathbf{E}+\mathbf{B})^{-\frac{1}{2}}\mathbf{B}(\mathbf{E}+\mathbf{B})^{-\frac{1}{2}}$ and since the conditional density of $\operatorname{tr}\mathbf{R}$ for $\mathbf{A}_{22}$ fixed was derived by Khatri and Pillai (1968) the result follows again by integrating over $\mathbf{A}_{22}$.

COROLLARY 2.2. *If* $\mathbf{P} = \mathbf{0}$, $\mathbf{X}^{(1)}$ *and* $\mathbf{X}^{(2)}$ *are independent and the density of* V *is then*

$$(2.12)\quad (\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}n-\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}rq))V^{\frac{1}{2}rq-1}\textstyle\sum_{m=0}^{\infty}\sum_{\eta}((\tfrac{1}{2}(q+r+1-n))_{\eta}V^m(\tfrac{1}{2}r)_{\eta} \cdot C_{\eta}(\mathbf{I})/(\tfrac{1}{2}rq)_m m!) \qquad \text{for } 0<V<1.$$

The density (2.12) was first derived by Khatri and Pillai (1968).

THEOREM 2.3. *The density function of* $Q = \operatorname{tr}(\mathbf{I}-\mathbf{R})\mathbf{R}^{-1} = \sum(1-r_i^2)/r_i^2$ *is given by*

$$(2.13)\quad \begin{aligned}&(\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}r)\Gamma(\tfrac{1}{2}(n-r)q))|\mathbf{I}-\mathbf{P}|^{\frac{1}{2}n}Q^{\frac{1}{2}(n-r)q-1}\textstyle\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\\ &\cdot\textstyle\sum_{\kappa}\sum_{\eta}\sum_{\delta}((-1)^m(\tfrac{1}{2}n)_{\kappa}(\tfrac{1}{2}n)_{\delta}(\tfrac{1}{2}(n-r))_{\eta}Q^m g_{\kappa,\eta}^{\delta}C_{\kappa}(\mathbf{P})C_{\delta}(\mathbf{I})/(\tfrac{1}{2}(n-r)q)_m\\ &\cdot(\tfrac{1}{2}r)_{\kappa}C_{\kappa}(\mathbf{I})m!k!) \qquad \text{for } |Q|<1.\end{aligned}$$

PROOF. Let $s_i^2 = (1-r_q^2)/r_q^2, \cdots, s_q = (1-r_1^2)/r_1^2$ then the joint density of $s_1, \cdots, s_q$, $0 < s_1 < \cdots < s_q < \infty$ is given by ($\mathbf{S} = \mathrm{diag}\,(s_1, \cdots, s_q)$).

$$(2.14)\quad (\textstyle\prod^{\frac{1}{2}q}\Gamma_q(\frac{1}{2}n)/\Gamma_q(\frac{1}{2}r)\Gamma_q(\frac{1}{2}n-\frac{1}{2}r)\Gamma_q(\frac{1}{2}q))|\mathbf{S}|^{\frac{1}{2}(n-r-q-1)}\alpha_q(\mathbf{S})|\mathbf{I}+\mathbf{S}|^{-\frac{1}{2}n} \cdot {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}r; \mathbf{P}, (\mathbf{I}+\mathbf{S})^{-1}).$$

Using Lemma 2.2 one can write (2.14) as

$$(2.15)\quad (\textstyle\prod^{\frac{1}{2}q}\Gamma_q(\frac{1}{2}r)/\Gamma_q(\frac{1}{2}r)\Gamma_q(\frac{1}{2}n-\frac{1}{2}r)\Gamma_q(\frac{1}{2}q))|\mathbf{S}|^{\frac{1}{2}(n-r-q-1)}\alpha_q(\mathbf{S})\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\kappa}\sum_{\eta}\sum_{\delta} \cdot(\tfrac{1}{2}n)_\kappa(\tfrac{1}{2}n)_\delta(-1)^m g_{\kappa,\eta}^{\delta} C_\kappa(\mathbf{P})C_\eta(\mathbf{S})C_\delta(\mathbf{I})/(\tfrac{1}{2}r)_\kappa C_\kappa(\mathbf{I})C_\eta(\mathbf{I})k!m!).$$

Integrating (2.15) over the surface $\sum s_i = Q$ and using Lemma 2.1 yields the density of Q which is convergent for $|Q| < 1$.

COROLLARY 2.3. *If* $\mathbf{P} = \mathbf{0}$, $\mathbf{X}^{(1)}$ *and* $\mathbf{X}^{(2)}$ *are independently distributed then the density of* Q *is given by*

$$(2.16)\quad (\Gamma_q(\tfrac{1}{2}n)/\Gamma_q(\tfrac{1}{2}r)(\tfrac{1}{2}(n-r)q))Q^{\frac{1}{2}(n-r)q-1}\textstyle\sum_{m=0}^{\infty}\sum_{\eta}((-1)^m(\frac{1}{2}n)_\eta(\frac{1}{2}n-\frac{1}{2}r)_\eta Q^m \cdot C_\eta(\mathbf{I})/(\tfrac{1}{2}(n-r)q)_m m!), \qquad |Q| < 1.$$

From the densities of U and Q the densities of $H^{(q)}$, $R^{(q)}$ and $T^{(q)}$ given by (1.7), (1.8) and (1.9) respectively can be derived.

Preliminary investigations have shown that, because of slow convergence of the series in the densities of U, V and Q, the above results appear not to be very useful. One should reject the null hypothesis of independence when the characteristic roots in some sense are large. Thus in each of the cases U and V one would reject the null hypothesis if the criterion exceeds some specified number and one is interested in the upper tails of these distributions. On the other hand in the case of Q one would reject the null hypothesis if this criterion is below some specified number (since $R^{(q)} = (1+Q/q)^{-1}$ and $T^{(q)} = q/Q$ one can just as well use Q as the test criterion). Thus one would be interested in the lower tails of the distribution. Since the density of $Q = \sum(1-r_i^2)/r_i^2$ is convergent for $|Q| < 1$ one is immediately inclined to think that some headway might well be made with this statistic. However, this statistic is so sensitive with respect to small values of some (or all) of the characteristic roots that it can only be used for certain restrictive alternatives of the null hypothesis. For example one would only be able to use the statistic Q for large deviations of the null hypothesis when the population characteristic roots (i.e. population canonical correlations) are all different from zero.

Comparisons of the power functions of the criteria W (Wilks), r_q^2 (Roy's largest root), U and V for testing the independence hypothesis have been made by Pillai & Jayachandran (1967), (1968) for the case where $q = 2$. They conclude that the three criteria U, V and W are all good tests of the independence hypothesis. For small deviations from the hypothesis the differences in power between the tests are slight. However, considering larger deviations also they conclude that V has greater power than the rest when the values of the population canonical correlations are close. But when these parameters are far apart and for larger values of N the power of U is greater than that of V and W. The power of the largest root test stays below those of the other tests except in the case of large deviations when there is only one nonzero population canonical correlation coefficient; the power then exceeds those of the other three tests.

It would be interesting to compare the power of Q with those of U, V and W and such investigations are presently being attempted by the author.

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