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ACCURACY OF CONVERGENCE OF SUMS OF DEPENDENT RANDOM VARIABLES WITH VARIANCES NOT NECESSARILY FINITE[1]

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Let $S_n = \sum_{k=1}^{k_n} X_{nk}$ and X be random variables with distribution functions $F_n(x)$ and $F(x)$. No assumptions are made that the (X_{nk}) have finite means or variances. Also, no independence conditions are assumed. A bound is found for

$$M_n = \sup_{-\infty<x<\infty} |F_n(x) - F(x)|.$$

This bound involves various truncated moments and conditional probabilities and expectations. A typical quantity involved is $\sum_{k=1}^{k_n} E|E(X_{nk}|\sum_{j=1}^{k-1} X_{nj}) - E(X_{nk})|$. Using this bound, particular conditions are found so that S_n converges in distribution to X.

1. Introduction and summary. Let (X_{nk}), $k = 1, 2, \cdots, k_n$, $n = 1, 2, \cdots$ be a system of random variables with distribution functions $F_{nk}(x)$. Let $S_n = \sum_{k=1}^{k_n} X_{nk}$ have distribution function $F_n(x)$ and X be a random variable with distribution function $F(x)$. If $\mathscr{L}(S_n) \to \mathscr{L}(X)$ (i.e., S_n converges in distribution to X), it has been of interest to investigate bounds on

$$M_n = \sup_{-\infty<x<\infty} |F_n(x) - F(x)|.$$

In this paper we obtain a bound on M_n and use the bound to give conditions for the convergence of S_n to X where neither finite variances of the X_{nk} nor any independence conditions are assumed. The convergence theorem is of a type considered by Loève in [5] and by the author in [1] assuming finite variances of the X_{nk}. Theorem 1, whose proof is given in Section 3, gives the bound on M_n. The main component of this bound, $g^a(n, m, r)$, is given at the end of Section 2.

THEOREM 1. *Let (X_{nk}) be a system of random variables and X be an infinitely divisible random variable with distribution function $F(x)$. Assume $F'(x) = dF(x)/d(x)$ exists for each x and is bounded by B. Then for each r and a such that $0 < r \leqq 1$, $a > 1$*

$$M_n = \sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leqq \sum_{k=1}^{k_n} (F_{nk}(-a) + 1 - F_{nk}(a)) + h(B) g^a(n, m, r)$$

if $\sigma_{nk}^2(a) \leqq 1$ for all n, k[2] *where $h(b)$ depends only on B.*

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[2] This is a very mild assumption as seen by the remark of the next to the last paragraph of [1].

Theorems of this type were discussed by Shapiro [6] and Boonyasombut and Shapiro [2] and were applied by them to specific limit theorems where X_{n1}, $X_{n2}, \cdots, X_{nk_n}$ were assumed to be independent for each n.

2. Notation. For $a > 0$, let

$$X_{nk}^a = X_{nk} \quad \text{if} \quad -a < X_{nk} \leqq a$$
$$= 0 \quad \text{otherwise}$$

with $S_n{}^a = \sum_{k=1}^{k_n} X_{nk}^a$, $F_{nk}^a(x)$ and $F_n{}^a(x)$ the corresponding distribution functions, means $\mu_{nk}(a)$ and $\mu_n(a)$, $\sigma_{nk}^2(a)$ the variance of X_{nk}^a and $\sigma_n{}^2(a) = \sum_{k=1}^{k_n} \sigma_{nk}^2(a)$. Also let

$$F_{nk}^{a'}(x) = P(X_{nk}^a \leqq x \mid \textstyle\sum_{j=1}^{k-1} X_{nj}^a)$$
$$E'(X_{nk}^a) = E(X_{nk}^a \mid \textstyle\sum_{j=1}^{k-1} X_{nj}^a)$$
$$K_{nk}^a(x) = \textstyle\int_{-\infty}^x u^2 dF_{nk}^a(u+\mu_{nk}(a)),\ K_n{}^a(x) = \sum_{k=1}^{k_n} K_{nk}^a(x)$$
$$K_{nk}^{a'}(x) = \textstyle\int_{-\infty}^x u^2 dF_{nk}^{a'}(u+\mu_{nk}(a)).$$

We say (X_{nk}^*) is the independent version of (X_{nk}) if for all n, k, X_{nk}^* and X_{nk} have the same distribution and for each n, $X_{n1}^*, X_{n2}^*, \cdots, X_{nk_n}^*$ are independent.

Let X be an infinitely divisible random variable with Lévy–Khintchine representation (see (2.1) of [2]) determined by the function $G(u)$ and the constant γ. Let

$$G^a(u) = 0 \quad \text{if} \quad u \leqq -a,$$
$$= G(u)-G(-a) \quad \text{if} \quad -a < u \leqq a, \qquad \gamma^a = \gamma - \textstyle\int_{|u|>a} u^{-1} dG(a)$$
$$= G(a)-G(-a) \quad \text{if} \quad u > a,$$

which by the above representation determines a unique infinitely divisible random variable X^a with distribution function $F^a(x)$, mean $\mu(a)$, variance $\sigma^2(a)$ (which can be shown to be finite), and Kolmogorov representation (see page 85 of [4]) given by a bounded non-decreasing function $K^a(x)$ and the constant $\mu(a)$.

For any $A > 0$ such that $\pm A$ are continuity points of $G(u)$, let $0 < \delta \leqq 2A$ and define $m = m(A, \delta) = [2A/\delta]+1$, $-A = x_0 < x_1 < \cdots < x_m = A$ with x_i continuity points of $G(u)$ and such that $\max(x_i - x_{i-1}) < \delta$. Then for $r > 0$ let

$$\begin{aligned} g^a(n, m, r) = {} & [\tfrac{5}{16}\sigma_n{}^2(a) \max_{1\leqq k\leqq k_n} \sigma_{nk}^2(a)]^{\frac{1}{5}} + [\tfrac{5}{6}\delta(3\sigma_n{}^2(a)+\sigma^2(a))]^{\frac{1}{3}} \\ & + [\tfrac{1}{2} \textstyle\sum_{i=0}^m \sum_{k=1}^{k_n} E|K_{nk}^{a'}(x_i) - K_{nk}^a(x_i)| + \tfrac{1}{2}\sum_{i=0}^m |K_n{}^a(x_i) - K^a(x_i)|]^{\frac{1}{3}} \\ & + [2 \textstyle\sum_{k=1}^{k_n} E|E'(X_{nk}^a) - \mu_{nk}(a)| + 2|\mu_n(a) - \mu(a)| \\ & + (4/A)(2\sigma_n{}^2(a) + K_n{}^a(\infty) - K_n{}^a(A) + K^a(\infty) - K^a(A) + K_n{}^a(-A) \\ & + K^a(-A))]^{\frac{1}{2}} \\ & + \left[\frac{8\int_{|u|>a} |u|^r dG(u)}{r}\right]^{1/1+r}. \end{aligned}$$

3. Proof of Theorem 1. The following proof combines the techniques of Theorem 1 of [2] and Theorem 2 of [1]. The crucial approximation of $|\phi_n{}^a(t)-\phi^a(t)|$ derives from Lemma 2 of [1].

Proof. To obtain the bound for M_n observe that

$$\begin{aligned}|F_n(x)-F(x)| &\leqq |F_n(x)-F_n{}^a(x)|+|F_n{}^a(x)-F(x)| \\ &\leqq \textstyle\sum_{k=1}^{k_n} (F_{nk}(-a)+1-F_{nk}(a))+|F_n{}^a(x)-F(x)|\end{aligned}$$

where the second inequality follows from Lemma 1 of [2]. A bound is now found for $|F_n{}^a(x)-F(x)|$. Let $\phi_n{}^a(t)$, $\phi^a(t)$ and $\phi(t)$ be the characteristic functions of $S_n{}^a$, X^a, and X respectively. Then

$$|\phi_n{}^a(t)-\phi(t)| \leqq |\phi_n{}^a(t)-\phi^a(t)|+|\phi^a(t)-\phi(t)|.$$

By the proof of Theorem 2 of [1] if $T_n = 1/g^a(n, m, r)$, then for $|t| \leqq T_n$

$$\begin{aligned}|\phi_n{}^a(t)-\phi^a(t)| \leqq\ & |t|^4[\tfrac{5}{8}\sigma_n{}^2(a) \max_{1\leqq k\leqq k_n} \sigma_{nk}^2(a)]+|t|^3[\tfrac{5}{4}\delta(3\sigma_n{}^2(a)+\sigma^2(a))] \\ &+|t|^2[\tfrac{1}{2}\textstyle\sum_{i=0}^{m}\sum_{k=1}^{k_n} E|K_{nk}^{a'}(x_i)-K_{nk}^a(x_i)|+\tfrac{1}{2}\sum_{i=0}^{m}|K_n{}^a(x_i)-K^a(x_i)|] \\ &+|t|[\textstyle\sum_{k=1}^{k_n} E|E'(X_{nk}^a)-\mu_{nk}(a)|+|\mu_n(a)-\mu(a)|+(2/A)\{2\sigma_n{}^2(a) \\ &+K_n{}^a(\infty)-K_n{}^a(A)+K^a(\infty)-K^a(-A)+K^a(-A)\}].\end{aligned}$$

From Lemma 2 of [2]

$$|\phi^a(t)-\phi(t)| \leqq 4|t|^r\textstyle\int_{|u|>a}|u|^r dG(u).$$

Now applying a result of Esseen [3], for any $p > 1$

$$\sup_{-\infty<x<\infty} |F_n{}^a(x)-F(x)| \leqq \frac{p}{2\pi}\int_{-T_n}^{T_n}\left|\frac{\phi_n{}^a(t)-\phi(t)}{t}\right| dt+c(p)\cdot\frac{B}{T_n}$$

where $c(p)$ is a constant depending only on p. It is easy to show that

$$\int_{-T_n}^{T_n}\left|\frac{\phi_n{}^a(t)-\phi(t)}{t}\right| dt \leqq g^a(n, m, r).$$

Thus

$$\sup_{-\infty<x<\infty} |F_n{}^a(x)-F(x)| \leqq \left(\frac{p}{2\pi}+c(p)\cdot B\right)g^a(n, m, r)$$

and so by fixing p and letting $h(B) = (p/2\pi+c(p)\cdot B)$ the theorem follows.

4. A convergence theorem. In [1], [2], and [6] theorems similar to Theorem 1 were proven for specific limit theorems. Then in each case the bounds obtained on M_n were shown to converge to zero under conditions of the particular limit theorems under discussion. In this paper, the bound of the previous section, obtained without reference to a particular limit theorem, yields immediate conditions for a limit theorem. At the same time these conditions are such that the bound on M_n converges to zero. A corollary to Theorem 1 is thus obtained with conditions similar to those in [5] for random variables whose variances are not necessarily finite.

THEOREM 2. *Let* (X_{nk}) *be a system of infinitesimal random variables with* X *as in Theorem* 1. *Also let* $\mathscr{L}(S_n^*) \to \mathscr{L}(X)$ *where* $S_n^* = \sum_{k=1}^{k_n} X_{nk}^*$ *and* (X_{nk}^*) *is the independent version of* (X_{nk}).[3] *Then* $\mathscr{L}(S_n) \to \mathscr{L}(X)$ *if*

(*i*) *there is an* $r > 0$ *such that* $\int_{-\infty}^{\infty} |u|^r dG(u) < \infty$, [4]

(*ii*) $\lim_{n\to\infty} \sum_{k=1}^{k_n} E|E'(X_{nk}^a) - \mu_{nk}(a)| = 0$,

(*iii*) $\lim_{n\to\infty} \sum_{k=1}^{k_n} E|K_{nk}^{a'}(x) - K_{nk}^a(x)| = 0$

for all x and a which are continuity points of $G(u)$, the Lévy–Khintchine function associated with X. Furthermore, the bound on M_n converges to zero as n goes to ∞.

PROOF. The proof of the last statement establishes the theorem. We first observe that if (X_{nk}) is infinitesimal so is (X_{nk}^a). It then follows that $\lim_{n\to\infty} \max_{1 \leq k \leq k_n} \sigma_{nk}^2(a) = 0$. From the fact that $\mathscr{L}(S_n^*) \to \mathscr{L}(X)$ we have by Theorem 3 of [8] that for each a which is a continuity point of $G(u)$ that $\mathscr{L}(S_n^{*a}) \to \mathscr{L}(X^a)$ where $S_n^{*a} = \sum_{k=1}^{k_n} X_n^{*a}$. Furthermore, by Theorem 6 of the same paper we have that $\lim_{n\to\infty} \sigma_n^2(a) = \sigma^2(a)$ and $\lim_{n\to\infty} \mu_n(a) = \mu(a)$. Since $\mathscr{L}(S_n^{*a}) \to \mathscr{L}(X^a)$ it follows from the proof of Theorem 2, page 100 of [4] and the remark following it that $K_n^a(x) \to K^a(x)$ and $K_n^a(+\infty) \to K^a(+\infty)$. We then have that as $n \to \infty$

$$K_n^a(\infty) - K_n^a(A) + K^a(\infty) - K^a(A) + K_n^a(-A) + K^a(-A) \to 2(K^a(\infty) - K^a(A) + K^a(-A)).$$

We now let $A = 1/\delta^{\frac{1}{2}}$ and realize that in $g^a(n, m(A, \delta), r)$, δ is a function of n so we write δ_n and $m(A, \delta) = m(\delta_n)$. Then we can find a sequence $\delta_n = \delta_n(a)$ so that $\pm\delta_n^{-\frac{1}{2}}$ are continuity points of $G(u)$ and such that

$$\left[\tfrac{1}{2} \sum_{i=1}^{m(\delta_n)} \sum_{k=1}^{k_n} E|K_{nk}^{a'}(x_i) - K_{nk}^a(x_i)| + \tfrac{1}{2} \sum_{i=0}^{m(\delta_n)} |K_n^a(x_i) - K^a(x_i)|\right] \to 0 \quad \text{as} \quad n \to \infty.$$

Thus it is clear that for each a which is a continuity point of $G(u)$

$$\lim_{n\to\infty} \left\{ g^a(n, m(\delta_n(a)), r) - \left[\frac{8\int_{|u|>a} |u|^r dG(u)}{r}\right]^{1/1+r} \right\} = 0$$

and so by Lemma 5 of [2] we can find $a_n \leq a_{n+1}$ so that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} g^{a_n}(n, m(\delta_n(a_n)), r) = 0$. Since by Lemma 4 of [2] $\lim_{n\to\infty} \sum_{k=1}^{k_n} (F_{nk}(-a_n) + 1 - F_{nk}(a_n)) = 0$, the result follows.

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[3] This condition is equivalent to assuming certain conditions on $F_{nk}(x)$ such as those given in Theorem 1, Section 25 of [4].

[4] This condition is satisfied by all stable laws, for example.

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