A COMPOSITE NONPARAMETRIC TEST FOR A SCALE SLIPPAGE ALTERNATIVE

BY MELVIN N. WOINSKY

Bell Telephone Laboratories, Inc.

Consider the 2-sample problem where the null cdf \( F(x) \) satisfies \( F(0) = 0 \) and the alternative is \( F_\theta(x) = F(x/(1 + \theta)) \) with \( \theta > 0 \). An asymptotically optimum statistic \( z \) is obtained for a parametric model where \( F(x) \) is a gamma distribution. The Mann-Whitney \( U \) and Savage \( T \) statistics are compared to \( z \) for several null densities. It is shown that the Pitman asymptotic relative efficiency, \( ARE(U/z) \), can approach zero if \( \mu/\sigma \to 0 \), where \( \mu \) is the mean and \( \sigma^2 \) the variance of the null distribution. However, a lower bound on \( ARE(U/z) \) is obtained as a function of \( \mu/\sigma \) for general \( F(x) \). Using the bound a composite test is constructed which has a specified minimum \( ARE \) of any desired value between 0 and .864. Densities exist for the composite test which result in arbitrarily large values of efficiency.

1. Introduction. Consider the two-sample problem,

\[
H: X_1, X_2, \ldots X_{n_1}, Y_1, Y_2 \ldots Y_{n_2} \text{ i.i.d. } \sim F(x) \]

\[
K: X_1, X_2, \ldots X_{n_1} \text{ i.i.d. } \sim F_\theta(x) \]

\[
Y_1, Y_2 \ldots Y_{n_2} \text{ i.i.d. } \sim F(x) \]

where \( F(x) \) is an absolutely continuous cdf with \( F(0) = 0 \) and corresponding density \( f(x) \) and mean \( \mu \) and variance \( \sigma^2 \). The \( X \) and \( Y \) data are independent. The alternative cdf is \( F_\theta(x) = F(x/(1 + \theta)) \) with \( \theta > 0 \). In a parametric model of interest \( f(x) \) is a gamma density,

\[
f(x) = (s^\lambda / \Gamma(\lambda))x^{s-1} \exp(-sx) \quad \lambda, s > 0, \quad x > 0,
\]

with known shape parameter \( \lambda \) and unknown scale parameter \( s \). This model arises in a target detection problem [19] where the \( X \) and \( Y \) data are obtained by spectral analysis of a stationary Gaussian time-series. The parameter \( \lambda \) is the time-bandwidth product used in the analyzer and \( s \) is inversely proportional to the input noise power in the analyzer band. The presence of an input sinusoid induces a noncentral gamma density which at small signal-to-noise ratio can be characterized as a scale alternative. If the form of the distribution of the input time-series data is unknown then the form of the distribution of the spectral data is unknown and a nonparametric formulation is appropriate.

In the parametric case (1.1) an asymptotically optimum statistic \( z \) defined in (2.2) is used. This statistic depends on the ratio of sample means. The restriction \( F(0) = 0 \) makes the scale alternative a one-sided slippage alternative, i.e.

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$F_\phi(x) \leq F(x)$, and the Mann-Whitney-Wilcoxon $U$ and Savage $T$ tests are suitable for use in the nonparametric model. $T$ is the locally most powerful rank test [4] for (1.1) when $\lambda = 1$. Pitman asymptotic relative efficiency (ARE) is used to make comparisons. ARE results are obtained for the gamma and other densities. For (1.1) and $\lambda = 1$ it follows from [2], [4] that $\text{ARE}(U/z) = \frac{2}{3}$ and $\text{ARE}(T/z) = 1$. It is shown that for $\lambda > 1$, $\text{ARE}(U/z) > \frac{2}{3}$ and $\text{ARE}(T/z) > .816$. For other densities such as a mixture of gamma densities, large values of ARE can be obtained.

Of particular interest is the result that for general $f(x)$ with $f(x) = 0$ for $x < 0$ and finite second moment,

$$\text{ARE}(U/z) \geq .864 \left(1 - .458 \frac{\sigma^2}{\mu^2}\right) \quad \text{if} \quad \frac{\mu}{\sigma} \geq 2^4;$$

$$\geq \frac{27}{4} \frac{(\mu/\sigma)^6}{(1 + \mu^2/\sigma^2)^4} \quad \text{if} \quad \frac{\mu}{\sigma} < 2^4.$$

Using this result a composite test can be designed which has a specified minimum ARE of any desired value between 0 and .864. It is shown that densities exist for the composite test which result in an arbitrarily large ARE. The composite test is constructed by forming an estimate of $\mu/\sigma$; if the estimate is smaller than a specified value, $z$ is used otherwise $U$ is used as the test statistic.

It should be noted that the literature contains several papers, for example, [4], [9], [15], concerning nonparametric tests against a scale alternative. The emphasis is usually on dispersion, i.e., $F(0) = \frac{1}{2}$. The statistics of Puri and Puri [13] and the statistic of Ansari and Bradley [1] reduce to the Mann-Whitney statistic if it is known that $F(0) = 0$. Sukhatme's $S$ statistic [17] appears efficient for the problem considered. However, although it is not mentioned the derivation of Sukhatme's [17] efficiency equations assumes $F(0) = \frac{1}{2}$. The dispersion statistic of Mood [16] is efficient for testing for a change in variance in a Gaussian distribution [1]. However, for $F(0) = 0$ this statistic appears to be very inefficient [3].

2. Parametric statistic. For the problem considered and the gamma density of (1.1) it can be shown in a lengthy but straightforward manner that a statistic equivalent to the likelihood ratio statistic for all known $\lambda$ is,

$$(2.1) \quad t = \bar{X}/\bar{Y},$$

where $\bar{X}$ and $\bar{Y}$ are the sample means. The critical region consists of large values of $t$. It has been shown [7] that in the case $\lambda = 1$, $t$ is uniformly most powerful. The ratio $t$, is $F$-distributed with $2\lambda n_1$ and $2\lambda n_2$ degrees of freedom under $H$ if $2\lambda$ is an integer. If $\lambda$ is unknown or if the density is not given by (1.1), $t$ cannot be used since the critical region cannot be specified, not even
asymptotically. Note also that a maximum likelihood estimator of \( \lambda \) is not available in closed form [6].

Consider the following statistic,

\[ z = \hat{\phi} \log \bar{X}/\bar{Y}, \]

where

\[ \hat{\phi} = n_{1}(n_{1} + n_{2})^{-1}\bar{X}/S_{x} + n_{1}(n_{1} + n_{2})^{-1}\bar{Y}/S_{y}, \]

and \( S_{x}^{2}, S_{y}^{2} \) are the sample variances of the \( X \) and \( Y \) sample, respectively. For the nonparametric formulation, \( F(x) \) continuous and \( F(0) = 0, \hat{\phi} \rightarrow_{n.a.} \mu/\sigma \) as \( \min(n_{1}, n_{2}) \rightarrow \infty \), for all \( \theta \). Also from Lehmann ([10] page 274) and the central limit theorem it follows that

\[ (r(1 - r)N)^{1/2}(\mu/\sigma)(\log \bar{X}/\bar{Y} - \log (1 + \theta)) \]

is asymptotically distributed according to \( \phi(x) \) the standard normal cdf, where \( N = n_{1} + n_{2} \) and \( r = n_{1}/N \) provided \( \lim_{N \to \infty} r \neq 0, 1 \). It follows from ([8] page 236) that (2.4) with \( \mu/\sigma \) replaced by \( \hat{\phi} \) is still asymptotically distributed according to \( \phi(x) \) and, therefore, \( r \) and \( z \) are asymptotically equi-efficient. Then from the properties of the likelihood ratio [18], \( z \) is asymptotically optimum for the gamma density and all values of \( \lambda \). Clearly this remains true if \( \hat{\phi} \) in (2.2) is replaced by any consistent estimate of \( \mu/\sigma \). The statistic \( z \) can be used when \( \lambda \) is unknown or for general \( F(x) \). The critical region consisting of large values of \( z \) can be specified asymptotically from (2.4).

3. Asymptotic relative efficiency. The nonparametric statistics can be defined in terms of the ranks \( R_{i}, i = 1, 2 \cdots n_{1} \), where \( R_{i} \) is the rank of \( X_{i} \) in the pooled \( Y, X \) data. The linearly equivalent Mann-Whitney-Wilcoxon [11] statistic is,

\[ U = (n_{1}n_{2})^{-1}\sum_{i=1}^{n_{1}} R_{i} - (n_{1} + 1)/2n_{2} \]

and the Savage statistic [14] is,

\[ T = n_{1}^{-1}\sum_{i=1}^{n_{1}} (\sum_{j=N-R_{i}+1}^{N} l^{-1}). \]

The Savage statistic is the optimum rank statistic [14] for an exponential distribution and a scale alternative. Tables of the null distribution of \( U \) and \( T \) are available and the critical regions can be specified approximately by using the asymptotic normality of \( U \) and \( T \).

Subject to the usual regularity conditions for Pitman efficiency [12], the ARE can be obtained from the efficacy of each test. The procedure is outlined below.

Let \( E_{\phi}(Q_{i}) \) and \( \sigma_{\phi}^{2}(Q_{i}) = \sigma_{\theta=0}^{2}(Q_{i}) \) be the moments of \( Q_{i} \) representing \( z, U \) or \( T \). The efficacy of \( Q_{i} \) is,

\[ \varepsilon(Q_{i}) = \left[ \frac{dE_{\phi}(Q_{i})}{d\theta} \right]^{-1/2}\sigma_{\phi}^{2}(Q_{i}) \]
and
\[ \text{ARE}(Q_1/Q_2) = \lim_{N \to \infty} \varepsilon(Q_1)/\varepsilon(Q_2). \]
From Section 2, \( z \) is asymptotically normal under \( H \) and \( K \) and it follows that the efficacy of \( z \) is,
\[ \varepsilon(z) = n_1 n_2 N^{-1}(\mu/\sigma)^3. \]  
From [11], \( \sigma^2(U) = (N + 1)/12n_1 n_2 \) and \( E_0(U) = \int_0^\infty [1 - F_0(x)] dF(x) \), using \( F_0(x) = F(x/(1 + \theta)) \) gives
\[ \varepsilon(U) = 12n_1 n_2 (N + 1)^{-1} [\int_0^\infty x f^2(x) \, dx]^2. \]
From Chernoff and Savage [5],
\[ E_0(T) = \int_0^\infty J[n_1 N^{-1} F_0(x) + n_2 N^{-1} F(x)] dF_0(x), \]
where \( J(x) = -\log(1 - x), 0 < x < 1 \), and \( \sigma^2(T) = n_2/(n_1 N) \) so that
\[ \varepsilon(T) = n_1 n_2 N^{-1} \left[ \int_0^\infty \frac{xf^2(x)}{1 - F(x)} \, dx \right]^2. \]
Note that Basu and Woodworth [3] give the efficacy of \( T \) for general \( f(x) \) as shown in (3.6) but with the lower limit of integration \(-\infty \) and \( 1 - F(x) \) incorrectly replaced by \( e^{-x} \). However, they only make a numerical calculation for an exponential \( f(x) \). In that case their result and (3.6) agree.

It follows from (3.3), (3.4) and (3.6) that
\[ \text{ARE}(U/z) = 12 \left( \frac{\alpha}{\mu} \right)^3 \left[ \int_0^\infty x f^2(x) \, dx \right]^2, \]
\[ \text{ARE}(T/z) = \left( \frac{\sigma}{\mu} \right)^3 \left[ \int_0^\infty \frac{xf^2(x)}{1 - F(x)} \, dx \right]^2. \]
For the gamma density of (1.1),
\[ \text{ARE}(U/z) = 12 \Gamma^*(2\lambda)/(\lambda 2^{2+1}\Gamma(\lambda)), \]
\[ \text{ARE}(T/z) = I/(\lambda 2^{2+1}\Gamma(\lambda)), \]
where
\[ I = \int_0^\infty dx e^{-x} x^{2+1} \left[ 1 - \frac{\Gamma(\lambda, x/2)}{\Gamma(\lambda)} \right] \]
and \( \gamma(\lambda, x/2) \) is the incomplete gamma function.

Using \( \lim_{\lambda \to 0} \lambda\Gamma(\lambda) = 1 \) yields \( \lim_{\lambda \to 0} \text{ARE}(U/z) = 0 \) and by numerical evaluation \( \text{ARE}(U/z) \) is a monotonically increasing function of \( \lambda \). For \( \lambda = \frac{1}{2} \) (density function has infinite discontinuity at the origin) \( \text{ARE}(U/z) = 6/\pi^2 \) and for \( \lambda = 1 \) (exponential density) \( \text{ARE}(U/z) = \frac{3}{2} \). Also if \( f(x) \) is the gamma density, \( \sigma f(\sigma x + \mu) \to \phi(x) \) the standard normal density as \( \lambda \to \infty \). Then from (3.7) with \( x = \sigma y + \mu \) and \( \mu/\sigma = \lambda t \),
\[ \text{ARE}(U/z) = 12[\lambda t \int_0^\infty y[\sigma f(\sigma y + \mu)]^3 \, dy + \int_0^\infty [\sigma f(\sigma y + \mu)]^3 \, dy]^2, \]
and

\[
\lim_{\lambda \to \infty} \text{ARE} \left( U/z \right) = 12 \left[ \int_{-\infty}^{\infty} \phi^2(y) \, dy \right]^2 = 3/\pi ,
\]

since \( \int_{-\infty}^{\infty} |y| \phi^2(y) \, dy < \infty \). This efficiency is the same as the translation value for \( U \) and a normal density.

Similarly, by numerical integration, \( \lim_{\lambda \to 0} \text{ARE} \left( T/z \right) = 0 \) and \( \text{ARE} \left( T/z \right) \) reaches its maximum at \( \lambda = 1 \). At \( \lambda = \frac{1}{2} \), \( \text{ARE} \left( T/z \right) = .978 \) and by direct evaluation \( \text{ARE} \left( T/z \right) = 1 \) at \( \lambda = 1 \). The function falls monotonically for \( \lambda > 1 \). As before, with \( \Phi(x) \) the standard normal cdf,

\[
\lim_{\lambda \to \infty} \text{ARE} \left( T/z \right) = \left[ \int_{-\infty}^{\infty} \frac{\phi^2(x)}{1 - \Phi(x)} \, dx \right]^2 ,
\]

since

\[
\int_{-\infty}^{\infty} \frac{|y| \phi^2(y)}{1 - \Phi(y)} \, dy < \infty .
\]

Expression (3.12) has the value .816 by numerical integration. The result of (3.12) corresponds to the translation value for \( T \) and a normal density.

It follows that \( \text{ARE} \left( T/z \right) \geq .816 \) and \( \text{ARE} \left( U/z \right) \geq \frac{3}{4} \) for \( \lambda \geq 1 \) if \( f(x) \) is a gamma density. Note that \( \text{ARE} \left( U/z \right) \) can be near zero and that this occurs for small \( \lambda \) or small values of \( \rho^2/\sigma^2 \). This will be shown to hold for general densities with a “large concentration” of mass near the origin resulting in small values of \( \text{ARE} \left( U/z \right) \).

If other densities are considered, large values for \( \text{ARE} \) can be obtained. For a mixture density of \( f(x) = (1 - \varepsilon)f(x: \lambda, s_1) + \varepsilon f(x: \lambda, s_2) \), the value of \( \text{ARE} \left( U/z \right) \) can be obtained by multiplying (3.9) by

\[
M = \frac{\left[ 1 + \varepsilon(R^2 - 1) + \lambda(1 - \varepsilon)\varepsilon(R - 1)^2 \right]}{\left[ 1 + \varepsilon(R - 1)^2 \right]}
\times \left[ 1 - 2\varepsilon(1 - \varepsilon) \left( 1 - \frac{2R^2}{(R + 1)^2} \right) \right] ,
\]

where \( R = s_1/s_2 > 1 \). The factor \( M \) is the relative improvement due to non-parametric processing when there is contamination of the underlying gamma density. Note that

\[
\lim_{\varepsilon \to 0, R \to \infty, \varepsilon R^2 \to \lambda} M = 1 + (1 + \lambda) \lambda ,
\]

so large improvements are possible. With \( \lambda = 8 \) and \( \lambda = \frac{3}{4} \) the limiting value of \( M \) is 4. For the Savage statistic the limiting value of \( M \) is the same as in (3.14) and the actual value approximately the same as (3.13).

Based on the examples, for the alternative \( F_\theta(x) = F(x/(1 + \theta)) \), the Savage statistic in general appears to perform better than the Mann-Whitney statistic. When the density has a very heavy upper tail or is concentrated far from the
origin there is a slight preference for the Mann-Whitney statistic. The Savage statistic does relatively well for densities with both heavy and sharp upper tails. It does particulary well when there is a sharp cut-off on this tail. For instance if \( f(x) \) is triangular (decreasing linearly from \( x = 0 \)), \( \text{ARE}(U/z) = \frac{2}{3} \) while \( \text{ARE}(T/z) = 2 \).

4. Lower bound on \( \text{ARE}(U/z) \). It is clear from the previous section that \( \text{ARE}(U/z) \) can approach zero. However it is possible to obtain a lower bound as a function of \( \mu/\sigma \).

Since all factors are positive, minimizing \( \text{ARE}(U/z) \) of (3.7) is equivalent to minimizing

\[
L = \int_0^\infty x f(x)^2 \, dx,
\]

subject to \( 1 = \int_0^\infty f(x) \, dx, \mu = \int_0^\infty x f(x) \, dx, \mu_2 = \int_0^\infty x^2 f(x) \, dx \) and \( f(x) \geq 0 \). Let, \( V = x f^2(x) - 2(\lambda_1 + \lambda_2 x + \lambda_3 x^2) f(x) \) where the \( \lambda \)'s are numbers determined by the integral constraints. The necessary Euler equations are \( \partial V / \partial f = 0 \) for \( f(x) > 0 \) and \( \partial V / \partial f \geq 0 \) for \( f(x) = 0 \). The first equation yields

\[
f(x) = \lambda_1/x + \lambda_2 + \lambda_3 x.
\]

Assume \( \lambda_1 \leq 0 \) so that the integral constraints can be satisfied with \( \lambda_2 > 0 \) and \( \lambda_3 < 0 \). The resulting \( f(x) \) intersects the \( x \) axis at \( r_1 \) and \( r_2 \), \( 0 \leq r_1 < r_2 \) where \( r_1 \) and \( r_2 \) are solutions of

\[
\lambda_1 + \lambda_2 x + \lambda_3 x^2 = 0.
\]

Taking \( f(x) = 0 \) outside of \( [r_1, r_2] \) allows \( f(x) \) of (4.2) to satisfy both Euler equations. From (4.3), \( \lambda_2/\lambda_3 = -(r_1 + r_2), \lambda_1/\lambda_3 = r_1 r_2 \) and if \( y = r_1/r_2 \) it is clear that \( 0 \leq y < 1 \). Using the integral constraints and \( \sigma^2 = \mu_2 - \mu^2 \) gives after much algebra,

\[
\frac{\sigma^2}{\mu^2} = \frac{3}{2} \left( 1 - 2y + 2y^2 - y^3 \right) \frac{(1 - y^2 + 2y \log y)}{(1 - 3y + 3y^2 - y^3)^2}
\]

and

\[
\left( \int_0^\infty x f^2(x) \, dx \right)^2 = \frac{(1 - 8y - 12y^2 \log y + 8y^3 - y^4)^2}{9(1 - y^2 + 2y \log y)^4}.
\]

The min \{\text{ARE}(U/z)\} is obtained by using (4.4) and (4.5) in (3.7). This calculation was performed on a computer for \( y \in [0, 1] \). As \( y \) goes from zero to one, \( \mu/\sigma \) monotonically increases from \( 2^\dagger \) to infinity. The min \{\text{ARE}(U/z)\} is a linearly decreasing function of \( \sigma^2/\mu^2 \) (to the accuracy of the plotting) with a value of .864 as \( \mu/\sigma \to \infty \) and \( \frac{2}{3} \) at \( \mu/\sigma = 2^\dagger \). It then follows that

\[
\text{ARE}(U/z) \geq .864(1 - .4586\sigma^2/\mu^2), \quad \text{if} \quad 2^\dagger \leq \mu/\sigma < \infty
\]

except for small computational error in the lower bound.
The result is a global minimum. This is easily verified by substituting an arbitrary density into (4.1), consisting of the minimizing density plus a term $\varepsilon(x)$ with $\int_0^\infty \varepsilon(x) = \int_0^\infty x \varepsilon(x) \, dx = \int_0^\infty x^2 \varepsilon(x) \, dx = 0$, and $\varepsilon(x) \geq 0$ for $x \in [r_1, r_2]$.

To obtain a solution for $0 \leq \mu/\sigma \leq 2^4$, assume that $\lambda_1 = \varepsilon_1 > 0$ with $\lambda_2 > 0$, $\lambda_3 < 0$. Taking $f(x) = 0$ outside of $(0, r_2)$ allows $f(x)$ of (4.2) to satisfy the Euler conditions if $r_2$ is the positive root of (4.3). In order to satisfy the constraint that the density integrates to one, let $f(x) = 0$ outside of $(\varepsilon_2, r_2)$. By letting $\varepsilon_1$ and $\varepsilon_2$ approach zero at an appropriate rate it is possible to satisfy both this constraint and in the limit the minimizing Euler equations with $0 \leq \mu/\sigma \leq 2^4$.

Using (4.2), $xf'(x) = x(\varepsilon_1/x + \lambda_2 + \lambda_3 x)f(x)$ so that from (4.1), $L = \varepsilon_1 + \lambda_2 \mu + \lambda_3 \mu x$. In the limit as $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, it follows from (4.3) and the constraints on the first and second moments that $r_2 = -\lambda_2/\lambda_3$, $\mu = \lambda_2 r_2^2/2 + \lambda_3 r_2^3/3$, and $\mu_2 = \lambda_2 r_2^3/3 + \lambda_3 r_2^4/4$. Then $r_2 = 2\mu_2/\mu$, $\lambda_2 \to 6\mu_2/r_2^2 = 3\mu_2/(2\mu_2)$, $\lambda_3 \to -3\mu_2/(4\mu_2^3)$, $L \to 3\mu_2/(4\mu_2^3) = 3\mu_2/(4(\sigma^2 + \mu^2))$ and from (3.7),

\[
\min \{\text{ARE}(U/z) \} = 12(\sigma^2/\mu^2)L^2
\]

or

\[
\min \{\text{ARE}(U/z) \} = \frac{27}{4} \frac{1}{(1 + \mu^2/\sigma^2)^{\gamma}}.
\]

For this procedure to be valid and consistent with (4.6) it is necessary to show that the constraint $1 = \int_0^\infty f(x) \, dx$ can be satisfied if and only if $0 \leq \mu/\sigma \leq 2^4$. Using the constraint yields,

\[
1 = B + \lim_{\varepsilon_1 \to 0, r_2 \to 0} (\varepsilon_1 + \lambda_3 r_2^2/2),
\]

where $B = \lim_{\varepsilon_1 \to 0, r_2 \to 0} (\varepsilon_2 \log \varepsilon_2)$. Equation (4.8) is equivalent to $1 = B + (3\mu_2/(\sigma^2)(2[1 + \mu^2/\sigma^2]))$ so that $\mu_2/\sigma^2 = 2(1 - B)/(1 + 2B)$. Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, it follows that $B \geq 0$. It is then possible to let $\varepsilon_1$ and $\varepsilon_2$ approach zero such that (4.7) is valid only for any given $\mu/\sigma \in [0, 2^4]$. Note that for $\mu/\sigma = 2^4$ both bounds give $\text{ARE}(U/z) = \frac{27}{4}$ and that the density resulting in this value is triangular, decreasing linearly from a peak at $x = 0$. The bounds of (4.6) and (4.7) are monotonically increasing functions of $\mu/\sigma$.

Relation (4.7) and its derivation points out what was indicated in Section 3 for the gamma density. $\text{ARE}(U/z)$ can be small when $\mu/\sigma$ is small due to a "great concentration" of mass near the origin.

5. Composite test. The results of Section 4 can be used to construct a test that has a lower bound but not an upper bound on its relative efficiency.

Let $z$ and $\phi$ be as defined in (2.2) and (2.3) and let

\[
W_z = 1 \quad \phi < k,
\]

\[
= 0 \quad \phi \geq k;
\]

\[
W_u = 1 \quad \phi \geq k,
\]

\[
= 0 \quad \phi < k.
\]
The number \( k \) is a design parameter for the test. A proper choice for \( k \) will be made clearer in the following discussion. The composite test rejects \( H \) if

\[
C = W_z z + W_U U \geq W_z z + W_U U ,
\]

where \( L_z = \Phi^{-1}(1 - \alpha)/(r(1 - r)N)^4 \), \( r = n_i/N \), \( N = n_i + n_s \) and \( \alpha \) is the desired size of the test. \( L_U \) is determined from the null distribution of \( U \) such that \( P[U \geq L_U] = \alpha \) or using the asymptotic normality of \( U \),

\[
L_U = \Phi^{-1}(1 - \alpha)/12r(1 - r)N + \frac{3}{2} .
\]

Since as \( N \to \infty \), \( \hat{\theta} \to_{a.s.} \mu/\sigma \), it follows that \( W_z \) and \( W_U \) approach 1 or 0 a.s. depending on whether \( \mu/\sigma \) is less than or greater than the chosen \( k \). Then it follows ([8], page 236) that for any \( k \geq 0 \), the test of (5.3) is asymptotically size \( \alpha \) and

\[
(5.4) \quad \text{ARE}(C/z) = \text{ARE}(U/z) \quad \mu/\sigma \geq k ,
\]

\[
= 1 \quad \mu/\sigma < k .
\]

From Section 4,

\[
\text{ARE}(C/z) \geq \min_{\mu/\sigma = k} \{ \text{ARE}(U/z) \}
\]

and

\[
\text{ARE}(C/z) \geq (27/4)k^4/(1 + k^2)^4 \quad \text{if} \quad 0 \leq k \leq 2^\dagger ,
\]

\[
\geq 0.864(1 - \frac{458}{k^2}) \quad \text{if} \quad 2^\dagger \leq k \leq \infty .
\]

The parameter \( k \) for the test can be chosen to give any desired lower bound between 0 and .864.

It can be shown that for any \( k \geq 0 \), \( \text{ARE}(C/z) \) does not have an upper bound. Let \( g(x) \) be a density with mean \( \mu_y \) and variance \( \sigma^2 \) such that \( g(x) = 0 \), \( x < 0 \). Take \( f(x) = g(x - m) \) \( m > 0 \) and from (3.7)

\[
\text{ARE}(U/z) = 12 \frac{\sigma^2}{(\mu + m)^2} \left[ \int_{-\infty}^{\infty} x g(x - m) \, dx \right]^2
\]

\[
= 12\sigma^2 \left[ \int_{-\infty}^{m} x g(x) \, dx + \frac{m}{\mu + m} \int_{\mu}^{\infty} g(x) \, dx \right]^2 ,
\]

\[
(5.5) \quad \text{ARE}(U/z) \geq \left[ \frac{m}{\mu + m} \right]^2 \cdot 12\sigma^2 \left[ \int_{-\infty}^{m} g(x) \, dx \right]^2 .
\]

For any fixed \( g(x) \), \( \mu/\sigma = (m + \mu_y)/\sigma \) can be made arbitrarily large and \( m/(\mu_y + m) \) arbitrarily close to one, by choosing a sufficiently large value of \( m \). The second term in (5.5) is the ARE value for a translation alternative and null density \( g(x) \). It is well known that densities \( g(x) \) exist which make this term arbitrarily large. Therefore for any \( k \geq 0 \), a density exists such that \( \text{ARE}(C/z) \) is arbitrarily large.

To implement the composite test a choice for \( k \) must be made. Although large values of \( k \) give a lower bound close to .864 and still allow the possibility of a large ARE value, in most cases this will result in essentially using the z-test.
A reasonable choice is $k = 2^t$ this gives a lower bound of $\frac{2}{3}$ and should frequently result in the use of the $U$-test.

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REFERENCES