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# AN INVARIANCE PRINCIPLE FOR MARTINGALES[1]

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Many discrete martingales with increments in $L_2$ can be normalized so that the resulting trajectory is distributed approximately like Brownian motion. This paper will find all such martingales, subject to a natural side condition. Two techniques of normalization are possible: The usual one involving the partial sums of conditional variances of the increments given the past, and the analogous method using the partial sums of squares of the increments. This result is applied to obtain a central limit theorem and an arc sin law for dependent random variables.

**0. Introduction.** Many discrete martingales with increments in $L_2$ can be normalised so that the resulting trajectory is distributed approximately like Brownian motion. This paper will find all such martingales, subject to a natural side condition. Two techniques of normalization are possible: The usual one involving the partial sums of conditional variances of the increments given the past, and the analogous method using the partial sums of squares of the increments. The main result is stated by (2). It is applied to obtain a central limit Theorem (16) and an arc sin law (17) for dependent random variables.

Formally, let $X_1, X_2, \cdots$ be a sequence of random variables on $(\Omega, \mathscr{A}, P)$. Let $\mathscr{A}_0, \mathscr{A}_1, \cdots$ be an increasing sequence of sub $\sigma$-fields of $\mathscr{A}$ such that $X_i$ is $\mathscr{A}_i$-measurable. Set $s_m^2 = \sum_{i=1}^m X_i^2$ and $v_m = \sum_{i=1}^m E(X_i^2 \mid \mathscr{A}_{i-1})$. Assume only

$$(1) \qquad E(X_{m+1} \mid \mathscr{A}_m) = 0\,, \qquad EX_m^2 < \infty\,, \qquad \text{and} \qquad v_m \to \infty \text{ a.s.}$$

Define $T_n = \inf\{m : v_m \geqq n\}$. Form the continuous (random) function $S$ on $[0, \infty)$ by the requirements that $S(0) = 0$, $S(v_m) = X_1 + \cdots + X_m$, and $S$ is linear on $[v_m, v_{m+1}]$. In the same way form $\hat{S}$, except use $s_m^2$ in place of $v_m$. Let $S^n$ and $\hat{S}^n$ be the two continuous functions on $[0, 1]$ defined at $t$ by $S^n(t) = S(nt)/n^{\frac{1}{2}}$ and $\hat{S}^n(t) = \hat{S}(nt)/n^{\frac{1}{2}}$.

Let $I$ be an interval of the real line. For Theorem 1 just think of $I = [0, 1]$, the case $I = [0, \infty)$ will be needed in Section 2. Let $C(I)$ be the space of real-valued continuous functions on $I$ with the sup metric $d$ and the Borel $\sigma$-field $\Sigma(I)$ generated by the $d$-open sets. For $t \in I$, let $B(t, \cdot)$, or simply $B(t)$, be the function on $C(I)$ whose value at $f$ is $f(t)$. Let $\pi_I$ be the probability on $(C(I), \Sigma(I))$ which makes $\{B(t) : t \in I\}$ standard Brownian motion starting from

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0 on $I$. Abbreviate $\pi_{[0,1]}$ by $\pi_1$, and $\pi_{[0,\infty)}$ by $\pi$. Denote by $\Phi$ the set of real-valued functions on $C[0, 1]$ which are bounded and continuous $\pi_1$-a.s.

Take $1_A$ to be the indicator of the set $A$, and $\rightarrow_{L_1}$ to mean convergence in $L_1$-norm with respect to the probability distribution of the random variables involved.

The main result is

(2) THEOREM 1. *Suppose* (1) *holds. Then* (a), (b), *and* (c) *are equivalent. If they are true, so is* (d).

(a) $$(1/n) \textstyle\sum_{i=1}^{T_n} X_i^2 1_{[X_i^2 > n\varepsilon]} \rightarrow_{L_1} 0 \quad as \quad n \rightarrow \infty, \qquad for\ all \quad \varepsilon > 0$$

(b) $$\sup_{1 \leq i \leq T_n} |s_i^2 - v_i|/n \rightarrow_{L_1} 0 \quad and \quad v_{T_n}/n \rightarrow_{L_1} 1 \qquad as \quad n \rightarrow \infty$$

(c) $$\int_\Omega \varphi(S^n)\, dP \rightarrow \int_{C[0,1]} \varphi\, d\pi_1 \quad and \quad v_{T_n}/n \rightarrow_{L_1} 1 \quad as \quad n \rightarrow \infty, \qquad for\ all \quad \varphi \in \Phi$$

(d) $$\sup_{0 \leq t \leq 1} |S^n(t) - \hat{S}^n(t)| \rightarrow_P 0 \quad and \quad \int_\Omega \varphi(\hat{S}^n)\, dP \rightarrow \int_{C[0,1]} \varphi\, d\pi_1 \quad as \quad n \rightarrow \infty, \qquad for\ all \quad \varphi \in \Phi.$$

When $X_1, X_2, \cdots$ are independent, $T_n$ and $v_m$ are nonrandom, and (a) reduces to the classical Lindeberg condition.

The asymptotic behavior of $S_n$ has long been studied under various conditions on the dependence among the $X_i$ and their growth rate. The main result is Donsker's invariance principle [1] which treats the case when $X_1, X_2, \cdots$ are i.i.d. Lévy [5] and Dvoretsky [3] have proved central limit theorems for martingales. These are generalized and unified by Theorem 1. Strassen [6] has showed that, under a growth condition which includes the case when $X_1, X_2, \cdots$ are uniformly bounded, the weak convergence asserted in (c) can be replaced by almost sure convergence.

$\hat{S}$ has not been investigated. The implication of (d) that the process $\{\hat{S}(nt)/n^{\frac{1}{2}} : 0 \leq t \leq 1\}$ converges in distribution to Brownian motion, can be thought of as the discrete time analog to a property of many continuous time martingales. Roughly speaking, if the time scale for these martingales is changed by mapping each $t \in [0, \infty)$ into the squared variation of the process up to time $t$, then a Brownian motion is obtained [2]. $\hat{S}$ is formed similarly, with $\sum_{i=1}^m X_i^2$ playing the role of squared variation. An interesting feature of $\hat{S}$ is that, unlike $S$, its construction does not involve $P$ or any parameters of $P$.

The present work presents a theory of the weak convergence of suitably normalized discrete stochastic processes to Brownian motion based entirely upon results about fair coin tossing. It does not involve characteristic functions, or the approach of embedding a discrete martingale in a Brownian motion.

**1. An invariance principle.** The proof of Theorem 1 will be based on the fact that for large $n$ $S^n$ and $\hat{S}^n$ each behave approximately like a fair coin random walk with small steps. In pinning this down there are two things to check. First, it must be shown that for any $\alpha > 0$ the sequence of changes in size of at least $\alpha$ completed by the trajectory $0, X_1/n^{\frac{1}{2}}, (X_1 + X_2)/n^{\frac{1}{2}}, \cdots$ are approximately i.i.d. as $\pm\alpha$ with probability $\frac{1}{2}$ each. Then it is necessary to prove that, for small $\alpha > 0$, the time transformations used to obtain $S^n$ and $\hat{S}^n$ make these changes occur at a nearly uniform rate, for large $n$. This is stated rigorously by (4). (5)–(7) will develop tools for determining when these two properties are satisfied. Their proofs are given in Section 3. The first property will be called asymptotic fairness. A formal definition is given by (3). The proof of Theorem 1 follows (11).

To state (3), let $\Gamma$ be the set of functions $y: \{0, 1, \cdots\} \to (-\infty, \infty)$ for which $y(0) = 0$. For $y \in \Gamma$ and $k \geqq 0$ identify $y_k$ with $y(k)$, and define $\sigma(0, m, y) = 0$, let $\sigma(j, m, y)$ be the first time $k$ after $\sigma(j-1, m, y)$ for which $|y_k - y_{\sigma(j-1,m,y)}| \geqq 2^{-m}$, and $\sigma(j, m, y) = \sigma(j-1, m, y)$ when no such $k$ exists, for $j \geqq 1$. For unbounded $y$, think of $\sigma(j, m, y)$ as the first time $y$ completes $j$ changes in size of at least $2^{-m}$. Abbreviate $y_{\sigma(j,m,y)}$ by $y_{\sigma(j,m)}$. In addition, for pairs of nonnegative integers $(k, m)$, let $c(k, m, \cdot): \Gamma \to \{0, 1, 2, \cdots\}$ be defined at $y$ to be the smallest nonnegative integer $j$ for which either $\sigma(j, m, y) \leqq k < \sigma(j+1, m, y)$ or $\sigma(j, m, y) = \sigma(j+1, m, y) \leqq k$. Think of $c(k, m, y)$ as the number of changes in size of at least $2^{-m}$ completed by $y$ up to time $k$.

For $m = 0, 1, \cdots$ and $N = 1, 2, \cdots$, let $G(N, m)$ be the set of $N$-tuplets whose entries are $-2^{-m}$ or $2^{-m}$. For each $\eta > 0$ and $a = (a_1, \cdots, a_N) \in G(N, m)$ let $a_i(\eta)$ be the interval $[-(2^{-m} + \eta), -2^{-m}]$ or $[2^{-m}, 2^{-m} + \eta]$ according to whether $a_i$ is $-2^{-m}$ or $2^{-m}$. Let $a(\eta) = a_1(\eta) \times \cdots \times a_N(\eta)$, an $N$-dimensional cube.

For $n = 1, 2, \cdots$ let $\{\xi_k^n : k = 0, 1, \cdots\}$ be a sequence of processes on a probability space $(\mathscr{X}, \mathfrak{F}, \mu)$. For all $x \in \mathscr{X}$ assume $\xi_0^n(x) = 0$, and hence $\xi^n(x) \in \Gamma$. So $\xi^n_{\sigma(j,m,\xi^n)} = \xi^n_{\sigma(j,m)}$ is a random variable. Say

(3) $\xi^n$ is asymptotically fair as $n \to \infty$ if for all $\eta > 0$, integers $m \geqq 0$ and $N \geqq 1$, and $a \in G(N, m)$

$$\mu[(\xi^n_{\sigma(1,m)}, \xi^n_{\sigma(2,m)} - \xi^n_{\sigma((1,m)}, \cdots, \xi^n_{\sigma(N,m)} - \xi^n_{\sigma(N-1,m)}) \in a(\eta)] \to (\tfrac{1}{2})^N \quad \text{as} \quad n \to \infty .$$

For (4), let $\{S_m : m = 0, 1, \cdots\}$ be a process on $(\Omega, \mathscr{A}, P)$ with $S_0 = 0$. Set $X_0 = 0$ and $X_i = S_i - S_{i-1}$. The lemma is completely general, it is not necessary to assume $E(X_{i+1} | X_0, \cdots, X_i) = 0$ or that $EX_i$ exists. Let $\{T_t : t \geqq 0\}$ be a time change for $\{S_m : m = 0, 1, \cdots\}$. That is, for all $\omega \in \Omega$ in a set of probability 1, $T_\cdot(\omega): [0, \infty) \to \{0, 1, \cdots\}$ is a non-decreasing, unbounded, right continuous step function with $T_0(\omega) = 0$. Impose the restriction that if $j - 1 =$

$T_{t-}$, $|X_j| > 0$, and $T_.$ has a jump at $t$ (i.e. $T_{t-} < T_t$), then $T_t = j$. Define the continuous (random) function $S^1$ on $[0, \infty)$ by the requirements $S^1(0) = 0$, $S^1(t) = S_{T_t}$ if $T_.$ has a jump at $t$, and $S^1$ is linearly interpolated between jumps of $T_.$. For each $t \in [0, 1]$ let $S^n(t) = S^1(nt)/n^{\frac{1}{2}}$. Set $Y_k{}^n = S_k/n^{\frac{1}{2}}$, and

$$U_k^{m,n} = \sum_{i=1}^{c(k,m,Y^n)} (Y^n_{\sigma(i,m,Y^n)} - Y^n_{\sigma(i-1,m,Y^n)})^2 ,$$

where $Y^n$ denotes the trajectory of $\{Y_k{}^n \mid k = 0, 1, 2, \cdots\}$. $\pi_1$ and $\Phi$ are defined in the introduction.

(4) LEMMA. (4a) *and* (4b) *are equivalent to* (4c).

(4a) $$Y^n \quad \textit{is asymptotically fair as} \quad n \to \infty ;$$

(4b) $$\sup_{0 \leq t \leq 1} |U_{T_{nt}}^{m,n} - t| \to_P 0 \quad \textit{as} \quad m, n \to \infty ,$$
*with* $n \geqq n(m)$, *for some sequence* $\{n(m) : m = 0, 1, \cdots\}$ ;

(4c) $$\int \varphi(S^n)\, dP \to \int \varphi\, d\pi_1 \quad \textit{as} \quad n \to \infty , \qquad \textit{for all} \quad \varphi \in \Phi .$$

For (5)–(7) assume a probability space $(\Omega, \mathscr{A}, P)$ is given and that all processes mentioned are 0 at 0, for all $\omega \in \Omega$.

(5) gives sufficient conditions for a sequence of processes to be asymptotically fair.

(5) LEMMA. *If* $\{Y_k{}^n : k = 0, 1, \cdots\}$ *is a martingale with* $P[\sup_k |Y_k{}^n| = \infty] = 1$ *for each* $n$, *and* $\sup_k |Y_k{}^n - Y_{k-1}^n| \to_{L_1} 0$ *as* $n \to \infty$, *then* $Y^n$ *is asymptotically fair as* $n \to \infty$.

The next lemma states that the property of asymptotic fairness is in a certain sense continuous. Assume $\{Y_k{}^n : k = 0, 1, \cdots\}$ and $\{\hat{Y}_k{}^n : k = 0, 1, \cdots\}$ are processes on $(\Omega, \mathscr{A}, P)$.

(6) LEMMA. *If* $Y^n$ *is asymptotically fair and* $\sup_k |Y_k{}^n - \hat{Y}_k{}^n| \to_P 0$ *as* $n \to \infty$, *then* $\hat{Y}^n$ *is asymptotically fair as* $n \to \infty$.

(7) will be used to show that certain time changes are asymptotically equivalent. Assume $\{Y_k{}^n : k = 0, 1, \cdots\}$, $\{A_k{}^n : k = 0, 1, \cdots\}$, and $\{\hat{A}_k{}^n : k = 0, 1, \cdots\}$ are processes on $(\Omega, \mathscr{A}, P)$ for each $n$. In addition, suppose that for each $n$ $0 \leqq A_k{}^n \uparrow$ and $0 \leqq \hat{A}_k{}^n \uparrow$ as $k \to \infty$, and $T_n$ is a nonnegative integer valued random variable such that $Y_k{}^n = A_k{}^n - \hat{A}_k{}^n$ for $k = 0, 1, \cdots, T_n$.

(7) LEMMA. *If* $Y^n$ *is asymptotically fair*, $E(A^n_{T_n}) \to 1$, *and* $E(\hat{A}^n_{T_n}) \to 1$, *then* $\sup_{0 \leq k \leq T_n} |Y_k{}^n| = \sup_{0 \leq k \leq T_n} |A_k{}^n - \hat{A}_k{}^n| \to_P 0$ *as* $n \to \infty$.

Recall the setup given in the introduction. In addition, without real loss, assume $(\Omega, \mathscr{A}, P)$ supports $Z_1, Z_2, \cdots$ which are i.i.d. with $P(Z_1 = 1) = P(Z_1 = -1) = \frac{1}{2}$, and such that $(Z_1, Z_2, \cdots)$ is independent of $\bigcup_k \mathscr{A}_k$. Otherwise one could construct a cross product space with analogous properties.

In order to match up the growth condition (a) in Theorem 1 with those of

(5) and (6) the following definition will be useful. Fix $n$, and recall $T_n$ is a stopping time w.r.t. $\{\mathscr{A}_k : k = 0, 1, \cdots\}$. Let $\{\zeta_k : k = 0, 1, \cdots\}$ be a process for which $\zeta_k$ is $\mathscr{A}_k$-measurable, and define $\zeta_k{}^* = \sum_{i=1}^k (\zeta_i - \zeta_{i-1}) 1_{[i \le T_n]} + (1/n) \sum_{i=1}^k Z_i 1_{[i > T_n]}$. Call $\{\zeta_k{}^* : k = 0, 1, \cdots\}$ the *n-extension* of $\{\zeta_k : k = 0, 1, \cdots\}$. Let $\mathscr{A}_k{}^*$ be the smallest $\sigma$-field containing $\mathscr{A}_k$ for which $\zeta_0{}^*, \cdots, \zeta_k{}^*$ are measurable. The following properties are easy to check:

(8) If $\{\zeta_k : k = 0, 1, \cdots\}$ is a martingale with respect to $\{\mathscr{A}_k : k = 0, 1, \cdots\}$, then $\{\zeta_k{}^* : k = 0, 1, \cdots\}$ is a martingale with respect to $\{\mathscr{A}_k{}^* : k = 0, 1, \cdots\}$,

(9) $$\zeta_k = \zeta_k{}^* , \qquad \text{for} \quad k = 0, 1, \cdots, T_n ,$$

(10) $$\sup_k |\zeta_k{}^* - \zeta_{k-1}^*| = \sup_{1 \le k \le T_n} |\zeta_k - \zeta_{k-1}| + 1/n , \qquad \text{and}$$

(11) $$\sup_k |\zeta_k{}^*| = \infty \text{ a.s.}$$

It is now possible to give the

PROOF OF THEOREM 1. (a) *implies* (b): For each $n$ and $k$ let $W_k{}^n = (s_k{}^2 - v_k)/n$, and observe $\{W_k{}^n : k = 0, 1, \cdots\}$ is a martingale with respect to $\{\mathscr{A}_k : k = 0, 1, \cdots\}$. For each $n$, denote the $n$-extension of $\{W_k{}^n : k = 0, 1, \cdots\}$ by $\{W_k{}^{n*} : k = 0, 1, \cdots\}$. Use (a) and (8)–(11) to check the hypothesis of (5), concluding that $W^{n*}$ is asymptotically fair as $n \to \infty$. This fact, together with (a), implies the hypothesis of (7), upon identifying $W_k{}^{n*}$, $s_k{}^2/n$, $v_k/n$ with $Y_k{}^n$, $A_k{}^n$, $\hat{A}_k{}^n$ of that lemma, and observing for any $\varepsilon > 0$

$$\begin{aligned} 1 &\le Es_{T_n}^2/n = Ev_{T_n}/n \le 1 + (1/n)E \max_{1 \le i \le T_n} E(X_i{}^2 \mid \mathscr{A}_{i-1}) \\ &\le 1 + \varepsilon + (1/n)E \textstyle\sum_{i=1}^{T_n} E(X_i{}^2 1_{[X_i{}^2 > n\varepsilon]} \mid \mathscr{A}_{i-1}) \\ &= 1 + \varepsilon + (1/n)E \textstyle\sum_{i=1}^{T_n} X_i{}^2 1_{[X_i{}^2 > n\varepsilon]} \to 1 + \varepsilon . \end{aligned}$$

Hence, $\sup_{1 \le k \le T_n} |W_k{}^{n*}| = \sup_{1 \le k \le T_n} |s_k{}^2 - v_k|/n \to_P 0$ as $n \to \infty$. Using this, the convergence in $L_1$ follows easily, proving (b).

(b) *implies* (c): For each $k$, $m$, $n$ let

(12) $$\begin{aligned} Y_k{}^n &= (X_1 + \cdots + X_k)/n^{\frac{1}{2}} , \\ U_k{}^{m,n} &= \textstyle\sum_{j=1}^{c(k,m,Y^n)} (Y^n_{\sigma(j,m)} - Y^n_{\sigma(j-1,m)})^2 , \qquad \text{and} \\ \hat{U}_k{}^{m,n} &= \textstyle\sum_{j=1}^{\infty} (Y^n_{\sigma(j,m) \wedge k} - Y^n_{\sigma(j-1,m) \wedge k})^2 . \end{aligned}$$

In view of (4), it is enough to verify conditions (4a) and (4b). (4b) will be checked first.

For each $m$, $n$, and $k$ let $\hat{D}_k{}^{m,n} = \hat{U}_k{}^{m,n} - s_k{}^2/n$, and observe $\{\hat{D}_k{}^{m,n} : k = 0, 1, \cdots\}$ is a martingale with respect to $\{\mathscr{A}_k : k = 0, 1, \cdots\}$. For each pair $(m, n)$, denote by $\{\hat{D}_k{}^{m,n*} : k = 0, 1, \cdots\}$ the $n$-extension of $\{\hat{D}_k{}^{m,n} : k = 0, 1, \cdots\}$. (b) implies $\max_{1 \le i \le T_n} (X_i{}^2/n) \to_{L_1} 0$, and hence $\max_{1 \le k \le T_n} |\hat{D}_k{}^{m,n} - \hat{D}_{k-1}^{m,n}| \to_{L_1} 0$ as $m, n \to \infty$. Therefore, $\hat{D}^{m,n*}$ is asymptotically fair as $m, n \to \infty$ by (5). For each $m, n, k$ let $D_k^{m,n} = U_k^{m,n} - s_k^2/n$, and denote by $\{D_k^{m,n*} : k = 0, 1, \cdots\}$ the $n$-extension of $\{D_k^{m,n} : k = 0, 1, \cdots\}$. Since $\sup_k |\hat{D}_k^{m,n*} - D_k^{m,n*}| = \sup_{1 \le k \le T_n} |\hat{U}_k^{m,n} - U_k^{m,n}| \le 4^{-m} \to 0$ as $m, n \to \infty$, (6) applies to get that $D^{m,n*}$ is asymptotically fair as $m, n \to \infty$. Now use (b) to check the hypotheses of (7), upon identifying $D_k^{m,n*}$, $U_k^{m,n}$, $s_k^2/n$ with $Y_k^n$, $A_k^n$, $\hat{A}_k^n$ of that lemma, and conclude

$$\max_{0 \le t \le 1} |U_{T_{nt}}^{m,n} - s_{T_{nt}}^2/n| = \max_{1 \le k \le T_n} |U_k^{m,n} - s_k^2/n| = \max_{1 \le k \le T_n} |D_k^{m,n}|$$
$$= \max_{1 \le k \le T_n} |D_k^{m,n*}| \to_P 0 \qquad \text{as} \quad m, n \to \infty .$$

Finally, (b) implies $\max_{0 \le t \le 1} |s_{T_{nt}}^2/n - t| \to_P 0$, proving $\max_{0 \le t \le 1} |U_{T_{nt}}^{m,n} - t| \to_P 0$ as $m, n \to \infty$.

To check (4a), fix integers $l$ and $N$, real numbers $\eta, \varepsilon > 0$, and $a = (a_1, \cdots, a_N) \in G(N, l)$. Without real loss, assume $l = 0$. For each $n$, let $\{Y_k^{n*} : k = 0, 1, \cdots\}$ be the $n$-extension of $\{Y_k^n : k = 0, 1, \cdots\}$. Since $\{Y_k^n : k = 0, 1, \cdots\}$ is a martingale w.r.t. $\{\mathscr{S}_k : k = 0, 1, \cdots\}$, and $\max_{1 \le k \le T_n} |Y_k^n - Y_{k-1}^n| = \max_{1 \le k \le T_n} |X_k|/n^{\frac{1}{2}} \to_{L_1} 0$ by (b), (5) makes $Y^{n*}$ asymptotically fair as $n \to \infty$. The inequality $c(T_n, m, Y^{n*})4^{-m} \le U_{T_n}^{m,n} \le c(T_n, m, Y^{n*})(2^{-m} + \max_{1 \le i \le T_n} |X_i|/n^{\frac{1}{2}})^2$, and the facts $\max_{1 \le i \le T_n} X_i^2/n \to_P 0$ as $n \to \infty$ and $\max_{0 \le t \le 1} |U_{T_{nt}}^{m,n} - t| \to_P 0$ as $m, n \to \infty$, imply there exists a sequence $\{n(m) : m = 0, 1, \cdots\}$ such that $|c(T_n, m, Y^{n*})4^{-m} - U_{T_n}^{m,n}| \to_P 0$, and hence $P[c(T_n, m, Y^{n*}) \ge \frac{1}{2} \cdot 4^{-m}] \to 1$, as $m, n \to \infty$ with $n \ge n(m)$. Accordingly, $P[\sigma(N, m, Y^{n*}) < T_n] \to 1$ as $m, n \to \infty$ with $n \ge n(m)$. Choose $m_0$ and $n_1$ such that $P[H_n] \ge 1 - \varepsilon$ whenever $n \ge n_1$, where $H_n = [\sigma(N, m_0\, Y^{n*}) < T_n]$. Since $Y^{n*}$ is asymptotically fair, there exists an $n_2$ such that $|P[L_n] - (\frac{1}{2})^N| < \varepsilon$ whenever $n \ge n_2$, where

$$L_n = [(Y_{\sigma(1,m_0)}^{n*}, \cdots, Y_{\sigma(N,m_0)}^{n*} - Y_{\sigma(N-1,m_0)}^{n*}) \in b(\eta \cdot 2^{-m_0})] ,$$

assuming $b = (a_1/2^{m_0}, a_2/2^{m_0}, \cdots, a_N/2^{m_0})$. Put $n_0 = n_1 \vee n_2$. Use (9) to check that on $H_n$

$$Y_{\sigma(k,0,Y^n)}^n = 2^{m_0} Y^{n \cdot 4^{m_0}*} \sigma(k, m_0, Y^{n \cdot 4^{m_0}*}) \qquad \text{for} \quad k = 1, 2, \cdots, N .$$

The conclusion $|P[(Y_{\sigma(1,0)}^n, \cdots, Y_{\sigma(N,0)}^n - Y_{\sigma(N-1,0)}^n) \in a(\eta)] - (\frac{1}{2})^N| < 2\varepsilon$ whenever $n \ge n_0$ now follows. $\varepsilon$ is arbitrary, so $Y^n$ is asymptotically fair as $n \to \infty$, finishing off (4 a) and hence (c).

(c) *implies* (a): (a) will follow from

$$\text{(13)} \qquad \max_{1 \le i \le T_n} X_i^2/n \to_P 0 \quad \text{as} \quad n \to \infty , \qquad \text{and}$$

$$\text{(14)} \qquad s_{T_n}^2/n \to_{L_1} 1 \quad \text{as} \quad n \to \infty .$$

For, (13) implies $(1/n) \sum_{i=1}^{T_n} X_i^2 1_{[X_i^2 > n\varepsilon]} \to_P 0$ and (14) makes this sequence uniformly integrable.

Let $Y_k^n$, $U_k^{m,n}$, and $\hat{U}_k^{m,n}$ be as in (12). The proof of (4) will show (c) implies (13). As for (14), first use (4) to get $U_{T_n}^{m,n} \to_P 1$ as $m, n \to \infty$ with $n \ge n(m)$, for some sequence $\{n(m): m = 0, 1, \cdots\}$. Then observe $|U_{T_n}^{m,n} - \hat{U}_{T_n}^{m,n}| \leqq 4^{-m} \to 0$, and $E\hat{U}_{T_n}^{m,n} = Es_{T_n}^2/n = Ev_{T_n}/n \to 1$ as $m, n \to \infty$, to obtain

$$(15) \qquad U_{T_n}^{m,n} \to_{L_1} 1 \quad \text{as} \quad m, n \to \infty \qquad \text{with} \quad n \geqq n(m).$$

(13), (15), and the inequality $\max_{1 \leqq i \leqq T_n} X_i^2/n \leqq U_{T_n}^{m,n} + (2 \cdot 2^{-m})^2$ imply $\max_{1 \leqq i \leqq T_n} X_i^2/n \to_{L_1} 0$. This is sufficient, as the proof of (b) implies (c) showed, to make $D^{m,n^*}$ asymptotically fair as $m, n \to \infty$, where $\{D_k^{m,n^*}: k = 0, 1, \cdots\}$ is the $n$-extension of $\{D_k^{m,n}: k = 0, 1, \cdots\}$ and $D_k^{m,n} = U_k^{m,n} - s_k^2/n$. Remember $Es_{T_n}^2/n = Ev_{T_n}/n \to 1$. Now (7) can be applied, upon identifying $D_k^{m,n^*}$, $U_k^{m,n}$, $s_k^2/n$ with $Y_k^n$, $A_k^n$, $\hat{A}_k^n$ of that lemma, to get

$$\max_{1 \leqq k \leqq T_n} |D_k^{m,n^*}| = \max_{1 \leqq k \leqq T_n} |U_k^{m,n} - s_k^2/n| \to_P 0$$

as $m, n \to \infty$ with $n \geqq n(m)$. Hence $s_{T_n}^2/n \to_P 1$ as $n \to \infty$ by (15). Finally, $Es_{T_n}^2/n \to 1$ and $s_{T_n}^2/n \to_P 1$ yield (14), since $s_{T_n}^2 \geqq 0$. (d) follows easily from (a), (b) and (c). □

To get the flavor of Theorem 1 two corollaries are presented. In addition to the setup for Theorem 1 given in the introduction, let $S_m = X_1 + \cdots + X_m$, and $\hat{T}_n = \inf\{m: s_m^2 \geqq n\}$.

(16) Corollary (central limit theorem). *If* $(1/n)\sum_{i=1}^{T_n} X_i^2 1_{[X_i^2 > n\varepsilon]} \to_{L_1} 0$ *for all* $\varepsilon > 0$, *then the P-distributions of* $S_{\hat{T}_n}/n^{\frac{1}{2}}$ *and* $S_{T_n}/n^{\frac{1}{2}}$ *each converge to the normal distribution with mean* 0 *and unit variance.*

Proof. Immediate from Theorem 1.

For each $n$, let

$$L_n = (1/n)\sum_{i=1}^{T_n} E(X_i^2 \mid \mathscr{A}_{i-1}) 1_{[S_i > 0]},$$

and $\hat{L}_n = (1/n)\sum_{i=1}^{\hat{T}_n} X_i^2 1_{[S_i > 0]}$.

(17) Corollary (arc sin law). *If* $(1/n)\sum_{i=1}^{T_n} X_i^2 1_{[X_i^2 > n\varepsilon]} \to_{L_1} 0$ *for all* $\varepsilon > 0$, *then the P-distributions of* $L_n$ *and* $\hat{L}_n$ *each converge to the* arc sin *distribution.*

The proof will be given at the end of Section 3.

**2. Brownian motion.** This section isolates a few properties of Brownian motion which will be needed to prove (4)–(7).

Recall the definitions of $B(t)$, $\pi_1$, $\pi$ and $\Phi$ given in the introduction. In addition, identify $B_t$ and $B(t)$ with $B(t, \cdot)$, and consider $B$ as the identity map on $C[0, \infty)$.

Being consistent with the definition of $\sigma(j, m, \cdot)$ on $\Gamma$ given in Section 1, for $f \in C[0, \infty)$ set $\sigma(0, m, f) = 0$, and let $\sigma(j, m, f)$ be the first time $t$ after $\sigma(j-1, m, f)$ for which $|f(t) - f(\sigma(j-1, m, f))| = 2^{-m}$, and $\sigma(j, m, f) = \sigma(j-1, m, f)$ when no such $t$ exists, if $j \geqq 1$. Abbreviate $\sigma(j, m, B)$ by $\sigma(j, m)$, and $f(\sigma(j, m, f))$ by $f(\sigma(j, m))$. Let $[t]$ be the largest integer less than $t$.

(18) Lemma. *On a set of $\pi$-probability* 1, $\sigma([4^m t], m) \to t$ *for all* $t \geqq 0$, *and*

$B_{\sigma([4^m t], m)} \to B_t$ *uniformly for* $t$ *in finite intervals, as* $m \to \infty$. *A short proof is given in* [2].

For (19) let $B^m : C[0, \infty) \to C[0, 1]$ be defined at $f$ by the requirements $B^m(f)(0) = 0$, $B^m(f)(k \cdot 4^{-m}) = f(\sigma(k, m, f))$, and $B^m(f)$ is linear on $[(k-1) \cdot 4^{-m}, k \cdot 4^{-m}]$, for $k = 1, 2, \cdots, 4^m$. Let $B' : C[0, \infty) \to C[0, 1]$ be defined at $f$ by the requirements $B'(f)(t) = f(t)$ for $0 \leqq t \leqq 1$.

(19) LEMMA. *For every* $\varphi \in \Phi$, $\int \varphi(B^m)\, d\pi \to \int \varphi(B')\, d\pi = \int \varphi\, d\pi_1$ *as* $m \to \infty$.

PROOF. Use (18) and dominated convergence, and then the definition of $B'$. □

The remainder of this section is concerned with establishing that $\sigma(j, m)$ and $B_{\sigma(j,m)}$ are continuous $\pi$-a.s., and other related facts. These will be needed to derive properties of sequences of processes which converge weakly to Brownian motion. Throughout this section continuity of real valued functions whose domain is $C[0, 1]$ or $C[0, \infty]$ is with respect to the sup metric.

To state (20) a few definitions will be necessary. For $t \geqq 0$ let $c(t, m, \cdot) : C[0, \infty) \to \{0, 1, 2, \cdots\}$ be defined at $f$ to be the smallest nonnegative integer $j$ for which either $\sigma(j, m, f) \leqq t < (j+1, m, f)$ or $\sigma(j, m, f) = \sigma(j+1, m, f) \leqq t$. So for each $f$, $c(\cdot, m, f)4^{-m}$ is the right continuous inverse of $\sigma([4^m \cdot], m, f)$.

For $i = 1, 2, \cdots$, $m = 0, 1, \cdots$, and $k \geqq m + 2$ let $\sigma_k(i, m, \cdot)$ and $\sigma^k(i, m, \cdot)$ map $C[0, \infty)$ into $[0, \infty)$ be defined at $f$ by $\sigma_k(i, m, f) = \inf\{t : t > \sigma(i-1, m, f)$ and $|f(t) - f(\sigma(i-1, m, f))| = 2^{-m} - 2^{-k}\}$, $\sigma^k(i, m, f) = \inf\{t : t > \sigma(i, m, f))$ and $|f(t) - f(\sigma(i, m, f))| = 2^{-k}\}$, and both equal $|\sigma(i, m, f)|$ if either set is empty. Use $\mathscr{R}$ to denote the set of nonnegative binary rationals. Let $A_0$ be the set of $f \in C[0, \infty)$ which satisfy the following 3 requirements for all $i$ and $m$:

$0 < \sigma(i, m, f) < \infty$, $\sigma(i, m, f) \notin \mathscr{R}$, and there are points arbitrarily close to $\sigma(i, m, f)$ on the right at which the value of $f$ is above and below its value at $\sigma(i, m, f)$.

Let $A_1(i, m, k)$ be the set of $f \in C[0, \infty)$ for which

$|f(t) - f(\sigma_k(i, m, f))| \leqq 2^{-(m+1)}$ for all $t \in [\sigma_k(i, m, f), \sigma(i, m, f)]$.

Let $A_2(i, m, k)$ be the set of $f \in C[0, \infty)$ for which

there are points $s$ and $t$ in $[\sigma(i, m, f), \sigma^k(i, m, f)]$ satisfying $f(s) \geqq f(\sigma(i, m, f)) + 4^{-k}$ and $f(t) \leqq f(\sigma(i, m, f)) - 4^{-k}$.

For $N \geqq 1$, take

$$A(N, m, k) = \bigcap_{i=1}^{N} [A_1(i, m, k) \cap A_2(i, m, k)],$$

and

$$A(N, m) = \liminf_{k \to \infty} A(N, m, k) \cap A_0 .$$

Finally, let

$$A = \bigcap_{N,m} A(N, m) .$$

(20) LEMMA.

(20a) $\pi(A) = 1$,
(20b) $f \to \sigma(i, m, f)$ *is continuous on* $A$, *for all* $i, m$;
(20c) $f \to f(\sigma(i, m, f))$ *is continuous on* $A$, *for all* $i, m$;
(20d) $f \to 1_{A(N,m,k)}(f)$ *is continuous on* $A$, *for all* $N, m, k$; *and*
(20e) $(t, f) \to c(t, m, f)$ *is jointly continuous on* $\mathscr{R} \times A$, *for all* $m$.

PROOF. (20a): The distribution function of $\sigma(1, 0)$ is absolutely continuous with respect to Lebesgue measure, $0 < \sigma(1, 0) < \infty$ $\pi$-a.s., and $\pi[\max\{f(u): 0 \leqq u \leqq t\} > 0 > \min\{f(u): 0 \leqq u \leqq t\}] = 1$ for all $t > 0$. Now use the strong Markov property and a scale change argument to get $\pi[A_0] = 1$. Again by the strong Markov property and a scale change argument in order to show $\pi[A(N, m)] = 1$ and hence $\pi[A] = 1$, it suffices to verify $\pi[\liminf_{k\to\infty} A_1(1, 0, k)] = 1 = \pi[\liminf_{k\to\infty} A_2(1, 0, k)]$. This follows from Borel-Cantelli, upon computing $\pi\{[A_1(1, 0, k)]^c\} = 2^{-k}/(2^{-1} + 2^{-k})$ and $\pi\{[A_2(1, 0, k)]^c\} = 2 \cdot 4^{-k}/(4^{-k} + 2^{-k})$ using the fairness of Brownian motion.

(20b): By the strong Markov property and a rescaling argument it is enough to prove $\sigma(1, 0)$ is continuous on $A$. Let $\tau = \sigma(1, 0)$, $\tau_k = \sigma_k(1, 0)$, and $\tau^k = \sigma^k(1, 0)$. It is easy to check that on $A$ $\tau_k \uparrow \tau$ and $\tau^k \downarrow \tau$ as $k \to \infty$. The continuity of $\tau$ now follows from the fact that if $f \in A_1(1, 0, k) \cap A_2(1, 0, k)$ and $|f - g| \leqq 4^{-k}$ then $\tau_k(f) \leqq \tau(g) \leqq \tau^k(f)$, and the relation $\tau_k(f) \leqq \tau(f) \leqq \tau^k(f)$. This gets (20b).

(20c): is immediate from (20b). (20d) and (20e) follow from (20b), (20c) and the fact that $A \subset A_0$. □

(21) will adapt (20) to $C[0, 1]$. In order to avoid using new symbols extend the definition of $\sigma(j, m)$ to $C[0, 1]$ by taking $\sigma(0, m, f) = 0$, and $\sigma(j, m, f)$ to be the first $t > \sigma(j-1, m, f)$ for which $|f(t) - f(\sigma(j-1, m, f))| = 2^{-m}$ and $\sigma(j-1, m, f)$ if no such $t$ exists, at $f \in C[0, 1]$, for $j \geqq 1$. Likewise, extend the definition of $c(t, m, \cdot)$ to $C[0, 1]$ by letting $c(t, m, f) = \inf\{j: \sigma(j, m, f) \leqq t < \sigma(j+1, m, f)$ or $\sigma(j, m, f) = \sigma(j+1, m, f) \leqq t\}$ for $0 \leqq t \leqq 1$. Let

$$E_0 = \{g \in C[0, 1]: g(t) = f(t) \quad \text{for all } t \in [0, 1]; \text{ for some } f \in A\} .$$

For $i$, $m$, and $k \geqq m + 2$ let

$$E_1(i, m, k) = \{g \in C[0, 1]: g(t) = f(t) \quad \text{for all } t \in [0, 1]; \text{ for some } f \in A_1(i, m, k)\} ,$$

and

$$E_2(i, m, k) = \{g \in C[0, 1]: g(t) = f(t) \quad \text{for all } t \in [0, 1]; \text{ for some } f \in A_2(i, m, k)\} ,$$

For $M = 1, 2, \cdots$ take

$$E(M, m, k) = [c(1, m) = M] \bigcap_{i=1}^{M} [E_1(i, m, k) \cap E_2(i, m, k)] ,$$
$$E(m, k) = \bigcup_{M=1}^{\infty} E(M, m, k) ,$$
$$E(m) = \liminf_{k\to\infty} E(m, k) \cap E_0 ,$$

and

$$E = \bigcap_{m=1}^{\infty} E(m) .$$

(21) LEMMA.

(21 a) $\pi_1(E) = 1$,

(21 b) $f \to \sigma(i, m, f)$ *is continuous on* $E$, *for all* $i, m$;

(21 c) $f \to f(\sigma(i, m, f))$ *is continuous on* $E$, *for all* $i, m$,

(21 d) $f \to 1_{E(m,k)}(f)$ *is continuous on* $E$, *for all* $m, k$; *and*

(21 e) $(t, f) \to c(t, m, f)$ *is jointly continuous on* $[0, 1] \cap \mathscr{R} \times E$, *for all* $m$.

PROOF. Immediate from (20). □

**3. Asymptotic fairness.** This section contains the proofs of the lemmas supporting Theorem 1, and (17). Recall (3), and the definitions preceding it. In addition, remember $\{B(t) : t \geqq 0\}$ is the coordinate function on $C[0, \infty)$, $\pi_1$ makes $\{B(t) : 0 \leqq t \leqq 1\}$ Brownian motion on $[0, 1]$, $\pi$ makes $\{B(t) : 0 \leqq t \leqq \infty\}$ Brownian motion on $[0, \infty)$, $\sigma(j, m, f)$ is the first time $f$ undergoes $j$ changes of size $2^{-m}$ for unbounded $f \in C[0, \infty)$, and $B^m$ is the piecewise linear continuous function on $[0, 1]$ whose value at $k\cdot 4^{-m}$ is $B_{\sigma(k,m)}$ for $k = 0, 1, \cdots, 4^m$. So, $B^m : C[0, \infty) \to C[0, 1]$.

For (22) and (23) let $\{Y_k^n : k = 0, 1, \cdots\}$ be a sequence of processes on $(\Omega, \mathscr{A}, P)$. Assume $\delta, \varepsilon > 0$ and $m, N$ are positive integers such that $2^{-m} < \varepsilon$. The Lemmas will involve $\pi$ and $B_{\sigma(k,m)}$ only in so far as $\{B_{\sigma(k,m)} - B_{\sigma(k-1,m)} : k = 1, 2, \cdots\}$ are i.i.d. as $\pm 2^{-m}$ with probability $\frac{1}{2}$ each. Of course, any random variables with these properties could be used instead.

(22) LEMMA. *If* $Y^n$ *is asymptotically fair and* $\pi[\max_{1\leq k\leq N} |B_{\sigma(k,m)}| \leqq \varepsilon] \geqq 1 - \varepsilon$, *then*

$$P[\max_{1\leq k\leq N} |Y^n_{\sigma(k,m)}| \leqq 2\varepsilon] \geqq 1 - 2\varepsilon \qquad \textit{for all large } n .$$

PROOF. Let $\eta = \varepsilon/N$. For each $a \in G(N, m)$, by the asymptotic fairness of $Y^n$, choose $n_a$ such that $n \geqq n_a$ implies $|P[(Y^n_{\sigma(1,m)}, \cdots, Y^n_{\sigma(N,m)} - Y^n_{\sigma(N-1,m)}) \in a(\eta)] - (\frac{1}{2})^N| \leqq \varepsilon(\frac{1}{2})^N$, and set $n_0 = \max_a n_a$. Let $F = \{a = (a_1, \cdots, a_N) \in G(N, m) : \max_{1\leq k\leq N} |\sum_{j=1}^{k} a_i| \leqq \varepsilon\}$, and observe that if $a \in F$ and $b = (b_1, \cdots, b_N) \in a(\eta)$ then $\max_{1\leq k\leq N} |\sum_{i=1}^{k} b_i - \sum_{i=1}^{k} a_i| \leqq N\eta \leqq \varepsilon$. Since $G(N, m)$ contains $2^N$ points it follows that if $n \geqq n_0$, then

$$\begin{aligned} P[\max_{1\leq k\leq N} |Y^n_{\sigma(k,m)}| \leqq 2\varepsilon] &\geqq \textstyle\sum_{a\in F} P[(Y^n_{\sigma(1,m)}, \cdots, Y^n_{\sigma(N,m)} - Y^n_{\sigma(N-1,m)}) \in a(\eta)] \\ &\geqq \textstyle\sum_{a\in F} [(\frac{1}{2})^N - \varepsilon(\frac{1}{2})^N] \\ &= \textstyle\sum_{a\in F} \{\pi[(B_{\sigma(1,m)}, \cdots, B_{\sigma(N,m)}) = a] - \varepsilon(\frac{1}{2})^N\} \\ &\geqq \pi[\max_{1\leq k\leq N} |B_{\sigma(k,m)}| \leqq \varepsilon] - \varepsilon \geqq 1 - 2\varepsilon , \end{aligned}$$

which proves the lemma.

(23) LEMMA. *If $Y^n$ is asymptotically fair and* $\pi[\max\{|B_{\sigma(k,m)}-B_{\sigma(j,m)}| : |k-j| \leqq 4^m\delta, j \text{ and } k \leqq N\} \leqq \varepsilon] \geqq 1-\varepsilon$, *then*

$$P[\max\{|Y^n_{\sigma(k,m)} - Y^n_{\sigma(j,m)}| : |k-j| \leqq 4^m\delta, j \text{ and } k \leqq N\} \leqq 2\varepsilon] \geqq 1-2\varepsilon \quad \text{for all large } n .$$

The proof is analogous to the one given for (22), and is omitted.

It is now possible to present a

PROOF OF (4). (4a) *and* (4b) *imply* (4c): Fix $\varphi \in \Phi$. Without real loss, by [4] assume $|\varphi|$ is bounded by $M < \infty$ and is uniformly continuous. Form the continuous (random) function $S^{m,n}$ on $[0, 1]$ by the requirements $S_0^{m,n} = 0$, $S^{m,n}_{k\cdot 4^{-m}} = Y^n_{\sigma(k,m)}$, and $S^{m,n}$ is linearly interpolated on $[(k-1)4^{-m}, k\cdot 4^{-m}]$, for $k = 1, \cdots, 4^m$. It suffices to show

$$(24) \qquad \int \varphi(B^m)\, d\pi \to \int \varphi\, d\pi_1 \quad \text{as} \quad m \to \infty ,$$

$$(25) \qquad \int \varphi(S^{m,n})\, dP \to \int \varphi(B^m)\, d\pi \quad \text{as} \quad n \to \infty , \qquad \text{for all } m, \text{ and}$$

$$(26) \qquad |\int \varphi(S^{m,n})\, dP - \int \varphi(S^n)\, dP| \to 0 \quad \text{as} \quad m, n \to \infty$$

with $n \geqq n'(m)$, for some sequence $\{n'(m) : m = 0, 1, \cdots\}$.

(24) was proved as (19). The proof of (25) follows. Fix an $\varepsilon > 0$, and a non-negative integer $m$. Let $l$ map the set of $4^m$-tuplets of real numbers into $C[0, 1]$ be defined at $a = (a_1, \cdots, a_{4^m})$ by the requirements $l_0(a) = 0$, $l_{k\cdot 4^{-m}}(a) = a_1 + \cdots + a_k$, and $l(a)$ is linearly interpolated on $[(k-1)4^{-m}, k\cdot 4^{-m}]$, for $k = 1, \cdots, 4^m$. Clearly,

$$(27) \qquad \begin{aligned} \int \varphi(B^m)\, d\pi &= \int \textstyle\sum_{a\in G(4^m,m)} \varphi(B^m) 1_{[B^m = l(a)]}\, d\pi \\ &= \textstyle\sum_{a\in G(4^m,m)} \varphi(l(a))(\frac{1}{2})^{4^m} . \end{aligned}$$

Use the asymptotic fairness of $Y^n$, and the finiteness of $G(4^m, m)$, to obtain a sequence $\eta(n)$ which decreases to 0 as $n \to \infty$ such that

$$P[(Y^n_{\sigma(1,m)}, \cdots, Y^n_{\sigma(4^m,m)} - Y^n_{\sigma(4^m-1,m)}) \in a(\eta(n))] \to (\tfrac{1}{2})^{4^m} \quad \text{for all} \quad a \in G(4^m, m) .$$

The properties of $\varphi$ now imply

$$(28) \qquad \int \varphi(S^{m,n}) 1_{[(Y^n_{\sigma(1,m)},\cdots,Y^n_{\sigma(4^m,m)}-Y^n_{\sigma(4^m-1,m)}) \in a(\eta(n))]}\, d\pi \to \varphi(l(a))(\tfrac{1}{2})^{4^m} \quad \text{as} \quad n \to \infty , \qquad \text{for all} \quad a \in G(4^m, m) .$$

Noting that $P[(Y^n_{\sigma(1,m)}, \cdots, Y^n_{\sigma(4^m,m)} - Y^n_{\sigma(4^m-1,m)}) \in a(\eta(n))$ for some $a \in G(4^m, m)] \to 1$, (25) follows from (28) by adding over $a$, and then using (27).

(26) follows from the fact $\sup_{0\leqq t\leqq 1} |S_t^{m,n} - S_t^n| \to_P 0$ as $m, n \to \infty$ with $n \geqq n'(m)$, for some sequence $\{n'(m) : m = 0, 1, \cdots\}$. To prove this let $Z_t^n = Y^n_{T_{nt}} = S_{T_{nt}}/n^{\frac{1}{2}}$, for $0 \leqq t \leqq \infty$. Plainly $Z_\cdot^n$, or simply $Z^n$, is a right continuous

step function on $[0, \infty)$. Since $S^n(t) = S^1(nt)/n^{\frac{1}{2}} = S_{T_{nt}}/n^{\frac{1}{2}}$ when $T$ has a jump at $nt$,

$$\sup_{0\leq t\leq 1} |S^n(t) - Z^n(t)| \leqq \max_{1\leq i\leq T_n} |X_i|/n^{\frac{1}{2}} , \tag{29}$$

using the properties of $T$. For each $k$, $m$, $n$ let $\sigma(k, m, Z^n) = \inf\{t \mid T_{nt} = \sigma(k, m, Y^n)\}$, and $c(t, m, Z^n) = c(T_{nt}, m, Y^n)$ for $t \geqq 0$. Think of $\sigma(j, m, Z^n)$ as the first time $Z^n$ undergoes $j$ changes in size of at least $2^{-m}$, and $c(t, m, Z^n)$ as the number of such changes up to time $t$, on the set where $Z^n$ is unbounded. Plainly,

$$Z^n_{\sigma(j,m,Z^n)} = Y^n_{\sigma(j,m,Y^n)} \qquad \text{for all} \quad j, m, n . \tag{30}$$

As usual abbreviate $Z^n_{\sigma(j,m,Z^n)}$ by $Z^n_{\sigma(j,m)}$. Since (4b) implies $\max_{1\leq i\leq T_n} |X_i|/n^{\frac{1}{2}} \rightarrow_P 0$ as $n \rightarrow \infty$, and $\sup_{0\leq t\leq 1} |S_t^{m,n} - Z^n_{\sigma([4^m t],m)}| \leqq 2^{-m} + \max_{1\leq i\leq T_n} |X_i|/n^{\frac{1}{2}}$ by the definition of $S^{m,n}$ and (30), in view of (29), (26) will be established once it is shown that

$$\sup_{0\leq t\leq 1} |Z^n_{\sigma([4^m t],m)} - Z_t^n| \rightarrow_P 0 \quad \text{as} \quad m, n \rightarrow \infty \\ \text{with} \quad n \geqq n'(m), \quad \text{for some sequence} \quad \{n'(m) : m = 0, 1, \cdots\} . \tag{31}$$

(31) is a consequence of (32) and (33):

$$\sup_{0\leq t\leq 1} |\sigma([4^m t], m, Z^n) - t| \rightarrow_P 0 \quad \text{as} \quad m, n \rightarrow \infty \tag{32}$$

with $n \geqq n'(m)$ for some sequence $\{n'(m) : m = 0, 1, \cdots\}$; and

$$\limsup_{n\to\infty} P[\sup\{|Z_r^n - Z_s^n| : |r - s| \leqq \delta, 0 \leqq r, s \leqq 1\} > \varepsilon] \rightarrow 0 \\ \text{as} \quad \delta \rightarrow 0 , \qquad \text{for all} \quad \varepsilon > 0 . \tag{33}$$

To prove (32) fix $\varepsilon > 0$. Let

$$H_{m,n} = [\sup_{0\leq t\leq 1} |U^{m,n}_{T_{nt}} - t| < \varepsilon] ,$$

and

$$L_{m,n} = [|\textstyle\sum_{j=1}^{2\cdot 4^m} (Z^n_{\sigma(j,m)} - Z^n_{\sigma(j-1,m)})^2 - 2| < 4^{-m}] .$$

The relation $c(1, m, Z^n)4^{-m} \leqq U^{m,n}_{T_n}$ implies

$$c(1, m, Z^n) \leqq 2\cdot 4^m \quad \text{on} \quad H_{m,n} . \tag{34}$$

For all $m$, by the asymptotic fairness of $Y^n$ as $n \rightarrow \infty$, one may choose $n_1(m)$ such that $P[L_{m,n}] > 1 - m^{-1}$ for all $n \geqq n_1(m)$. Then using (4b),

$$P[H_{m,n}] \rightarrow 1 \quad \text{and} \quad P[L_{m,n}] \rightarrow 1 \quad \text{as} \quad m, n \rightarrow \infty \quad \text{with} \quad n \geqq n'(m), \\ \text{where} \quad n'(m) = n(m) \vee n_1(m) . \tag{35}$$

If $m$ is large enough to satisfy $4^{-m} < \varepsilon$ then, in view of (34), on the set $H_{m,n} \cap L_{m,n}$, for all $t \in [0, 1 - \varepsilon]$, the two relations $t - \varepsilon \leqq U^{m,n}_{T_{nt}} \leqq t + \varepsilon$ and

$c(t, m, Z^n)4^{-m} \leqq U^{m,n}_{T_{nt}} \leqq [c(t, m, Z^n) + 1]4^{-m}$ hold, and hence

$$\sigma([4^m(t - 2\varepsilon)], m, Z^n) \leqq \sigma(c(t, m, Z^n), m, Z^n) \leqq t$$
$$\leqq \sigma(c(t, m, Z^n) + 1, m, Z^n) \leqq \sigma([4^m(t + 2\varepsilon)], m, Z^n) .$$

Accordingly, $t - 4\varepsilon \leqq \sigma([4^m t], m, Z^n) \leqq t + 4\varepsilon$ for all $t \in [0, 1 - \varepsilon]$ on $H_{m,n} \cap L_{m,n}$, when $m$ is large. This fact, along with (35), implies (32).

Verification of (33) remains. Actually, (33) follows from (4a) and (32). To see this fix $\varepsilon > 0$, and use (18) to choose $\delta$ such that

$$\sup_m \pi[\sup \{|B_{\sigma(k,m)} - B_{\sigma(j,m)}| : |k - j| \leqq 4^m\delta, j \text{ and } k \leqq 2\cdot 4^m\} > \varepsilon] < \varepsilon .$$

Then use (4a), (23) and (30) to choose, for each $m$, an $n_2(m)$ such that

(36) $$P[\sup \{|Z^n_{\sigma(k,m)} - Z_{\sigma(j,m)}| : |k - j| \leqq 4^m\delta, j \text{ and } k \leqq 2\cdot 4^m\} > 2\varepsilon] \leqq 2\varepsilon \quad \text{for all} \quad n \geqq n_2(m) .$$

Use (32) to choose $m_1$ such that $2^{-(m_1-1)} < \varepsilon$ and

(37) $$\text{if} \quad n \geqq n'(m_1), \quad \text{then} \quad P[\sup_{0\leqq t\leqq 1} |\sigma([4^{m_1}t], m_1, Z^n) - t| > \tfrac{1}{3}\delta] < \varepsilon.$$

Put $n_0 = n'(m_1) \vee n_2(m_1)$. Now combine (36) and (37) to check if $n \geqq n_0$, then

$$P[\sup \{|Z_r{}^n - Z_s{}^n| : |r - s| \leqq \tfrac{1}{2}\delta, 0 \leqq r, s \leqq 1\} > 3\varepsilon]$$
$$\leqq P[\sup \{|Z^n_{\sigma(k,m_1)} - Z^n_{\sigma(j,m_1)}| : |k - j| \leqq 4^{m_1}\delta, k \text{ and } j \leqq 2\cdot 4^{m_1}\} \geqq 2\varepsilon] + \varepsilon \leqq 3\varepsilon .$$

This proves (33), finishing off (26), and hence (4c).

(4c) *implies* (4a) *and* (4b): (4a) will be proved first. Fix a positive integer $N$, a nonnegative integer $m'$, an $a = (a_1, \cdots, a_N) \in G(N, m')$, and $\eta > 0$. Without real loss assume $0 < \eta < 1$ and $m' = 0$. For each $m$, let $b_m = (a_1\cdot 2^{-m}, \cdots, a_N\cdot 2^{-m}) \in G(N, m)$. For each $m$ and $n$, let

$$K_{m,n} = [(Y^n_{\sigma(1,m)}, \cdots, Y^n_{\sigma(N,m)} - Y^n_{\sigma(N-1,m)}) \in b_m(\eta \cdot 2^{-m})]$$

and

$$K'_{m,n} = [(S^n_{\sigma(1,m)}, \cdots, S^n_{\sigma(N,m)} - S^n_{\sigma(N-1,m)}) = b_m] .$$

Let $p_{m,n} = P(K_{m,n})$ and $p'_{m,n} = P(K'_{m,n})$. Since $Y^n_{\sigma(i,0,Y^n)} = 2^m \cdot Y^{n4^m}_{\sigma(i,m,Y^{4^m n})}$ for all $i, m, n$, $p_{0n} = p_{m,n4^m}$. Thus, in order to show $p_{0n} \to (\frac{1}{2})^N$ as $n \to \infty$ it suffices to show

(38) $$p'_{m,n} \to (\tfrac{1}{2})^N ,$$

and

(39) $$|p_{m,n} - p'_{m,n}| \to 0 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq \hat{n}(m), \quad \text{for some sequence} \quad \{\hat{n}(m) : m = 0, 1, \cdots\} .$$

By (18) $\pi_1[\sigma(N, m) < 1] \to 1$ as $m \to \infty$. Since $\pi$ makes $\{B_{\sigma(i,m)} - B_{\sigma(i-1,m)} : i = 1, 2, \cdots\}$ i.i.d. as $\pm 2^{-m}$ with probability $\frac{1}{2}$ each, for all $m$,

$$\pi_1[\{f : (f(\sigma(1, m)), \cdots, f(\sigma(N, m)) - f(\sigma(N - 1, m))) = b_m\}] \to (\tfrac{1}{2})^N$$

as $m \to \infty$. Hence by (21c), there exists a sequence $\{\hat{n}_1(m): m = 0, 1, \cdots\}$ for which (38) holds.

To prove (39), it is enough to verify that there exists a set $R_{m,n}$ such that

$$K_{m,n} \cap R_{m,n} = K'_{m,n} \cap R_{m,n}, \tag{40}$$

and

$$P(R_{m,n}) \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq \hat{n}(m), \quad \text{for some sequence} \quad \{\hat{n}(m): m = 0, 1, \cdots\}. \tag{41}$$

The construction of $R_{m,n}$ follows.

Recall the definitions preceding (21). Since $E(m, k) \uparrow E(m)$ on $E$, and $\pi_1[E(m)] = 1$, it is possible to choose, for each $m$, a $k_m$ such that

$$\pi_1[E(m, k_m)] \to 1 \quad \text{and} \quad m/k_m \to 0 \quad \text{as} \quad m \to \infty. \tag{42}$$

For each $m$ and $n$, let $R_{m,n}$ be the set on which $\sigma(N+1, m, S^n) < 1$, $S^n \in E(m, k_m)$, and $\sup_{1 \leqq i \leqq T_n} |Y_i^n - Y_{i-1}^n| < \eta/(N \cdot 2^m \cdot 4^{k_m})$.

$R_{m,n}$ satisfies (40). To see this use induction on $i = 1, \cdots, N$ to show that the following 3 relations hold on $R_{m,n}$, keeping (29) and (30) in mind:

$$|Z^n_{\sigma(i,0,Z^n)} - S^n_{\sigma(i,0,S^n)}| \leqq (i \cdot \eta)/(N \cdot 2^m \cdot 4^{k_m})$$

$$\sigma_{k_m}(i, 0, S^n) \leqq \sigma(i, 0, Z^n) \leqq \sigma^{k_m}(i, 0, S^n),$$

and

$$\sup_{0 \leqq t \leqq 1} |S_t^n - Z_t^n| \leqq \eta/(N \cdot 2^m \cdot 4^{k_m}).$$

(40) now follows easily from the first of these relations, noting that

$$(i \cdot \eta)/(N \cdot 2^m \cdot 4^{k_m}) \leqq \eta \cdot 2^{-m} \quad \text{for} \quad i = 1, \cdots, N.$$

For (41), upon examining the proof of (38), it is easy to see that

$$P[\sigma(N+1, m, S^n) < 1] \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq \hat{n}_1(m). \tag{43}$$

Use (21d), (42), and (4c) to obtain

$$P[S^n \in E(m, k_m)] \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq \hat{n}_2(m), \quad \text{for some sequence} \quad \{\hat{n}_2(m): m = 0, 1, \cdots\}. \tag{44}$$

Also,

$$\text{(4c) implies} \quad \max_{1 \leqq i \leqq T_n} |X_i|/n^{\frac{1}{2}} = \max_{1 \leqq i \leqq T_n} |Y_i^n - Y_{i-1}^n| \to_P 0 \quad \text{as} \quad n \to \infty. \tag{45}$$

Hence,

$$P[\sup_{1 \leqq i \leqq T_n} |Y_i^n - Y_{i-1}^n| < \eta/(N \cdot 2^m \cdot 4^{k_m})] \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n > \hat{n}_3(m), \quad \text{for some sequence} \quad \{\hat{n}_3(m): m = 0, 1, \cdots\}. \tag{46}$$

To verify (45), for each $f \in C[0, 1]$, and positive integer $m$, let

$$\lambda_\delta(f) = \sup_{0 \leqq |r-s| \leqq \delta} |f(r) - f(s)|, \quad \text{and} \quad \varphi_m(f)$$
$$= \sup \{k/m : f \text{ is strictly monotone on } k \text{ successive points in } \{0, 1/m, \cdots, m/m\}\}.$$

$\lambda_\delta$ is continuous. Although $\varphi_m$ is not continuous on $C[0, 1]$, it is continuous at those $f$ satisfying $f(k/m) \neq f(j/m)$ for $0 \leqq k \neq j \leqq m$ and $m = 1, 2, \cdots$, a set of $\pi_1$-probability 1. Observe that for each $\delta, \varepsilon > 0$,

$$[\sup_{1 \leqq i \leqq T_n} |X_i|/n^{\frac{1}{2}} > \varepsilon] \subset [\lambda_\delta(S^n) > \varepsilon] \cup [\varphi_m(S^n) > \delta] .$$

Clearly, (4c) implies $P[\lambda_\delta(S^n) > \varepsilon] \to \pi_1[\lambda_\delta > \varepsilon]$, and $P[\varphi_m(S^n) > \delta] \to \pi_1[\varphi_m > \delta]$ as $n \to \infty$. Since $\pi_1[\lambda_\delta > \varepsilon] \to 0$ as $\delta \to 0$ for each $\varepsilon$, and $\pi_1[\varphi_m > \delta] \to 0$ as $m \to \infty$ for each $\delta$, (45) now follows by a routine argument. Finally, put $\hat{n}(m) = \hat{n}_1(m) \vee \hat{n}_2(m) \vee \hat{n}_3(m)$ and combine (43) and (44), and (46) to check (41), finishing off (39) and hence (4a).

For (4b) it is enough to show

$$\sup_{0 \leqq t \leqq 1} |U^{m,n}_{T_{nt}} - c(t, m, Z^n)4^{-m}| \to_P 0 , \tag{47}$$

$$\sup_{0 \leqq t \leqq 1} |c(t, m, Z^n)4^{-m} - c(t, m, S^n)4^{-m}| \to_P 0 , \tag{48}$$

and

$$\sup_{0 \leqq t \leqq 1} |c(t, m, S^n)4^{-m} - t| \to_P 0, \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq n(m), \text{ for some sequence } \{n(m) : m = 0, 1, \cdots\} . \tag{49}$$

(49) will be proved first, then (48), and (47) last.

(18) implies $\pi_1[c(t, m)4^{-m} \to t$ as $m \to \infty, 0 \leqq t \leqq 1] = 1$. This together with (21e), the hypotheses (4c), and the fact that $c(t, m)$ is non-decreasing in $t$ for each $m$, implies the existence of a sequence $\{\hat{n}_4(m) : m = 0, 1, \cdots\}$ for which (49) holds.

For (48), recall the meaning of $\{k_m : m = 0, 1, \cdots\}$ from (42). Now use (45) to choose $\{\hat{n}_5(m) : m = 0, 1, \cdots\}$ such that $P[\sup_{1 \leqq i \leqq T_n} |Y_i^n - Y_{i-1}^n| < 1/(2 \cdot 4^m \cdot 4^{k_m})] \to 1$ as $m, n \to \infty$ with $n \geqq \hat{n}_5(m)$. Recall (44), and the fact that (49) holds for $\{\hat{n}_4(m) : m = 0, 1, \cdots\}$. Upon putting $\hat{n}_6(m) = \hat{n}_2(m) \vee n_4(m) \vee \hat{n}_5(m)$ it follows that

$$P[Q_{m,n}] \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad n \geqq \hat{n}_6(m) , \tag{50}$$

where $Q_{m,n}$ is the set on which $c(1, m, S^n) \leqq 2 \cdot 4^m$, $S^n \in E(m, k_m)$, and

$$\sup_{1 \leqq i \leqq T_n} |Y_i^n - Y_{i-1}^n| < 1/(2 \cdot 4^m \cdot 4^{k_m}) .$$

Now, by an induction argument analogous to the proof of (40), it is easy to show that on $Q_{m,n}\, \sigma(i-1, m, S^n) \leqq \sigma(i, m, Z^n) \leqq \sigma(i+1, m, S^n)$. Hence, $\sup_{0 \leqq t \leqq 1} |c(t, m, Z^n) - c(t, m, S^n)| \leqq 1$ on $Q_{m,n}$. This fact, combined with (50) implies (48) for the sequence $\{\hat{n}_6(m) : m = 0, 1, \cdots\}$.

To get (47), since the asymptotic fairness of $Y^n$ has already been established, one can choose a sequence $\{\hat{n}_7(m) : m = 0, 1, \cdots\}$ such that

$$P[|\sum_{i=1}^{2\cdot 4^m} (Y^n_{\sigma(k,m)} - Y^n_{\sigma(k-1,m)})^2 - 2| < 4^{-m}] \to 1 ,$$

as $m, n \to \infty$ with $n \geqq \hat{n}_7(m)$. Use this fact, along with (48) and (49), to verify

$$P[c(t, m, Z^n)4^{-m} \leqq U^{m,n}_{T_{nt}} \leqq (c(t, m, Z^n) + 1)4^{-m} \quad \text{for all} \quad t \in [0, 1]] \to 1$$

as $m, n \to \infty$ with $n \geqq \hat{n}_4(m) \vee \hat{n}_6(m) \vee \hat{n}_7(m)$. This implies (47), and hence (4b) upon letting $n(m) = \hat{n}_4(m) \vee \hat{n}_6(m) \vee \hat{n}_7(m)$, concluding the proof of (4).

PROOF OF (5). Fix nonnegative integers $m$ and $N$, an $\eta > 0$, and $a = (a_1, \cdots, a_N) \in G(4^m, m)$. Let $V_i^n = Y^n_{\sigma(i,m)} - Y^n_{\sigma(i-1,m)}$ for $i = 1, \cdots, N$. It must be shown that

$$P[\bigcap_{i=1}^N [V_i^n \in a_i(\eta)]] \to (\tfrac{1}{2})^N$$

as $n \to \infty$. By induction this reduces to showing that

$$P\{\bigcap_{i=1}^N [V_i^N \in a_i(\eta)]\} - \tfrac{1}{2}P\{\bigcap_{i=1}^{N-1} [V_i^n \in a_i(\eta)]\} \to 0$$

as $n \to \infty$. Only the case of $N = 2$ and $m = 0$ with $a_2(\eta) = [1, 1 + \eta]$ will be considered, the general case being analogous.

By the optional sampling theorem, the hypothesis implies $Y^n_{\sigma(1,0)}$, $Y^n_{\sigma(2,0)}$, is a martingale, for large $n$. Let $A_n = \{V_1^n \in a_1(\eta)\}$, $B_n = \{V_2^n \in [1, 1 + \eta]\}$, $C_n = \{V_2^n \in [-(1 + \eta), -1]\}$, and $D_n\{|V_2^n| > 1 + \eta\}$. Since $A_n$ is measurable w.r.t. the $\sigma$-field generated by $Y^n_{\sigma(1,0)}$, the martingale property says

$$0 = \int_{A_n} V_2^n \, dP = \int V_2^n 1_{[A_n \cap B_n]} \, dP + \int V_2^n 1_{[A_n \cap C_n]} \, dP + \int V_2^n 1_{[A_n \cap D_n]} \, dP .$$

The hypothesis forces $\int V_2^n 1_{[A_n \cap D_n]} \, dP \to 0$, and hence $\int V_2^n 1_{[A_n \cap B_n]} \, dP + \int V_2^n 1_{[A_n \cap C_n]} \, dP \to 0$ as $n \to \infty$. Considering the range of values possible for $V_2^n$ on the sets $B_n$ and $C_n$, standard argumentation yields $P[A_n \cap B_n] - P[A_n \cap C_n] \to 0$ and $P[A_n \cap B_n] + P[A_n \cap C_n] - P[A_n] \to 0$ as $n \to \infty$. It now follows that $P[A_n \cap B_n] - \frac{1}{2}P[A_n] \to 0$ as $n \to \infty$.

PROOF OF (6): Fix $\eta > 0$, nonnegative integers $m$ and $N$, and $a \in G(N, m)$. Without real loss assume $m = 0$, and $N \geqq 1$. It is enough to show

(51) $(Y^n_{\sigma(1,0)}, \cdots, Y^n_{\sigma(N,0)} - Y^n_{\sigma(N-1,0)}) \in a(\eta)$ iff $(\hat{Y}^n_{\sigma(1,0)}, \cdots, \hat{Y}^n_{\sigma(N,0)} - \hat{Y}^n_{\sigma(N-1,0)}) \in a(\eta)$ on a set of arbitrarily large probability, for all sufficiently large $n$.

Informally, the asymptotic fairness of $Y^n$ entails that, like Brownian motion, it makes its changes of size 1, or does not make them, with room to spare, for large $n$. This forces $Y^n$ and $\hat{Y}^n$ to complete their successive changes of size 1 at about the same time, since they are close for large $n$. The first job is to construct the set where (51) holds.

Fix $\varepsilon > 0$. Recall the definition of $\pi$ given in the introduction and that of $A(N, m, k)$ preceding (20). Use (20) to choose $k_0$ such that $4^{-k_0} < (\eta/2) \wedge \varepsilon$ and

(52) $$\pi[A(N, 0, k_0)] > 1 - \varepsilon .$$

Using (18), choose $M_0$ and $m_0$ to satisfy $M_0 > N$, $2^{-m_0+3} \leqq 4^{-k_0}$, and

(53) $$\pi[\sigma(M_0, m_0) > \sigma(N + 1, 0)] > 1 - \varepsilon .$$

For each $n$ and $b \in G(M_0, m_0)$, let

$$F_n(b) = [(Y^n_{\sigma(1,m_0)}, \cdots, Y^n_{\sigma(M_0,m_0)} - Y^n_{\sigma(M_0-1,m_0)}) \in b(1/M_0 \cdot 2^{m_0})] .$$

Put $F_n = \cup \{F_n(b) : b \in G(M_0, m_0)\}$. For each $b$, choose $n_b$ such that

(54) $$n \geqq n_b \quad \text{implies} \quad |P[F_n(b)] - (\tfrac{1}{2})^{M_0}| < \varepsilon(\tfrac{1}{2})^{M_0}, \quad \text{and setting}$$
$$n_1 = \max_b n_b,\ n \geqq n_1 \quad \text{implies} \quad P[F_n] \geqq 1 - \varepsilon .$$

Let $H_n = [\sup_k |Y_k{}^n - \hat{Y}_k{}^n| < 1/M_0 \cdot (2^{m_0+1})]$, and choose $n_2$ such that

(55) $$P[H_n] > 1 - \varepsilon \qquad \text{whenever} \quad n \geqq n_2 .$$

Set $n_0 = n_1 \vee n_2$. For $k \geqq 1$ and $y \in \Gamma$ let

$$r_k(y) = \textstyle\sum_{i=1}^k 2^{-m_0} 1_{[y_{\sigma(i,m_0)} - y_{\sigma(i-1,m_0)} \geqq 2^{-m_0}]}$$
$$- \textstyle\sum_{i=1}^k 2^{-m_0} 1_{[y_{\sigma(i,m_0)} - y_{\sigma(i-1,m_0)} \leqq -2^{-m_0}]} .$$

For $y \in \Gamma$ such that $\sup \sigma(k, m_0, y) = \infty$, let $\gamma_.(y)$, or simply $\gamma(y)$, be the continuous function satisfying the requirements $\gamma_0(y) = 0$, $\gamma_{\sigma(k,m_0,y)}(y) = r_k(y)$, and $\gamma(y)$ is linearly interpolated on $[\sigma(k, m_0, y), \sigma(k + 1, m_0 y)]$ for $k = 0, 1, \cdots$. For other $y \in \Gamma$, define $\gamma(y)$ as above except make $\gamma_t(y)$ constant for all $t$ after the final change in size of at least $2^{-m}$ completed by $y$. Let

$$Q_n = [\gamma(Y^n) \in A(N, 0, k_0)] ,$$

and

$$R_n = [\sigma(M_0, m_0, \gamma(Y^n)) > \sigma(N + 1, 0, \gamma(Y^n))] .$$

Since $\pi$ makes $B_{\sigma(1,m_0)}, \cdots, B_{\sigma(M_0,m_0)} - B_{\sigma(M_0-1,m_0)}$ i.i.d. as $\pm 2^{-m_0}$ with probability $\frac{1}{2}$ each, (52) and (53) imply that there are at least $(1 - 2\varepsilon)2^{M_0} - 1$ points $a$ among the $2^{M_0}$ points in $G(M_0, m_0)$ such that if

$$(\gamma_{\sigma(1,m_0)}(y), \cdots, \gamma_{\sigma(M_0,m_0)}(y) - \gamma_{\sigma(M_0-1,m_0)}(y)) = a$$

for some $y \in \Gamma$, then $\gamma(y) \in A(N, 0, k_0)$ and $\sigma(M_0, m_0, \gamma(y)) > \sigma(N + 1, 0, \gamma(y))$. In view of (54) and the choice of $M_0$, it now follows that

$$P[Q_n \cap R_n] \geqq [(1 - 2\varepsilon)2^{M_0} - 1][(\tfrac{1}{2})^{M_0} - \varepsilon(\tfrac{1}{2})^{M_0}] \geqq 1 - 4\varepsilon ,$$

for $n \geqq n_0$. This, together with (54) and (55), forces

(56) $$P[W_n] \geqq 1 - 6\varepsilon , \qquad \text{whenever} \quad n \geqq n_0 ,$$

where $W_n = F_n \cap H_n \cap Q_n \cap R_n$.

(51) holds on $W_n$. To see this first observe that $W_n \subset F_n$, so on $W_n$

$$\sup\{|\gamma_t(Y^n) - Y^n_{[t]}| : 0 \leqq t \leqq \sigma(M_0, m_0, Y^n)\} \leqq 2^{-m_0+1} .$$

Then use induction on $i = 1, \cdots, N$ to verify that on $W_n$

$$|\gamma_{\sigma(i,0)}(Y^n) - Y^n_{\sigma(i,0)}| \leqq i/(M_0 \cdot 2^{m_0}) \leqq 4^{-k_0} \leqq \eta/2 ,$$

and

$$\sigma_{k_0}(i, 0, \gamma(Y^n)) \leqq \sigma(i, 0, Y^n) \leqq \sigma^{k_0}(i, 0, \gamma(Y^n)) .$$

It follows that on $W_n$, the first $N$ successive changes in size of at least 1 completed by $\gamma(Y^n)$ and $Y^n$ are of the same sign, and by similar argument using the fact

$$\sup\{|\gamma_t(Y^n) - \hat{Y}^n_{[t]}| : 0 \leqq t \leqq \sigma(M_0, m_0, Y^n)\} \leqq 2^{-m_0+2} ,$$

so are those of $\gamma(Y^n)$ and $\hat{Y}^n$. This implies (51), in view of $W_n \subset F_n \cap H_n$ and (56).

PROOF OF (8). For each $k \geqq 0$ and $y \in \Gamma$, let $V_k(y) = \sum_{j=1}^k |y_j - y_{j-1}|$. Think of $V_k(y)$ as the variation of $y$ up to time $k$. For each $n$, $V_{T_n}(Y^n)$ is a random variable. Plainly

$$V_{T_n}(Y^n) \leqq V_{T_n}(A^n) + V_{T_n}(\hat{A}^n) = A^n_{T_n} + \hat{A}^n_{T_n} ,$$

using the monotonicity of $A_k^n$ and $\hat{A}_k^n$ as $k$ increases. Therefore for any $K > 0$,

$$\begin{aligned} P[V_{T_n}(Y^n) > K] &\leqq P[V_{T_n}(A^n) > K/2] + P[V_{T_n}(\hat{A}^n) > K/2] \\ &\leqq 2/K + 2/K = 4/K , \end{aligned}$$

for large $n$, by Chebychev.

Fix $\varepsilon > 0$, and choose $K_0$ such that $4/K_0 < \varepsilon$. So

$$(57) \qquad P[V_{T_n}(Y^n) \leqq K_0] > 1 - \varepsilon , \qquad \text{for all large } n .$$

Since $\{\sigma(1, m, Y^n), \cdots, \sigma(c(T_n, m, Y^n), m, Y^n)\}$ is a subset of $(0, 1, 2, \cdots, T_n\}$ it follows from the triangle inequality and the definition of $\sigma(j, m)$ that, for all $m$ and $n$,

$$V_{T_n}(Y^n) \geqq \sum_{j=1}^{c(T_n,m,Y^n)} |Y^n_{\sigma(j,m)} - Y^n_{\sigma(j-1,m)}| \geqq c(T_n, m, Y^n)2^{-m} .$$

This, together with (57), implies

$$(58) \qquad P[c(T_n, m, Y^n) \leqq 2^m K_0] > 1 - \varepsilon , \qquad \text{for large } n .$$

Use (18) to choose $m_0$ such that

$$\pi[\max_{1 \leqq k \leqq 2^{m_0}K_0} |B_{\sigma(k,m_0)}| \leqq \varepsilon] \geqq 1 - \varepsilon, \quad \text{and} \quad 2^{-m_0} < \varepsilon .$$

Finally, use (22), (58), and the asymptotic fairness of $Y^n$, to get

$$\begin{aligned} P[\sup_{1 \leqq k \leqq T_n} |Y_k^n| \leqq 3\varepsilon] &\geqq P[\max\{|Y^n_{\sigma(j,m_0)}| : j = 1, \cdots, c(T_n, m, Y^n)\} \leqq 2\varepsilon] \\ &\geqq P[\max\{|Y^n_{\sigma(j,m_0)}| : j = 1, \cdots, 2^{m_0}K_0\} \leqq 2\varepsilon] - \varepsilon \\ &\geqq 1 - 3\varepsilon , \qquad \text{for large } n . \end{aligned}$$

PROOF OF (17). Only the case of $L_n$ will be proved, $\hat{L}_n$ is analogous. Let $l$ be Lebesgue measure on [0, 1], with the Borel $\sigma$-field. Let

$$\lambda(f) = l\{t : f(t) > 0\} \qquad \text{for } f \in C[0, 1].$$

Observe $\lambda$ is continuous on $E_0$ (defined just prior to (21)), and $\pi_1[E_0] = 1$ by (21). So, Theorem 1 implies the $P$-distribution of $\lambda(S^n)$ converges to the $\pi$-distribution of $\lambda$. Since the $\pi$-distribution of $\lambda$ is arc sin, it suffices to show $\lambda(S^n) - L_n \to_P 0$ as $n \to \infty$. For each $n$, $|\lambda(S^n) - L_n| \leqq U_n$, where

$$U_n = (1/n) \textstyle\sum_{i=1}^{T_n} E(X_i^2 \mid \mathscr{A}_{i-1}) 1_{[S_i<0<S_{i-1} \text{ or } S_i>0>S_{i-1}]} .$$

The fact that $U_n \to_P 0$ as $n \to \infty$ completes the proof.

To see this last point, for each $m \geqq 0$ let $h_m(f)$ be the Lebesgue measure of all intervals $[\sigma(j-1, m, f), \sigma(j, m, f)]$ in [0, 1] for which either $f(\sigma(j, m, f))$ or $f(\sigma(j-1, m, f))$ is 0, at $f \in C[0, 1]$. When $\max_{1\leqq i \leqq T_n} X_i^2/n < 2^{-m}$, an event having arbitrarily high probability as $n \to \infty$ for each $m$ by (45), $h_m(S^n) > U_n$. Hence, $P[h_m(S^n) > U_n] \to 1$ as $n \to \infty$, for each $m$. Now, $h_m$ is bounded, and is $\pi_1$-continuous since $\sigma(j, m)$ is by (21b). So, for each $m$ $\int h_m(S^n)\, dP \to \int h_m\, d\pi_1$ as $n \to \infty$. Use the fact $\int h_m\, d\pi_1 \to 0$ as $m \to \infty$ to choose a sequence $\{n(m) : m = 0, 1, \cdots\}$ for which $P[h_m(S^n) > U_n] \to 1$ and $h_m(S^n) \to_P 0$ as $m, n \to \infty$ with $n \geqq n(m)$.

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