The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 651-655

RELATIONSHIPS AMONG SOME CONCEPTS OF BIVARIATE DEPENDENCE

BY J. D. ESARY[1] AND F. PROSCHAN[2]

Boeing Scientific Research Laboratories

We consider some unresolved relationships among various notions of bivariate dependence. In particular we show that $P[T > t \mid S > s] \uparrow$ in s (or alternately, $P[T \leqq t \mid S \leqq s] \downarrow$ in s) implies S, T are associated, i.e. $\mathrm{Cov}\,[f(S, T), g(S, T)] \geqq 0$ for all non-decreasing f and g.

1. Introduction. In this note we consider some unresolved relationships among various notions of bivariate dependence studied by Lehmann (1966) and Esary, Proschan, and Walkup (1967). Applications of these notions and their multivariate extensions have been considered in the papers mentioned and in Jogdeo (1968), Harris (1970), and Esary and Proschan (1970).

Lehmann defines two random variables S, T to be *positively quadrant dependent* (we write PQD $\{S, T\}$) if $P[S \leqq s, T \leqq t] \geqq P[S \leqq s] \cdot P[T \leqq t]$ for all s, t; and T to be *positively regression dependent* on S (written PRD $\{T \mid S\}$) if $P[T \leqq t \mid S = s]$ is non-increasing in s for all t (with reference for the latter definition to Tukey (1958)). Esary, Proschan, and Walkup define S, T to be *associated* (written $A\{S, T\}$) if $\mathrm{Cov}\,[f(S, T), g(S, T)] \geqq 0$ for all pairs of functions f, g which are non-decreasing in each argument, and such that $Ef(S, T)$, $Eg(S, T)$, $Ef(S, T)g(S, T)$ exist. Lehmann also mentions the type of dependence characterized by

$$(1.1) \qquad P[T \leqq t \mid S \leqq s] \quad \text{is non-increasing in} \quad s \quad \text{for all} \quad t\,.$$

If condition (1.1) holds, we say that T is *left tail decreasing* in S (written LTD $\{T \mid S\}$). A condition similar to (1.1) is

$$(1.2) \qquad P[T > t \mid S > s] \quad \text{is non-decreasing in} \quad s \quad \text{for all} \quad t\,.$$

If condition (1.2) holds, we say that T is *right tail increasing* in S (written RTI $\{T \mid S\}$).

Lehmann shows that the implications

$$\mathrm{PRD}\,\{T \mid S\} \Rightarrow \mathrm{LTD}\,\{T \mid S\} \Rightarrow \mathrm{PQD}\,\{S, T\}$$

hold. The implications are strict, i.e., no two of the conditions are equivalent. Esary, Proschan, and Walkup show that the strict implications

$$\mathrm{PRD}\,\{T \mid S\} \Rightarrow A\{S, T\} \Rightarrow \mathrm{PQD}\,\{S, T\}$$

hold.

Received August 11, 1970.

[1] Now at the Naval Postgraduate School, Monterey, California.

[2] Now at The Florida State University.

We study the relationship of LTD $\{T|S\}$ and RTI $\{T|S\}$ to $A\{S, T\}$, and extend the structure of strict implications to

$$\mathrm{PRD}\{T|S\} \begin{smallmatrix} \nearrow \mathrm{LTD}\{T|S\} \searrow \\ \searrow \mathrm{RTI}\{T|S\} \nearrow \end{smallmatrix} A\{S, T\} \Rightarrow \mathrm{PQD}\{S, T\}\,.$$

2. LTD, RTI, and PRD. Condition (1.1) for LTD $\{T|S\}$ can be restated as $P[T > t \mid S \leqq s]$ is non-decreasing in s for all t. Then by elementary manipulation, condition (1.1) is equivalent to

$$(2.1)\qquad P[T > t \mid S \leqq s_1] \leqq P[T > t \mid s_1 < S \leqq s_2] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2\,.$$

Condition (1.2) for RTI $\{T|S\}$ is equivalent to

$$(2.2)\qquad P[T > t \mid s_1 < S \leqq s_2] \leqq P[T > t \mid s_2 < S] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2\,.$$

Combining, we have for [LTD $\{T|S\}$ and RTI $\{T|S\}$],

$$(2.3)\qquad P[T > t \mid S \leqq s_1] \leqq P[T > t \mid s_1 < S \leqq s_2] \leqq P[T > t \mid S > s_2] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2\,.$$

This expression gives a convenient way of viewing the joint condition [LTD $\{T|S\}$ and RTI $\{T|S\}$].

Using conditions (2.1) and (2.2) it is immediate that PRD $\{T|S\} \Rightarrow$ [LTD $\{T|S\}$ and RTI $\{T|S\}$], since for any interval I

$$P[T > t \mid S \in I] = \textstyle\int_{s \in I} P[T > t \mid S = s]\, dP[S \leqq s]/P[S \in I]\,.$$

(cf. Lehmann, 1966.)

It is known (e.g., see Esary, Proschan, and Walkup (1967)) that all of the conditions for bivariate dependence considered in this note are equivalent for 2×2 distributions (we say that S, T have an $n \times m$ distribution if S has n values, T has m values). To show that the implication PRD $\{T|S\} \Rightarrow$ LTD $\{T|S\}$ is strict Lehmann uses a 3×3 example. To show that the implication PRD $\{T|S\} \Rightarrow$ [LTD $\{T|S\}$ and RTI $\{T|S\}$] is strict we must use a 4×2 example, since PRD $\{T|S\} \Leftrightarrow$ [LTD $\{T|S\}$ and RTI $\{T|S\}$] for any $3 \times m$ distribution by condition (2.3). We let S take values $s_1 < s_2 < s_3 < s_4$, each with probability $\frac{1}{4}$. We let T take values $t_1 < t_2$, with $P[T = t_2 \mid S = s_i] = p_i$. If $p_1 = .4$, $p_2 = .6$, $p_3 = .5$, $p_4 = .7$, we have [LTD $\{T|S\}$ and RTI $\{T|S\}$] but not PRD $\{T|S\}$.

If in the example above $p_1 = .4$, $p_2 = .6$, $p_3 = .5$, $p_4 = .5$, we have LTD $\{T|S\}$ but not RTI $\{T|S\}$. If $p_1 = .5$, $p_2 = .5$, $p_3 = .4$, $p_4 = .6$, we have RTI $\{T|S\}$ but not LTD $\{T|S\}$.

3. LTD, RTI, and A. By elementary manipulation condition (2.1) for LTD $\{T|S\}$ is equivalent to

$$(3.1)\qquad P[T > t, S \leqq s_1] \cdot P[T \leqq t, s_1 < S \leqq s_2] \leqq P[T \leqq t, S \leqq s_1] \cdot P[T > t, s_1 < S \leqq s_2] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2\,.$$

Condition (2.2) for RTI $\{T \mid S\}$ is equivalent to

$$(3.2) \qquad P[T>t, s_1<S\leqq s_2]\cdot P[T\leqq t, S>s_2]\leqq P[T\leqq t, s_1<S\leqq s_2]\cdot P[T>t, S>s_2]$$
for all t and $s_1 < s_2$.

In Esary, Proschan, and Walkup (1967) it is shown that association ($A\{S, T\}$) is equivalent to

$$(3.3) \qquad P[\gamma(S, T) = 1, \delta(S, T) = 0]\cdot P[\gamma(S, T) = 0, \delta(S, T) = 1]$$
$$\leqq P[\gamma(S, T) = 0, \delta(S, T) = 0]\cdot P[\gamma(S, T) = 1, \delta(S, T) = 1]$$
for all pairs γ, δ of binary functions which are non-decreasing in each argument.

(A function is *binary* if it takes only the values 0 and 1.)

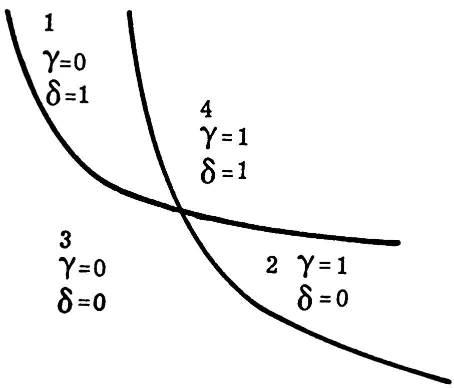

FIG. 1. $A\{S, T\} \Leftrightarrow P[1]P[2] \leqq P[3]P[4]$.

We consider the 3×3 distribution

TABLE 1

$S \backslash T$	t_1	t_2	t_3
s_1	$\frac{1}{4}$	0	p_{13}
s_2	0	$\frac{1}{4}$	0
s_3	p_{31}	0	$\frac{1}{4}$

where $s_1 < s_2 < s_3$ and $t_1 < t_2 < t_3$. If $p_{13} = p_{31} = \frac{1}{8}$, we have $A\{S, T\}$ but neither LTD $\{T \mid S\}$ nor RTI $\{T \mid S\}$ (cf. Esary, Proschan, and Walkup, 1967). If $p_{13} = 0$, $p_{31} = \frac{1}{4}$, we have LTD $\{T \mid S\}$ but not RTI $\{T \mid S\}$. If $p_{13} = \frac{1}{4}$, $p_{31} = 0$, we have RTI $\{T \mid S\}$ but not LTD $\{T \mid S\}$ (cf. Lehmann, 1966).

We now proceed to prove the implication RTI $\{T \mid S\} \Rightarrow A(S, T)$. Once this is accomplished, the implication LTD $\{T \mid S\} \Rightarrow A\{S, T\}$ follows, since LTD $\{T \mid S\} \Leftrightarrow$ RTI $\{-T \mid -S\} \Rightarrow A\{-S, -T\} \Leftrightarrow A\{S, T\}$.

Given random variables S, T we choose fixed $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_m$. We define discrete random variables S^*, T^* by

$$\begin{aligned} S^* &= i \quad \text{if} \quad s_i < S \leqq s_{i+1}\,, \\ T^* &= i \quad \text{if} \quad t_i < S \leqq t_{i+1}\,, \qquad i = 0, \cdots, n, \end{aligned}$$

where $s_0 = t_0 = -\infty$, $s_n = t_n = +\infty$. It is shown in Esary, Proschan, and Walkup (1967) that $A\{S, T\}$ is equivalent to $A\{S^*, T^*\}$ for all choices of n, m and $s_1, \cdots, s_n, t_1, \cdots, t_m$. It is clear that RTI $\{T \mid S\} \Rightarrow$ RTI $\{T^* \mid S^*\}$. Thus we only need to show that RTI $\{T^* \mid S^*\} \Rightarrow A\{S^*, T^*\}$.

Justified by the preceding remarks, *we assume from now on that S is discrete with the values* $0, 1, \cdots, n$ *and that T is discrete with the values* $0, 1, \cdots, m$. Also from now on we make the convention that γ, δ are binary, non-decreasing functions of $s = 0, 1, \cdots, n$ and $t = 0, 1, \cdots, m$.

Since if either γ or δ is identically 0 or identically 1, then $\mathrm{Cov}\,[\gamma(S, T), \delta(S, T)] = 0$, we make the further convention that neither γ or δ is identically 0 or 1.

We shall need the following lemmas; we omit the easy proofs.

We say that (s_0, t_0) is *a boundary point* of $\{\gamma = 0\} \equiv \{(s, t) \mid \gamma(s, t) = 0\}$ if $\gamma(s_0, t_0) = 0$ and $\gamma(s_0 + 1, t_0 + 1) = 1$.

LEMMA 1. *Let (s_2, t_2) be a boundary point of both $\{\gamma = 0\}$ and $\{\delta = 0\}$. Then, by weakening* (3.2), RTI $\{T \mid S\}$ *implies*

$$\begin{aligned} (3.4) \quad & P[\gamma(S, T) \neq \delta(S, T), s_1 < S \leqq s_2] \cdot P[\gamma(S, T) \neq \delta(S, T), s_2 < S] \\ & \leqq P[\gamma(S,T)=0, \delta(S,T)=0, s_1 < S \leqq s_2] \cdot P[\gamma(S,T)=1, \delta(S,T)=1, s_2 < S] \end{aligned}$$

for all $s_1 < s_2$.

For fixed s either (a) $\gamma(s, t) \geqq \delta(s, t)$ for all t, or (b) $\gamma(s, t) \leqq \delta(s, t)$ for all t. It is clear that we can find an *alternating partition* of $[0, n]$, i.e., a partition of $[0, n]$ into intervals $I_1, I_2, \cdots, I_k$ such that either (a) or (b) holds for each $s \in I_j$, $j = 1, \cdots, k$, and such that if (a) holds on I_j (or (b) holds on I_j), then (b) holds on I_{j+1} ((a) holds on I_{j+1}), $j = 1, \cdots, k - 1$.

LEMMA 2. *Let $I_1, I_2, \cdots, I_k$ be an alternating partition of* $[0, n]$. *Let* $s_j = \max\{s \mid s \in I_j\}$, $t_j = \max\{t \mid \gamma(s_j, t) = \delta(s_j, t) = 0\}$. *Then the points* (s_j, t_j), $j = 1, \cdots, k - 1$, *are boundary points of both $\{\gamma = 0\}$ and $\{\delta = 0\}$.*

THEOREM. RTI $\{T \mid S\}$ *implies* $A\{S, T\}$.

PROOF. With reference to condition (3.3) for $A\{S, T\}$, let $p_{ij} = P[\gamma(S, T) = i, \delta(S, T) = j]$, $i, j = 0, 1$. Let $I_1, I_2, \cdots, I_k$ be an alternating partition of $[0, n]$. Let $a_j = P[\gamma(S, T) \neq \delta(S, T), S \in I_j]$, $b_j = P[\gamma(S, T) = 0, \delta(S, T) = 0, S \in I_j]$, and $c_j = P[\gamma(S, T) = 1, \delta(S, T) = 1, S \in I_j]$, $j = 1, \cdots, k$. In view of Lemma 2 we can apply Lemma 1 (with the interval $(s_1, s_2]$ of Lemma 1 taken to be

I_j) to obtain

$$a_j(a_{j+1} + \cdots + a_k) \leqq b_j(c_{j+1} + \cdots + c_k)\,, \qquad j = 1, \cdots, k-1\,.$$

Now $p_{10} = \sum_{j=1}^k e_j a_j$, $p_{01} = \sum_{j=1}^k (1 - e_j)a_j$, where $e_j = 1$ if $\gamma \geqq \delta$ on I_j, $e_j = 0$ if $\gamma \leqq \delta$ on I_j. Also $p_{00} = \sum_{j=1}^k b_j$ and $p_{11} = \sum_{j=1}^k c_j$. Then

$$p_{10}\,p_{01} \leqq \sum\sum_{i<j} a_i a_j \leqq \sum\,\sum_{i<j} b_i c_j \leqq p_{00}\,p_{11}\,.$$

Thus condition (3.3) for $A\{T \mid S\}$) is verified. □

REFERENCES

ESARY, J. D., PROSCHAN, F., and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–74.

ESARY, J. D. and PROSCHAN, F. (1970). A reliability bound for systems of maintained, interdependent components. *J. Amer. Statist. Assoc.* **65** 329-338.

HARRIS, ROBERT (1970). A multivariate definition for increasing hazard rate distribution functions. *Ann. Math. Statist.* **41** 713–717.

JOGDEO, KUMAR (1968). Characterizations of independence in certain families of bivariate and multivariate distributions. *Ann. Math. Statist.* **39** 433–441.

LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

TUKEY, JOHN W. (1958). A problem of Berkson, and minimum variance orderly estimators. *Ann. Math. Statist.* **29** 588-592.