JOINT ASYMPOTOTIC DISTRIBUTION OF THE ESTIMATED REGRESSION FUNCTION AT A FINITE NUMBER OF DISTINCT POINTS

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As an approximation to the regression function \( m \) of \( Y \) on \( X \) based upon empirical data, E.A. Nadaraya and G.S. Watson have studied estimates of \( m \) of the form

\[
m_n(x) = \sum_{i=1}^{n} Y_i k((x - X_i)/a_n)/\sum_{i=1}^{n} k((x - X_i)/a_n)
\]

For distinct points \( x_1, \ldots, x_k \), we establish conditions under which

\[
(na_n)(m_n(x_1) - m(x_1), \ldots, m_n(x_k) - m(x_k))
\]

is asymptotically multivariate normal.

1. Introduction and summary. Let \((X, Y)\) be a bivariate random variable having a joint density function \( f \) and let \( g \) be the marginal density function of \( X \). If \( E Y \) is finite then the regression function \( m \) (of \( Y \) on \( X \)) may be defined as

\[
m(x) = E[Y|X = x].
\]

As an approximation to \( m \) based upon empirical data, Nadaraya (1964) and Watson (1964) have considered estimates of the form

\[
m_n(x) = \sum_{i=1}^{n} Y_i k((x - X_i)/a_n)/\sum_{i=1}^{n} k((x - X_i)/a_n)
\]

where \( k \) is a univariate density function, \( \{a_n\} \) is a sequence of positive numbers converging to zero and \((X_1, Y_1, \ldots, X_n, Y_n)\) is a random sample of size \( n \) from \( f \).

Nadaraya (1964) indicates that if \( Y \) is a bounded random variable and \( na_n \to \infty \), then \((na_n)^2(m_n(x) - Em_n(x))\) is asymptotically normal with mean zero and variance \( E[Y^2|X = x] \cdot k^2(u) du/g(x) \).\(^1\) Of more interest is the asymptotic distribution of \((na_n)^2(m_n(x) - m(x))\). One would normally approach the asymptotic distribution of this statistic by attempting to establish that \((na_n)^2(Em_n(x) - m(x)) = o(1)\), from which one could conclude that the asymptotic distribution of \((na_n)^2(m_n(x) - m(x))\) is the same as that of \((na_n)^2(m_n(x) - Em_n(x))\). Instead, we approach the problem by proving that both the numerator and denominator of \( m_n \) are asymptotically normal and hence so is \( m_n \). In fact for distinct points \( x_1, \ldots, x_k \) we will establish conditions under which \((na_n)^2(m_n(x_1) - m(x_1), \ldots, m_n(x_k) - m(x_k))\) is asymptotically multivariate normal with mean vector zero and diagonal covariance matrix \( C = [C_{ij}] \) with

\[
C_{ij} = \text{Var} [Y|X = x_i] \cdot k^2(u) du/g(x_i).
\]

We note that this asymptotic variance disagrees with that of Nadaraya unless \( m(x_i) = 0 \). The fact that Nadaraya’s result is incorrectly stated can be observed by noting that the

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\(^{1}\) Whenever the integration extends over \((-\infty, \infty)\) no limits of integration will be given.
asymptotic distribution should be invariant with respect to translations of $Y$. If one applies Nadaraya’s theorem to bivariate samples from each of $(X, Y)$ and $(X, Y - c)$, for constant $c$, one obtains different asymptotic variances for the same statistic.

2. **Statement of the theorem.** We assume the kernel $k$ and the sequence $\{a_n\}$ are chosen to satisfy the conditions:

(i) $k(u)$ and $|uk(u)|$ are bounded.
(ii) $\int u^t k(u) \, du = 0$.
(iii) $\int u^t k(u) \, du < \infty$.
(iv) $\lim na_n^3 = \infty$ and $\lim na_n^4 = 0$.

For convenience we write $V[Y|X = x] = v(x)/g(x) - w^2(x)/g^2(x)$, where $g(x), w(x)$, and $v(x)$ are defined by $\int f(x, y) \, dy$, $\int yf(x, y) \, dy$, and $\int y^2 f(x, y) \, dy$, respectively.

**Theorem.** Suppose $x_1, \ldots, x_k$ are distinct points and $g(x_i) > 0$ for $i = 1, 2, \ldots, k$. If $E_r Y$ is finite and if $g'$, $w'$, $v'$, $g''$ and $w''$ exist and are bounded, then $\left[na_n^3(m_n(x_i) - m(x_i)), \ldots, m_n(x_k) - m(x_k)\right]^t$ converges in distribution to $Z^*$ where $Z^*$ is multivariate normal with mean vector $0$ and diagonal covariance matrix $C = [C_{ii}]$ where

$$C_{ii} = V[Y|X = x_i] \int k^2(u) du/g(x_i) \quad (i = 1, 2, \ldots, k).$$

3. **Proof of the theorem.** For simplicity we shall prove the theorem for the special case when $k = 2$. The method of proof remains valid in the more general case.

For brevity we define for $i = 1, 2, \ldots, n$ and $s = 1, 2$:

$$U_{ns}(x_i) = k((x_i - X_i)/a_n)/a_n, \quad U_{ns}(x_i) = (a_n)^t(U_{ns}(x_i) - EU_{ns}(x_i)),$$
$$V_{ns}(x_i) = Y_n U_{ns}(x_i), \quad V_{ns}(x_i) = (a_n)^t(V_{ns}(x_i) - EV_{ns}(x_i)),$$
$$U_n(x_s) = \sum_{i=1}^n U_{ns}(x_s), \quad V_n(x_s) = \sum_{i=1}^n V_{ns}(x_s),$$
$$W_{ns} = (U_{ns}(x_s), V_{ns}(x_s), U_{ns}(x_s), V_{ns}(x_s))^t,$$
$$(n)^t Z_n = (U_n(x_1), V_n(x_1), U_n(x_2), V_n(x_2))^t.$$

$$A = \int k^2(u) du \begin{bmatrix} g(x_1) & w(x_1) & 0 & 0 \\ w(x_1) & v(x_1) & 0 & 0 \\ 0 & 0 & g(x_2) & w(x_2) \\ 0 & 0 & w(x_2) & v(x_2) \end{bmatrix}.$$

Let $Z$ be bivariate normal with mean vector $0$ and covariance matrix $A$. We first prove two lemmas.

**Lemma 1.** Suppose the density $k$ satisfies the conditions (i) and (ii) above and suppose $na_n^3 \rightarrow \infty$. Let $E_r Y$ be finite and let $g'$, $w'$, and $v'$ exist and be bounded. If $x_1 \neq x_2$ and $g(x_i) > 0$ for $i = 1, 2$, then $Z_n$ converges in distribution to $Z$. 
Proof. Using the Cramér-Wold Theorem (e.g., Theorem (xi) on page 103 of [3]), it will be sufficient to prove that \( c \cdot Z_n^i \) converges in distribution to \( c \cdot Z^i \) for any \( c = (c_1, d_1, c_2, d_2) \) in \( R^4 \).

The following hold for \( s = 1, 2 \) and \( r = 1, 2 \) under the assumption that \( s \neq r \) whenever \( s \) and \( r \) appear in the same expression:

\[
\begin{align*}
(1) & \quad EU_n^s(x_n) = g(x_n) \int k^2(u) \, du + O(a_n) . \\
(2) & \quad EV_n^s(x_n) = v(x_n) \int k^3(u) \, du + O(a_n) . \\
(3) & \quad EU_n^s(x_n) V_n^s(x_n) = w(x_n) \int k^3(u) \, du + O(a_n) . \\
(4) & \quad EU_n^s(x_n) U_n^s(x_n) = O(a_n) . \\
(5) & \quad EV_n^s(x_n) V_n^s(x_n) = O(a_n) . \\
(6) & \quad EU_n^s(x_n) V_n^s(x_n) = O(a_n) .
\end{align*}
\]

We will be sketch the proofs of (1) and (4) to illustrate the method. To obtain (1), we see

\[ EU_n^s(x_n) = a_n \left[ \int k^2(u) g(x_n - a_n u) \, du / a_n - \left( \int k(u) g(x_n - a_n u) \, du \right)^2 \right]. \]

Since \( g' \) and \( |yk(y)| \) are bounded and \( \int |u| k(u) \, du \) is finite, it follows that

\[ |\int k(u) [g(x_n - a_n u) - g(x_n)] \, du| \leq \sup_x |g'(x)| a_n \int |u| k(u) \, du = O(a_n) \]

and

\[ |\int k^2(u) [g(x_n - a_n u) - g(x_n)] \, du| \leq \sup_x |g'(x)| a_n \int |u| k^2(u) \, du = O(a_n). \]

Thus we have

\[ EU_n^s(x_n) = g(x_n) \int k^2(u) \, du + O(a_n). \]

As for (4), suppose \( x_n > x_1 \). Let \( \delta = x_n - x_1 \) and \( \delta_n = \delta / a_n \). Then

\[ EU_n^s(x_n) U_n(x_2) = \int k((x_1 - u)/a_n) k((x_2 - u)/a_n) g(u) \, du / a_n + O(a_n) \]

\[ = \int k(u) k(\delta_n + u) g(x_1 - a_n u) \, du + O(a_n) \]

\[ = \int_{|u| < \delta_n} k(u) k(\delta_n + u) g(x_1 - a_n u) \, du + \int_{|u| > \delta_n} k(u) k(\delta_n + u) g(x_1 - a_n u) \, du + O(a_n) \]

\[ \leq \sup_{|u| < \delta_n / 2} k(\delta_n + u) \cdot \int k(z) g(x_1 - a_n z) \, dz \]

\[ + \sup_{|u| > \delta_n / 2} k(u) \cdot \int k(\delta_n + z) g(x_1 - a_n z) \, dz + O(a_n) \]

\[ \leq 2 \sup_{|u| \geq \delta_n / 2} k(u) \cdot O(1) + O(a_n) \leq 4 \delta_n^{-1} \]

\[ \times \sup_{|u| \geq \delta_n / 2} |uk(u)| \cdot O(1) + O(a_n) = O(a_n) \]

which was to be shown.

Now let \( \sigma_n^2 = \text{Var}(c \cdot Z_n^i) \) so that by (1)–(6) above, we have
\[ \sigma_n^2 = \frac{1}{k} \int k(u) \, du \cdot \sum_{i=1}^n \left[ c_i^2 g(x_i) + d_i^2 v(x_i) + 2c_i d_i w(x_i) \right] + O(a_n). \]

Put \( \rho_n = E[(c \cdot W_n)/n^{-1}] \) and \( \rho_n^s = \sum_{i=1}^n \rho_{ni}^s \) so that
\[
\rho_n^s = n^{-1} E |c \cdot W_n| |^s \leq n^{-1} |c|^s E |W_n|^s \\
\leq 8n^{-1} |c|^s \max_{r=1,3} \{ E |U_n(x_i)|^r, E |V_n(x_i)|^r \}.
\]

Since \( g', w', w' \) and \( k \) are bounded and \( E |Y|^s \) is finite it follows by arguments similar to those above that
\[
E |U_n(x_i)|^s = O(a_n^{-s}) \quad \text{and} \quad E |V_n(x_i)|^s = O(a_n^{-s})
\]
\( (s = 1, 2) \) so that \( \rho_n^s = O(a_n^{-s} n^{-s}). \)

Since \( g(x)v(x) - w(x) = g(x)w[Y | X = x] \) we can deduce that \( A \) is positive definite whenever \( g(x_1) > 0 \) and \( g(x_2) > 0 \). Thus for \( c \neq 0 \)
\[
\lim_{n \to \infty} \sigma_n^2 = cAe^i > 0
\]
since \( cAe^i \) is a quadratic form associated with the positive definite matrix \( A \). Hence it follows that \( \lim_{n \to \infty} \rho_n^s / \sigma_n = 0 \) (recall that \( na_n^s \to \infty \)) whenever \( c \neq 0 \).


Let us write
\[
Z_n^s = a_n^{-s} n^{-1} \sum_{i=1}^n \left[ U_{ni}^s(x_i) - g(x_i), \sum_{i=1}^n \left[ V_{ni}^s(x_i) - w(x_i) \right] \right], \sum_{i=1}^n \left[ U_{ni}^s(x_i) - g(x_i), \sum_{i=1}^n \left[ V_{ni}^s(x_i) - w(x_i) \right] \right].
\]

**Lemma 2.** Suppose \( \int u k(u) \, du = 0, \int u^2 k(u) \, du \) is finite and \( na_n^s \to 0 \). If \( g' \) and \( w' \) exist and are bounded then, under the conditions of Lemma 1, \( Z_n^s \) converges in distribution to \( Z \).

**Proof.** Let \( B_n = (g(x_i) - EU_n^s(x_i), w(x_i) - EV_n^s(x_i), g(x_i) - EU_n^s(x_i), w(x_i) - EV_n^s(x_i))^t \). Since \( \int u k(u) \, du = 0, \int u^2 k(u) \, du \) is finite and \( g'' \) is bounded, it follows that
\[
|EU_n^s(x_i) - g(x_i)| = \left| \int k(u)[g(x_i - a_n u) - g(x_i)] \, du \right| \\
\leq \sup_x |g''(x)||a_n^2 \int u^2 k(u) \, du/2 = O(a_n^2) \quad (i = 1, 2).
\]
Similarly \( |EV_n^s(x_i) - w(x_i)| = O(a_n^2) \) so that \( B_n = O(a_n^2) \). Then \( Z_n - Z_n^s = (na_n^s)^t B_n = O(na_n^s)^t = o(1) \) since \( na_n^s \to 0 \). The desired result now follows from standard large sample theory and Lemma 1.

We are now in a position to complete the proof of the theorem. Let the function \( H \) from \( R^d \) to \( R^d \) be defined by
\[
H(y_1, y_2, y_3, y_4) = (H_1(y_1, y_2, y_3, y_4), H_2(y_1, y_2, y_3, y_4))^t,
\]
where \( H_1(y_1, y_2, y_3, y_4) = y_2/y_1 \), and \( H_2(y_1, y_2, y_3, y_4) = y_4/y_3 \), and let \( \theta = (g(x), w(x_i), g(x), w(x_i)) \).

Let us now write \( Z_n^* = (na_n)^t(T_n - \theta)^t \) where \( T_n = (T_{n1}, T_{n2}, T_{n3}, T_{n4}) \) with
\[ T_{n1} = \frac{\sum_{i=1}^{n} U_{ni}^{*}(x_i)}{n}, \quad T_{n2} = \frac{\sum_{i=1}^{n} V_{ni}^{*}(x_i)}{n}, \]
\[ T_{n3} = \frac{\sum_{i=1}^{n} U_{ni}^{*}(x_i)}{n}, \quad T_{n4} = \frac{\sum_{i=1}^{n} V_{ni}^{*}(x_i)}{n}, \]

Then the Mann-Wald Theorem (e.g., Theorem (ii) on page 321 of [3]), with \((n)^i\) replaced by \((na_n)^i\) may be applied, together with Lemma 2, to, conclude that \((na_n)^i(H(T_n) - H(\theta))\) converges in distribution to \(Z^*\) where \(Z^*\) is \(N(O, DAD^t)\) and where \(D\) is the matrix of partial derivatives of \(H\), evaluated at \(\theta\). It is readily verified that \(DAD^t = C\), and that

\[ H(T_n) - H(\theta) = (m_n(x_1) - m(x_1), m_n(x_2) - m(x_2))^t \]

completing the proof.

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REFERENCES