EXISTENCE AND CONSISTENCY OF MODIFIED MINIMUM CONTRAST ESTIMATES

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It is the purpose of this paper to explore the efficiency of a modified definition for maximum likelihood estimates which depends on the whole equivalence class of densities only and not—as in the classical case—on the particular choice of versions. We prove the existence of measurable maximum likelihood estimates in the new sense for compact metrizable families of probability measures without any continuity assumption for the densities. For appropriate families of probability measures the modified asymptotic maximum likelihood estimates are exactly the strongly consistent estimates. The paper uses Huber’s concept of minimum contrast estimates which covers maximum likelihood estimates as a special case.

Introduction. Maximum likelihood estimates in the usual sense are based on fixed versions of the densities. In [6] and [4] it was shown that the particular choice of the versions essentially influences existence as well as consistency of the maximum likelihood estimates.

It is the purpose of this paper to explore the efficiency of a modified definition\(^1\) of maximum likelihood and asymptotic maximum likelihood estimates which depends on the whole equivalence class of densities only and not on the particular choice of versions any more. It turns out that this new definition has a number of advantages:

(i) For a compact metrizable family of probability measures measurable maximum likelihood estimates exist without any continuity assumption for the densities (Theorem 3.1), whereas measurable maximum likelihood estimates in the usual sense exist in general only under the assumption that the densities are upper-semicontinuous (see [6], page 253).

(ii) Under appropriate regularity conditions (e.g. exactly the conditions of the "classical" consistency theorem) the property of being a sequence of asymptotic maximum likelihood estimates is not only sufficient but also necessary for the strong consistency of the sequence of estimates (Theorem 3.2).

Example 3.3 shows that strongly consistent estimates (and hence asymptotic maximum likelihood estimates in our sense) are, however, not necessarily asymptotic maximum likelihood estimates in the usual sense.

(iii) The proofs for the main theorems are simpler than in the "classical" case since with modified maximum likelihood estimates no measurability

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Received July 22, 1970.

\(^1\) This definition was suggested to the author by J. Pfanzagl.
problems (such as sup $h_C$ is measurable for all compact sets $C$, see [6]) arise.

(iv) The presented concept covers the "classical" concept in all important cases.

Since this creates no additional difficulties we use in Section 2 Huber's [3] concept of minimum contrast and asymptotic minimum contrast estimates. In Section 3 we apply the results of Section 2 to the case of maximum likelihood estimates. Section 4 collects auxiliary lemmas for Section 2 and Section 3.

For references concerning the "classical" theory of minimum contrast and maximum likelihood estimates we refer the reader to [6].

1. Preliminaries. Let $(T, \mathcal{U})$ be a topological space. The Borel-field $\sigma(\mathcal{U})$ is the $\sigma$-field, generated by $\mathcal{U}$. A denotes the closure of a set $A \subset T$. Let $(L, \preceq)$ be a complete lattice, i.e. a partially ordered set such that $\inf L_0$ and $\sup L_0$ exist for each $L_0 \subset L$. If $t \to g_t \in L$, $t \in T$, is any map, we write for every $S \subset T$: $\inf g_S := \inf \{g_t : t \in S\}$, $\sup g_S := \sup \{g_t : t \in S\}$. The map $t \to g_t \in L$, $t \in T$, is lower semicontinuous (l.s.c.) [upper semicontinuous (u.s.c.)] if $g_t = \sup_{s \in V \in W} \inf g_V[g_t = \inf_{t \in U \in W} \sup g_U]$ for every $t \in T$. If $L$ is the real line this is the usual definition of l.s.c. and u.s.c.

Let $\mu$ be a $\sigma$-finite measure on a $\sigma$-field $\mathcal{F}$ on $X$. Denote by $M(\mathcal{F}, \mu)$ the set of all $\mu$-equivalence classes of $\mathcal{F}$-measurable functions on $X$ with values in $[-\infty, +\infty]$. Elements of $M(\mathcal{F}, \mu)$ will be denoted by $\bar{f}$. It is known that $M(\mathcal{F}, \mu)$ endowed with the natural ordering (i.e. $f \leq g$ iff $f(x) \leq g(x)$ $\mu$-a.e.) is a complete lattice ([2] page 335). Furthermore each subfamily of $M(\mathcal{F}, \mu)$ contains a countable subfamily with the same 'infimum' and 'supremum' (loc. cit.). We remark that the infimum or supremum over a countable family, say $M_0$, can be taken pointwise, i.e. $\bar{f} \in f \in M_0$ implies $\inf_{\mu} f \in \inf M_0$.

Let $\mathbb{N}$ be the set of natural numbers. For each $n \in \mathbb{N}$ let $\mu^n$ on the product $\sigma$-field $\mathcal{F}^n$ be the Cartesian product of $n$ identical components $\mu$ on $\mathcal{F}$. For any function $f: X \to [-\infty, +\infty]$ we denote by $f^{(n)}: X^n \to [-\infty, +\infty]$ the function defined by $f^{(n)}(x_1, \ldots, x_n) = n^{-1} \sum_{i=1}^{n} f(x_i)$, $(x_1, \ldots, x_n) \in X^n$, using the convention $\infty - \infty = -\infty$. Sometimes $f^{(n)}$ is considered in a natural way as function defined on the countable product space $X^\mathbb{N}$. If $\bar{f} \in M(\mathcal{F}, \mu)$, denote by $\bar{f}^{(n)} \in M(\mathcal{F}^n, \mu^n)$ the $\mu^n$-equivalence class of functions on $X^n$ containing the functions $f^{(n)}$, $f \in \bar{f}$. For $\bar{f}_t \in M(\mathcal{F}, \mu)$, $t \in T$, and $f_t \in \bar{f}_t$, $t \in T$, we have—as introduced above—the following denotations:

$$\inf \bar{f}_t = \inf \{f_t : t \in S\} \quad \text{and} \quad \inf \bar{f}_t = \inf \{f_t : t \in S\},$$

where $S \subset T$.

The following notion is basic for our concept of minimum contrast estimates.

DEFINITION. Let $\bar{f}_t \in M(\mathcal{F}, \mu)$, $t \in T$. A subset $T_0 \subset T$ is a separant for $\bar{f}_t$, 

$t \in T$, iff $T_0$ is countable and $\inf f_{U \cap T_0} = \inf f_U$ for all $U \in \mathcal{U}$. $T_0$ is a strong separant for $f_t$, $t \in T$, iff for every $n \in \mathbb{N}$, $T_0$ is a separant for the family $f_t^{(n)}$, $t \in T$, (i.e. $\inf f_{U \cap T_0}^{(n)} = \inf f_U^{(n)}$ for all $U \in \mathcal{U}, n \in \mathbb{N}$).

A family of versions $f_t \in f_t^*$, $t \in T$, is separable (see Doob (1953) page 52) iff there exist a countable set $T_0 \subset T$ and a $\mu$-null set $X_0 \in \mathcal{F}$ such that $\inf f_{U \cap T_0}(x) = \inf f_{X_0}(x)$ for all $x \in X_0, U \in \mathcal{U}$.

If $(T, \mathcal{U})$ has a countable base, for every family $f_t \in M(\mathcal{F}, \mu), t \in T$, there exist a strong separant (Lemma 4.1) and a separable family of versions $f_t \in f_t^*$, $t \in T$ (see [1] page 57).

2. Minimum contrast estimates. Throughout this section let $(T, \mathcal{U})$ be a topological space, $\mu$ a $\sigma$-finite measure on a $\sigma$-field $\mathcal{F}$ and $f_t \in M(\mathcal{F}, \mu), t \in T$.

Definition 2.1. A function $\varphi : X \to T$ is a minimum contrast (m.c.) estimate for $f_t$, $t \in T$, iff there exist a separant $T_0 \subset T$, versions $f_t \in f_t^*$, $t \in T$, and a $\mu$-null set $X_0 \in \mathcal{F}$ such that

$$\inf f_{U \cap T_0}(x) = \inf f_{T_0}(x)$$

for all $x \in X_0, U \in \mathcal{U}$ with $\varphi(x) \in U$. $\varphi_n$ is a m.c. estimate for the sample size $n$, iff $\varphi_n$ is a m.c. estimate for the family $f_t^{(n)}, t \in T$.

If $(T, \mathcal{U})$ has a countable base, the definition of m.c. estimates is independent of the special separant $T_0$ and the particular choice of versions $f_t \in f_t^*, t \in T$ (see Lemma 4.2 (i)).

We remark that in contrast to the classical case we do not require a m.c. estimate to be measurable, because strong consistency of sequences of m.c. estimates can be proved without such measurability assumptions (see Theorem 2.8).

The following proposition shows that minimum contrast estimates in the sense of Huber [3] and Pfanzagl [6], derived from separable families of versions, are also minimum contrast estimates in our sense.

Proposition 2.2. If $(T, \mathcal{U})$ has a countable base and $f_t \in f_t^*, t \in T$, is a separable family of versions, then each function $\varphi : X \to T$ with $f_{\varphi(x)}(x) = f_{f_t}(x), x \in X, t \in T$.

Proof. As the family $f_t$, $t \in T$, is separable, there exists a countable $T_0 \subset T$ and a $\mu$-null set $X_0 \in \mathcal{F}$ such that $\inf f_0(x) = \inf f_{U \cap T_0}(x)$ for all $x \in X_0, U \in \mathcal{U}$. Then $T_0$ is a separant for $f_t^*, t \in T$, and $x \notin X_0, \varphi(x) \notin U \in \mathcal{U}$ imply $\inf f_{U \cap T_0}(x) = \inf f_0(x) = \inf f_{T_0}(x) = \inf f_{X_0}(x)$ which proves the assertion.

Theorem 2.3. Let $(T, \mathcal{U})$ be compact metrizable, and $f_t \in M(\mathcal{F}, \mu), t \in T$. Then $\mathcal{F}^*, \sigma(\mathcal{U})$-measurable minimum contrast estimates exist for every sample size $n \in \mathbb{N}$.

Proof. Let $T_0$ be a strong separant for $f_t^*, t \in T$, and let $f_t \in f_t^*, t \in T$. Apply now Lemma 4.5 for each $n \in \mathbb{N}$ to $f_t^{(n)}, t \in T$, and $T$. 

Now we consider sequences of asymptotic minimum contrast estimates (for short: a.m.c. sequences). We remind the reader that a sequence \( \varphi_n : X^N \to T, \ n \in \mathbb{N}, \) is a 'classical' a.m.c. sequence for versions \( f_t \in \bar{f}_t, \ t \in T, \) iff for all \( x \in X^N \)

\[
\lim_{n \to \infty} (\exp f^n_{\varphi_n}(x) - \exp f^n_T(x)) = 0 \tag{2.4}
\]

(see [6] page 251). Equivalently this means

\[
\lim \sup_{n \in \mathbb{N}_0} f^n_{\varphi_n}(x) = \lim \sup_{n \in \mathbb{N}_0} \inf f^n_T(x) \tag{2.5}
\]

for all \( x \in X^N \) and all subsequences \( N_0 \subset \mathbb{N}. \)

To obtain a concept which is independent of the special choice of versions \( f_t, \ t \in T, \) we do not consider the values of the versions pertaining to \( \varphi_n(x) \) any more, we are only interested in their behavior in the neighborhoods of accumulation points of \((\varphi_n(x))_{n \in \mathbb{N}}. \) This leads to the following modification of (2.5):

**DEFINITION 2.6.** Let \( P \ll \mu \) be a probability measure on \( \mathcal{F}. \) A sequence \( \varphi_n : X^N \to T, \ n \in \mathbb{N}, \) is a P-a.m.c. sequence for \( f^*_t, \ t \in T, \) iff there exist a strong separant \( T_0, \) versions \( f_t \in f^*_t, \ t \in T, \) and a \( P^N\)-null set \( X_0 \in \mathcal{F}^N \) such that

\[
\lim \sup_{n \in \mathbb{N}_0} \inf f^n_{U \cap T_0}(x) = \lim \sup_{n \in \mathbb{N}_0} \inf f^n_{T_0}(x)
\]

if \( x \notin X_0, \ U \in \mathcal{U} \) and \( \varphi_n(x) \in V \) for all \( n \in \mathbb{N}_0 \) and some \( V \in \mathcal{U} \) with \( V \subset U. \)

If \( (T, \mathcal{U}) \) has a countable base this definition is independent of the special strong separant \( T_0 \) and the special choice of versions \( f_t \in f^*_t, \ t \in T. \) (see Lemma 4.2 (ii)).

**PROPOSITION 2.7.** If \( (T, \mathcal{U}) \) has a countable base and if \( \varphi_n \) is m.c. estimate (for \( f_t, \ t \in T \) for each sample size \( n \in \mathbb{N}, \) then \((\varphi_n)_{n \in \mathbb{N}} \) is a P-a.m.c. sequence for every probability measure \( P \ll \mu \) on \( \mathcal{F}. \)

**Proof.** Let \( T_0 \) be a strong separant for \( f^*_t, \ t \in T, \) and \( f_t \in f^*_t, \ t \in T. \) As \( \varphi_n \) is m.c. estimate for \( f^{(n)}_t, \ t \in T, \) there exists a \( \mu^N\)-null set \( A_n \in \mathcal{F}^N \)(10,14),(986,981) such that for all \( x \in X^N \) which are not element of the cylinder set over \( A_n, \) say \( X_n \in \mathcal{F}^N, \) we have

\[
\inf f^{(n)}_{U \cap T_0}(x) = \inf f^{(n)}_{T_0}(x) \text{ if } \varphi_n(x) \in U \in \mathcal{U}. \]

Then \( X_0 = \bigcup_{n \in \mathbb{N}} X_n \) is a \( P^N\)-null set for each probability measure \( P \ll \mu, \) and for each \( n \in \mathbb{N} \) we have

\[
\inf f^{(n)}_{U \cap T_0}(x) = \inf f^{(n)}_{T_0}(x) \text{ if } x \notin X_0 \text{ and } \varphi_n(x) \in U \in \mathcal{U}. \]

This immediately implies the assertion.

The 'if part' of the following theorem is closely related to Theorem 1.12 of Pfanzagl [6]. As Example 3.3 shows, we prove, however, strong consistency for a larger class of estimates. Moreover our proof is simpler than that of Pfanzagl because no measurability problems such as those found there arise.

**THEOREM 2.8.** Let \( \mu \) be a \( \sigma\)-finite measure on a \( \sigma\)-field \( \mathcal{F} \) and \( P \ll \mu \) a probability measure on \( \mathcal{F}. \) Let \( (T, \mathcal{U}) \) be a compact metrizable space, \( \bar{f}_t \in M(\mathcal{F}, \mu), \ t \in T, \) and \( S \subset T \) dense in \( T. \) Assume that

(i) \( t \to \bar{f}_t, \ t \in T, \) is l.s.c. If for some \( t_0 \in S \)
(ii) \( P(f_{t_0}) < P(f_t) \) for all \( t \neq t_0, t \in T \),
(iii) \( P(\inf x^{-} c) > -\infty \) for all compact sets \( C \subset T \) with \( t_0 \in C \), then a sequence of estimates \( \phi_n : X^N \rightarrow S, n \in \mathbb{N} \), converges \( P^N \)-a.e. to \( t_0 \) if and only if \((\phi_n)_{n \in \mathbb{N}}\) is a \( P\)-a.m.c. sequence for \( f_t, t \in S \).

**Proof.** Let \( f_t \in f_t^+, t \in T \). Then, according to the strong law of large numbers, there exists a \( P^N\)-null set \( X_1 \in \mathcal{F}^N \), such that for all \( x \in X_1 \) and all \( S_i \subset T \) with \( t_0 \in S_i \)

\[
\limsup_{x \in X_1} f_{t_0}^{(n)}(x) \leq \limsup_{x \in X_1} f_{t_0}^{(n)}(x) = P(f_{t_0}).
\]

Let \( U_k \in \mathcal{U}, k \in \mathbb{N} \), be a base for the neighborhood system of \( t_0 \) such that \( U_{k+1} \subset U_{k+1} \subset U_k, k \in \mathbb{N} \). By Lemma 4.1 there exists a countable \( T_0 \subset T \), \( t_0 \in T_0 \), which contains a strong separant for all the families \( f_t, t \in Z \), where \( Z = S, T, T - U_k, S - U_k, k \in \mathbb{N} \). As for each \( k \in \mathbb{N} \) the assumptions of Lemma 4.3 are fulfilled for \( T - U_k \), there exists a \( P^N\)-null set \( X_2 \in \mathcal{F}^N \) such that for all \( x \in X_2 \) and all \( k \in \mathbb{N} \)

\[
P(f_{t_0}) < \inf_{x \in T \backslash U_k} P(f_t) = \limsup_{x \in X_2} \inf_{x \in (T \backslash U_k) \cap T_0} f_{t_0}^{(n)}(x)
\]

where the first inequality follows from (i) and the fact that a l.s.c. function attains its infimum on a compact set.

(a) Assume that \((\phi_n)_{n \in \mathbb{N}}\) is a \( P\)-a.m.c. sequence for \( f_t, t \in S \). As \( S_0 = S \cap T_0 \) is a strong separant for \( f_t, t \in S \), there exists a \( P^N\)-null set \( X_3 \in \mathcal{F}^N \) such that for all \( x \in X_3 \), \( N_0 \subset \mathbb{N} \) with \( \phi_n(x) \in V \subset \mathcal{V} \subset U \) for all \( n \in N_0(U, V \in \mathcal{U}) \)

\[
\limsup_{x \in X_3} f_{t_0}^{(n)}(x) = \limsup_{x \in X_3} \inf_{x \in S_0} f_{t_0}^{(n)}(x).
\]

Let \( X_0 = X_1 \cup X_2 \cup X_3 \). Then \( P^N (X_0) = 0 \). If \( x \in X_0 \) and \( \phi_n(x) \in T - U_{k-1} \subset T - U_{k-1} \subset T - U_k \) for infinitely many \( n \in \mathbb{N} \), say \( N_0 \), and some \( k \geq 1 \), we have by (2.11)

\[
\limsup_{x \in X_0} \inf_{x \in S_0} f_{t_0}^{(n)}(x) = \limsup_{x \in N_0} \inf_{x \in S_0} f_{t_0}^{(n)}(x).
\]

As \( T - U_k \subset T - U_{k+1} \) and \( S_0 \subset T_0 \), we have by (2.10)

\[
P(f_{t_0}) < \limsup_{x \in N_0} \inf_{x \in S_0} f_{t_0}^{(n)}(x).
\]

This, however, contradicts (2.9). Hence for all \( x \in X_0(\phi_n(x))_{n \in \mathbb{N}} \) is eventually in each \( U_k, k \in \mathbb{N} \), whence \( \lim_{x \in X_0} \phi_n(x) = t_0 \) \( P^N \)-a.e.

(b) Assume conversely that the sequence of functions \( \phi_n : X^N \rightarrow T, n \in \mathbb{N} \), converges \( P^N \)-a.e. to \( t_0 \). Hence there exists a \( P^N \)-null set \( X'_0 \in \mathcal{F}^N \) such that \( \lim_{x \in X'_0} \phi_n(x) = t_0 \) for all \( x \in X'_0 \).

Let \( X'_0 = X_1 \cup X_2 \cup X'_0 \). Then \( P^N (X'_0) = 0 \) and we have by (2.9) and (2.10) for all \( x \in X'_0, N_0 \subset \mathbb{N}, k \in \mathbb{N} \):

\[
\limsup_{x \in N_0} \inf_{x \in \mathcal{F}^N \cap U_k} (x) = \lim_{x \in N_0} \inf_{x \in S_0} f_{t_0}^{(n)}(x).
\]
and therefore
\[ \lim \sup_{n \in \mathbb{N}_0} \inf f_s^{(n)}(x) = \lim \sup_{n \in \mathbb{N}_0} \inf f_{S_0 \cup U_k}(x) . \]
Since \( \lim_{n \in \mathbb{N}} \varphi_n(x) = t_0 \) for all \( x \in X \), since \( S_0 \) is a strong separant for \( f_s \), \( t \in S \), and since \( U_k \cap S, k \in \mathbb{N} \), is a base for the neighborhood system of \( t_0 \) in \( S \), this implies that \( (\varphi_n)_{n \in \mathbb{N}} \) is a \( \mu \)-a.m.c. sequence for \( f_t \), \( t \in S \).

3. Maximum likelihood estimates. In this section we shall apply the results obtained in Section 2 to maximum likelihood estimation. Let \( \mathcal{P} \) be a family of probability measures endowed with a topology \( \mathcal{U} \) and dominated by a \( \sigma \)-finite measure \( \mu \) on the \( \sigma \)-field \( \mathcal{F} \). For each \( P \in \mathcal{P} \) let \( h_p \) be the equivalence class of densities of \( P \) with respect to \( \mu \).

A function \( \varphi_n : X^n \rightarrow \mathcal{P} \) is a maximum likelihood estimate for \( P \) at sample size \( n \), iff \( \varphi_n \) is m.c. estimate at sample size \( n \) for the family \( f_p = - \log h_p, P \in \mathcal{P} \). A sequence of functions \( \varphi_n : X^N \rightarrow \mathcal{P}, n \in \mathbb{N} \), is an asymptotic maximum likelihood (a.m.l.) sequence for \( \mathcal{P} \), iff for every \( P \in \mathcal{P} \) the sequence \( \varphi_n, n \in \mathbb{N} \), is \( \mu \)-a.m.c. sequence for \( f_p = - \log h_p, P \in \mathcal{P} \).

A sequence of functions \( \varphi_n : X^N \rightarrow \mathcal{P}, n \in \mathbb{N} \), is strongly consistent for \( \mathcal{P} \), iff for every \( P \in \mathcal{P} : \lim_{n \in \mathbb{N}} \varphi_n(x) = P \) \( P \)-N-a.e.

**Theorem 3.1.** If \( \mathcal{P} \) is endowed with a compact metrizable topology \( \mathcal{U} \), for each \( n \in \mathbb{N} \) there exists an \( \mathcal{F}^n \), \( \sigma(\mathcal{U}) \)-measurable maximum likelihood estimate for \( \mathcal{P} \) at sample size \( n \).

**Proof.** Apply Theorem 2.3 to \( T = \mathcal{P} \) and \( f_t = - \log h_t, t \in T \).

**Theorem 3.2.** Let \( \mathcal{P} \) be endowed with a compact metrizable topology and assume that \( P \rightarrow h_p, P \in \mathcal{P} \), is upper semicontinuous. If for every \( P \in \mathcal{P} \)

(i) \( P(\log h_p) > - \infty \),

(ii) \( P(\log \sup h_t) < + \infty \) for each compact \( C \subset \mathcal{P} \) with \( P \in C \), then a sequence of functions \( \varphi_n : X^N \rightarrow \mathcal{P}, n \in \mathbb{N} \), is strongly consistent for \( \mathcal{P} \) if and only if it is a sequence of asymptotic maximum likelihood estimates for \( \mathcal{P} \).

**Proof.** As by (i) and (ii) \( P(\log h_p) > - \infty \) and \( P(\log h_Q) < \infty \) if \( Q \neq P \), we have for all \( P, Q \in \mathcal{P} \) with \( P \neq Q \)

\[ P(\log h_Q) - P(\log h_p) < \log P(h_Q|h_p) \leq \log \mu(h_Q) = 0 , \]

and hence \( P(\log h_Q) < P(\log h_p) \). Therefore we may apply Theorem 2.8 for each \( P \in \mathcal{P} \) to \( S = T = \mathcal{P}, P = P_0, t_0 = P_0 \) and the family \( f_p = - \log h_p, P \in \mathcal{P} \). This implies the assertion.

We remark that—similar as Theorem 2.6 of Pfanzagl [6]—Theorem 3.2 can be formulated with a compact metric space \( T \supset \mathcal{P} \), where for \( t \in T - \mathcal{P} \) an equivalence class \( h_t \geq 0 \) with \( \mu(h_t) \leq 1 \) is given. Then we have in addition to assume that \( t \rightarrow h_t, t \in T \), is u.s.c. and that 3.2 (ii) holds with compact \( C \subset T \).
instead of \( C \subset P \). In this case the assertion follows from Theorem 2.8., applied to \( T, S = P \) and \( f_t = - \log h_t, t \in T \).

According to Theorem 3.2 each sequence of 'classical' a.m.l. estimates for \( P \) which turns out to be strongly consistent is a sequence of a.m.l. estimates for \( P \), if \( P \) fulfills the assumptions of Theorem 3.2. The following example shows that even for a compact metric \( P \) with uniformly bounded upper semi-continuous densities there exist large classes of a.m.l. (and hence strongly consistent) sequences for \( P \), which cannot be obtained as classical a.m.l. estimates from a suitable family of versions of the densities. Example 3.3 improves an example of Lüpsen [5].

**Example 3.3.** Let \( X = (0, 2) \), \( \mathcal{F} \) the Borel-field on \( X \) and \( \mu \) the Lebesgue measure on \( \mathcal{F} \). For each \( t \in T = [0, 1] \) let \( P_t \) be the probability measure on \( \mathcal{F} \) with density \( h_t = 1_{(t, t+1)} \) with respect to \( \mu \); where \( 1_A \) denotes the indicator function of the set \( A \subset X \). Let \( P = \{ P_t : t \in T \} \), and let \( \mathcal{U} \) be the topology on \( P \) induced by the \( 1-1 \) map \( t \rightarrow P_t, t \in T \), and the usual topology on \( T = [0, 1] \). Then the assumptions of Theorem 3.2 are fulfilled for \( (P, \mathcal{U}) \). Let

\[
X_0 = \{(x_i)_{i \in N} \in X^N : \sup(x_1, \ldots, x_n) - \inf(x_1, \ldots, x_n) < 1 \quad \text{for all} \quad n \geq 2 \}.
\]

Then \( P_t^N(X_0) = 1 \) for all \( t \in T \). Let \( T_0 = \{ t_i : i \in N \} \) be dense in \( T \). Then we have for all \( (x_i)_{i \in N} \in X_0 \) and all \( n \geq 2 \)

\[
\sup_{t \in T} \prod_{i=1}^n h_t(x_i) = \sup_{t \in T_0} \prod_{i=1}^n h_t(x_i) = 1.
\]

Now define for all \( x = (x_i)_{i \in N} \in X^N \) and all \( n \in N \): \( \varphi_n(x) = 1 \) iff \( \inf(x_1, \ldots, x_n) = 1 \), and \( \varphi_n(x) = t_k \) otherwise, where \( k \) is the smallest index \( i \in N \) such that \( t_i \in (\inf(x_1, \ldots, x_n), \inf(x_1, \ldots, x_n) + 1/n) \). It is easy to see that \( \varphi_n : X^N \rightarrow T \) is measurable for each \( n \in N \). As for all \( t \in T \) there exists \( X_t \in \mathcal{F}^N \) with \( P_t^N(X_t) = 1 \) such that \( \lim_{n \to \infty} \inf(x_1, \ldots, x_n) = t \) for all \( (x_i)_{i \in N} \in X_t \), the sequence \( (\varphi_n)_{n \in N} \) is strongly consistent for \( P \) and hence a an.m.l. sequence for \( P \) (Theorem 3.2).

Now we shall show that for no choice of versions \( (\varphi_n)_{n \in N} \) is an a.m.l. sequence in the usual sense. Let \( h_t' \in \hat{h}_t, t \in T \). As \( T_0 \) is countable there exists \( X_t \in \mathcal{F} \) with \( \mu(X - X_t) = 0 \) such that

\[
h_t(x) = h_t'(x) \quad \text{for all} \quad x \in X_t, \quad t \in T_0.
\]

Let \( X_t = X_t' \cap X_0 \cap X_t^N, t \in T \). As \( P_t(X_t) = 1 \) we have \( P_t^N(X_t^N) = 1 \) and hence \( P_t^N(X_t) = 1 \) for all \( t \in T \). Let \( f_t = - \log h_t, f_t' = - \log h_t', t \in T \). Since \( \varphi_n \) assumes its values only in \( T_0 \cup \{ 1 \} \), we obtain by (3.5) for all \( t \in [0, 1) \) and all \( x = (x_i)_{i \in N} \in X_t \):

\[
\frac{1}{n} \sum_{i=1}^n f_{\varphi_n(i)}(x_i) = \frac{1}{n} \sum_{i=1}^n f_{\varphi_n(i)}(x_i) = + \infty
\]

for all sufficiently large \( n \in N \).
By (3.4) and (3.5) we have for all \((x_i)_{i \in \mathbb{N}} \in X_n \cap X_1^N\) and all \(n \geq 2\)

\[
(3.7) \quad \inf \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) \leq \inf \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) = 0.
\]

According to (3.6) and (3.7) relation 2.5 is violated for all \(x \in X_0, t \in [0, 1)\)
for both the families \(f_i, t \in T,\) and \(f_i^*, t \in T\). Hence \((\varphi_n)_{n \in \mathbb{N}}\) is not an a.m.l.
sequence in the usual sense.

4. Auxiliary Lemmas. In this section we collect lemmas which are auxiliary
for the results in Section 2 and Section 3. Throughout this section let \(\mu\) be a
\(\sigma\)-finite measure on a \(\sigma\)-field \(\mathcal{F}\) over \(X\).

**Lemma 4.1.** Let \((T, \mathcal{C})\) be a topological space with countable base and
\(f_t \in M(\mathcal{F}, \mu), t \in T\). Then for each countable family \(\mathcal{C}\) of subsets of \(T\) there
exists a countable \(T_0 \subset T\) which contains for every \(C \in \mathcal{C}\) a strong separant for
the family \(f_t, t \in C\).

**Proof.** As each subset of \(T\) with the relative topology has a countable base,
for each \(C \in \mathcal{C}, n \in \mathbb{N}\), there exists a separant, say \(T(C, n) \subset C\), for the family
\(f_t^{(n)}, t \in C\) (see [6], Corollary 3.2). Then \(T_0 = \bigcup \{T(C, n) : C \in \mathcal{C}, n \in \mathbb{N}\}\)
is countable and fulfills the assertion.

**Lemma 4.2.** Let \((T, \mathcal{C})\) be a topological space with countable base, \(f_t \in \quad M(\mathcal{F}, \mu), t \in T\) and \(f_t, g_t \in \mathcal{F}, t \in T\).

(i) If \(T_0, T_1 \subset T\) are separants, then there exists a \(\mu\)-null set \(X_0 \in \mathcal{F}\) such
that for all \(x \in X_0\) and all \(U \in \mathcal{C}\)

\[
\inf f_{U \cap T_0}(x) = \inf g_{U \cap T_1}(x).
\]

(ii) If \(T_0, T_1 \subset T\) are strong separants, then there exists \(X_0 \in \mathcal{F}^N,\) which is
\(P\)-null set for each probability measure \(P \ll \mu\) on \(\mathcal{F},\) such that for all \(x \in X_0,\)
\(U \in \mathcal{C}, n \in \mathbb{N}\)

\[
\inf f_{U \cap T_0}^{(n)}(x) = \inf g_{U \cap T_1}^{(n)}(x).
\]

**Proof.** We only prove (ii). The proof for (i) is similar. As \(T_0, T_1\) are
strong separants we have for all \(n \in \mathbb{N}, U \in \mathcal{C}\)

\[
\inf f_{U \cap T_0}^{(n)} = \inf f_{U \cap T_1}^{(n)}.
\]

Hence for each \(U \in \mathcal{C}, n \in \mathbb{N}\), there exist a \(\mu\)-null set \(A_{U,n} \in \mathcal{F}^N\) such that
for all \(x \in X_N,\) which are not element of the cylinder set over \(A_{U,n},\) say
\(X_{U,n} \in \mathcal{F}^N,\) we have

\[
\inf f_{U \cap T_0}^{(n)}(x) = \inf g_{U \cap T_1}^{(n)}(x).
\]

Let \(\mathcal{C}_0\) be a countable base for \(\mathcal{C}\) and define \(X_0 = \bigcup \{X_{U,n} : U \in \mathcal{C}_0, n \in \mathbb{N}\}\).
If \(P \ll \mu\) is a probability measure on \(\mathcal{F}\) then \(P^N(X_{U,n}) = 0\) for all \(U \in \mathcal{C},\)
\( n \in \mathbb{N} \), whence \( P^N(X_0) = 0 \). As \( \inf_{U \in \mathcal{U}_0} f_U^{(n)}(x) = \inf_{U \in \mathcal{U}_1} g_U^{(n)}(x) \) for all \( U \in \mathcal{U}_0, x \in X \), \( n \in \mathbb{N} \), and each \( U \in \mathcal{U} \) is (countable) union of elements of \( \mathcal{U}_0 \), this implies the assertion.

The following lemma is a slight modification of Lemma 3.11 of Pfanzagl [6]:

**Lemma 4.3.** Let \((T, \mathcal{U})\) be a compact metric space and \( f_t \in M(\mathcal{F}, \mu), t \in T \). Assume that

(i) \( t \to f_t, t \in T \), is l.s.c.

Let furthermore \( P \ll \mu \) be a probability measure on \( \mathcal{F} \) such that

(ii) \( P(\inf f_T) > -\infty \).

Then

(a) \( t \to P(f_t), t \in T \), is l.s.c.

(b) If \( f_t \in f_t, t \in T \), and \( T_0 \) is a strong separant for \( f_t, t \in T \), then

\[
\inf_{t \in T} P(f_t) = \lim_{n \to \infty} \inf_{f_t^{(n)}} P(f_t) = \inf_{x \in T} P(f_t) \quad \text{P}^N\text{-a.e.}
\]

**Proof.** The proof for (a) and for the relation

(4.4) \( \inf_{t \in T} P(f_t) \leq \lim_{n \to \infty} \inf_{f_t^{(n)}} P(f_t) \quad \text{P}^N\text{-a.e.}\)

runs analogously to the proof of the corresponding assertions in Lemma (3.11) of Pfanzagl [6].

As the function \( t \to P(f_t), t \in T \), is l.s.c. by (a), it attains its infimum for some \( t_0 \in T \). Since \( \inf_{f_t^{(n)}} \leq \inf_{f_t^{(n)}} \) for each \( n \in \mathbb{N} \), we have by the strong law of large numbers \( \lim_{n \to \infty} \sup_{f_t^{(n)}} \inf_{f_t^{(n)}} P(f_t^{(n)}) = P(f_t^{(n)}) = \inf_{t \in T} P(f_t) \quad \text{P}^N\text{-a.e.}\). Together with (4.4) this implies (b).

**Lemma 4.4.** Let \((T, \mathcal{U})\) be a compact metric space and let \( f_t : X \to [-\infty, +\infty] \), \( t \in T \), be a family of \( \mathcal{F} \)-measurable functions. Then for every non void \( T_0 \subset T \) there exists a \( \mathcal{F} \), \( \sigma(\mathcal{U}) \)-measurable function \( \varphi : X \to T \) such that \( \varphi(x) \in U \in \mathcal{U} \) implies \( \inf_{t \in T_0} f_t(x) = \inf_{t \in U \cap T_0} f_t(x) \).

**Proof.** For each \( x \in X \) let \( Q_x = \{ t \in T : \inf_{t \in T_0} f_t(x) = \inf_{t \in U \cap T_0} f_t(x) \} \) if \( t \in U \in \mathcal{U} \).

In order to apply Theorem 4.1 of Sion (1960)—which admits a measurable choice \( \varphi(x) \in Q_x, x \in X \)—we prove the following:

(a) \( Q_x \neq \emptyset \) for each \( x \in X \)

Let \( t \in T_0 \) and \( n \in \mathbb{N} \), be such that \( \lim_{n \to \infty} f_t(x) = \inf_{t \in T_0} f_t(x) \). Then every accumulation point of \( t_n, n \in \mathbb{N} \), belongs to \( Q_x \). As a compact metric space is sequentially compact this implies \( Q_x \neq \emptyset \).

(b) \( Q_x \) is closed for each \( x \in X \)

Let \( U \) be the limit point of the sequence \( t_n \in U \), \( n \in \mathbb{N} \). Then \( t \in U \in \mathcal{U} \) implies \( t \in U \) for some \( n \in \mathbb{N} \); hence \( \inf_{t \in U \cap T_0} f_t(x) = \inf_{t \in T_0} f_t(x) \), whence \( t \in Q_x \).

(c) \( \{ x \in X : Q_x \cap C = \emptyset \} \) \( \in \mathcal{F} \) for each compact \( C \subset T \)

Let \( \mathcal{U}_0 \) be a countable base of \((T, \mathcal{U})\) which is closed under finite unions. We shall prove

(4.6) \( \{ x : Q_x \cap C = \emptyset \} = \emptyset \) \( \cup \{ x : \inf_{t \in U \cap T_0} f_t(x) > \inf_{t \in T_0} f_t(x) \} : C \subset U \in \mathcal{U}_0 \).
The inclusion "\( \supset \)" in (4.6) is trivial. Assume, conversely, \( Q_x \cap C = \emptyset \). Then for every \( t \in C \) there exists \( U_t \in \mathcal{U}_0, t \in U_t \), with \( \inf_{U \cap T_0} f_t(x) > \inf_{T_0} f(x) \). As \( \{U_t : t \in C\} \) is an open cover of the compact set \( C \), there exists a finite subcover, say \( U_t, \ldots, U_{t_n} \). Let \( U = \{ \cup U_t : i = 1, \ldots, n \} \). Then \( C \subset U \in \mathcal{U}_0 \) and \( \inf_{U \cap T_0} f_t(x) > \inf_{T_0} f(x) \). This proves (4.6). As \( T_0 \) is countable and each \( f_t \), \( t \in T \), is measurable, (4.6) implies (c).

Because of (a), (b), (c), Theorem 4.1 of Sion (1960) is applicable and we obtain the existence of an \( F \), \( \sigma(\mathcal{U}) \) measurable function \( \varphi : X \to T \) such that \( \varphi(x) \in Q_x \) for all \( x \in X \). This function has the asserted properties.

REFERENCES


