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STRONG RATIO LIMIT THEOREMS FOR MARKOV PROCESSES[1]

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Let P be a conservative and ergodic Markov operator on $L_\infty(X, \Sigma, m)$ (where $m(X) = 1$). It is proved that if for $A \in \Sigma$ with $m(A) > 0$ and $\mu \ll m$ a finite measure with $\mu(A) > 0$ $\lim_{n\to\infty} \langle \mu, P^{n+1} 1_B \rangle / \langle \mu, P^n 1_A \rangle$ exists for every $B \subset A$, then P has a σ-finite invariant measure λ and there is a sequence $A_k \uparrow X$ with $A_0 = A$ such that for $0 \leqq f, g \in L_\infty(A_k)$ $\langle \mu, P^n f\rangle / \langle \mu, P^n g \rangle \to \langle \lambda, f \rangle / \langle \lambda, g \rangle$. The result is used to study the convergence of $\langle \mu, P^n f \rangle / \langle \eta, P^n g \rangle$ for $\mu, \eta \ll m$, with applications to Harris processes and strong mixing point transformations. An analogous result for a positive contraction of $C(X)$ is given.

1. Definitions and notation. Let (X, Σ, m) be a measure space with $m(X) = 1$. A Markov process is a positive contraction P on $L_1(X, \Sigma, m)$. P will be written to the right of its variable, while its adjoint, defined in $L_\infty(X, \Sigma, m)$, will be denoted by P and written to the left of its variable. Thus $\langle uP, f \rangle = \langle u, Pf \rangle$ for $f \in L_\infty$, $u \in L_1$. P also acts on the space of finite signed measures absolutely continuous with respect to m: $\mu P(A) = \int P 1_A \, d\mu$ for $\mu \ll m$, $A \in \Sigma$. The same formula defines μP for a σ-finite positive measure $\mu \ll m$. A positive σ-finite measure μ is *invariant* if $\mu P = \mu$. The process is *conservative* if $m(A) > 0$ implies $\sum_{n=0}^\infty P^n 1_A(x) = \infty$ a.e. on A. The process is *conservative and ergodic* if $m(A) > 0$ implies $\sum_{n=0}^\infty P^n 1_A(x) = \infty$ a.e.

In this paper (Section 3) we study the convergence of ratios of the form $\langle \mu P^n, f \rangle / \langle \mu P^n, g \rangle$ and $\langle \mu P^n, f \rangle / \langle \eta P^n, g \rangle$, where $\mu, \eta \ll m$ and $0 < f, g \in L_\infty$, generalizing Orey's strong ratio limit theorem [12] to more general conservative and ergodic processes.

For $0 \leqq \alpha \leqq 1$ in L_∞ we define the operator T_α by $uT_\alpha(x) = u(x)\alpha(x)$, so $T_\alpha f(x) = \alpha(x) f(x)$ and $\mu T_\alpha(A) = \langle \mu, \alpha 1_A \rangle$. For $\alpha = 1_A$ we denote T_α by T_A. The complement of a set A is denoted by A'. For $A \in \Sigma$ we denote by A^* the set $\{x: \sum_{n=0}^\infty P^n 1_A(x) = \infty\}$. If $m(A) > 0$, we define $P_A = T_A \sum_{n=0}^\infty (PT_{A'})^n PT_A$. If P is conservative, then P_A defines a Markov process in $L_1(A, \Sigma, mT_A)$, see [4], Lemma VI. B.

Theorem 1.1. *Let P be a conservative Markov process, and $A \in \Sigma$ with $m(A) > 0$ and $X = A^*$. If $\mu \ll m$ is a finite measure satisfying $\mu P_A = \mu$, then*

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$\lambda = \sum_{n=0}^{\infty} \mu(PT_{A'})^n$ is a σ-finite measure weaker than m and $\lambda P = \lambda$. If λ is a σ-finite measure satisfying $\lambda P = \lambda$ and $0 < \lambda(A) < \infty$, then $\mu = \lambda T_A$ satisfies $\mu P_A = \mu$.

PROOF. The first part is proved in [4] Theorem VI. C. The second, is proved in [8], Lemma 3.2 (reading T_A instead of I_β. The proof does not use the topological assumptions there).

LEMMA 1.1. *Let P be a conservative and ergodic Markov process and $A \in \Sigma$ with $m(A) > 0$. If $\mu \ll m$ is a finite measure such that for $B \subset A$ $\lim_{N\to\infty} \sum_{n=1}^{N} \mu P^n(B)/\sum_{n=1}^{N} \mu P^n(A)$ exists, then there exists a σ-finite invariant measure $\lambda \sim m$, $0 < \lambda(A) < \infty$.*

The limit serves to define a measure on A which is invariant for P_A as in [4], page 72.

2. Harris processes.

DEFINITION 2.1. An ergodic and conservative Markov process is a *Harris process* if for some $n > 0$ there exist a $\Sigma \times \Sigma$ measurable function $0 \leqq q(x, y)$, $q \not\equiv 0$, satisfying for every $0 \leqq f \in L_\infty$:

$$(*) \qquad \int q(x, y) f(y) m(dy) \leqq P^n f(x) \quad \text{a.e.}$$

We denote by q_k the maximal function q satisfying (*) for $n = k$, by Q_k the integral operator corresponding to q_k. (See [4] Chapter V.)

We define, for a Harris process,

$$S = \{A \in \Sigma\colon \exists\, k > 0 \ \inf_{x,y\in A} q_k(x, y) > 0\}\,.$$

It has been proved by Levitan [13], Theorems 2.1 and 2.2 that there is a sequence $A_i \in S$ with $X = \bigcup A_i \pmod m$ and if P is *aperiodic* then S is closed under the formation of finite unions. (The fact that P is a Harris process is not affected by changing m by an equivalent σ-finite measure, but S depends on m.) It is known that a Harris process has a σ-finite invariant measure $\lambda \sim m$. We assume that λ replaces m.

3. Strong ratio limit theorems. In this section we look for conditions which imply the convergence of $\langle \mu P^{n+r}, f\rangle/\langle \mu P^n, g\rangle$ when $\mu \ll m$, $f, g \in L_\infty$. Our main result is Theorem 3.1, and its method of proof is a direct generalization of Orey's [12]. In Theorem 3.1 P is not necessarily Harris, and we give an example of a point transformation satisfying its hypothesis and preserving an infinite σ-finite measure. We do not assume the existence of an invariant measure, which is shown to exist under the assumptions of Theorem 3.1.

The main theorem is then applied to study the convergence of $\langle \mu P^n, f\rangle/\langle \eta P^n, g\rangle$ with $\mu, \eta \ll m$ probability measures and $f, g \in L_\infty$. The strong ratio limit theorem (Theorem 3.4) is then used to yield a result about strong mixing processes with a finite invariant measure (which, when P is induced by a point transformation, are *not* Harris processes but have the strong ratio limit property).

Our methods are entirely analytic and can be extended to processes on a topological space, using the results of [3], [11]. (See [6].)

The application of the results to Harris processes was suggested by the referee's comments.

LEMMA 3.1. *For $0 \leqq \alpha \leqq 1$ define $\beta = 1 - \alpha$. Then for every integer $n \geqq 1$*

$$P^n = \sum_{j=1}^{n-1} P^j T_\alpha (PT_\beta)^{n-j-1} P + (PT_\beta)^{n-1} P .$$

PROOF. By induction: For $n = 1$ the sum is zero, having no terms. Use the induction hypothesis and $T_\alpha + T_\beta = 1$.

THEOREM 3.1. *Let P be a conservative and ergodic Markov process, and let $\mu \ll m$ be a finite measure. If $A \in \Sigma$ with $\mu(A) > 0$ is such that for every $B \subseteq A$ $\lim_{n\to\infty} \mu P^{n+1}(B)/\mu P^n(A)$ exists, then there exists a σ-finite invariant measure λ equivalent to m, and there is a sequence $A_k \uparrow X$, with $A_0 = A$ and $0 < \lambda(A_k) < \infty$, such that*

$$\lim_{n\to\infty} \frac{\langle \mu P^{n+r}, f\rangle}{\langle \mu P^n, g\rangle} = \frac{\langle \lambda, f\rangle}{\langle \lambda, g\rangle}$$

for every integer r and $f, g \in L_\infty(A_k, \Sigma, m)$.

PROOF. Since P is ergodic and conservative, $\sum_{n=1}^{\infty} \mu P^n(A) = \infty$. Thus it is easy to check that the conditions of Lemma 1.1 are satisfied, whence the existence of λ follows. Furthermore, the construction shows that necessarily

$$\lim_n \mu P^{n+1}(B)/\mu P^n(A) = \lambda(B)/\lambda(A) .$$

By a standard approximation argument we obtain:

$$\lim_n \langle \mu P^{n+1}, h\rangle/\mu P^n(A) = \langle \lambda, h\rangle/\lambda(A) \qquad h \in L_\infty(A) .$$

Hence

$$\langle \mu P^{n+r}, h\rangle/\mu P^n(A) \to \langle \lambda, h\rangle/\lambda(A) \qquad \text{for} \quad h \in L_\infty(A) ,$$

since

$$\mu P^{n+1}(A)/\mu P^n(A) \to 1 . \qquad (r \text{ is a positive integer.})$$

We now define the sets A_k. P is ergodic and conservative, so $\sum_{i=0}^{\infty} (PT_{A'})^i PT_A 1 = 1$ a.e. $-$ m [4], (3.2) and (3.3). Using Egoroff's theorem we can find a sequence $\{B_k\}$ of sets, $B_k \uparrow X$, such that on B_k $\sum_{i=0}^{\infty} (PT_{A'})^i PT_A 1$ converges uniformly to 1. If $C_k \uparrow X$ with $\lambda(C_k) < \infty$, we put $A_0 = A$ and $A_k = (B_k \cap C_k) \cup A$. We now proceed to the proof: Let r be fixed, and $1 \leqq N < n$. Using Lemma 3.1 we have

$$\begin{aligned} a_{nr}(f) &\equiv \langle \mu P^{n+r}, f\rangle / \mu P^n(A) \\ &= \{\langle \mu, \textstyle\sum_{j=0}^{n+r-2} P^{n+r-1-j} T_A(PT_{A'})^j Pf\rangle + \langle \mu, (PT_{A'})^{n+r-1} Pf\rangle\} / \mu P^n(A) \\ &= b_{Nnr}(f) + c_{Nnr}(f) \end{aligned}$$

with $b_{Nnr}(f)$ defined as

$$b_{Nnr}(f) = \{\mu P^n(A)\}^{-1} \textstyle\sum_{j=0}^{N} \langle \mu, P^{n+r-1-j} T_A(PT_{A'})^j Pf\rangle .$$

$T_A(PT_{A'})^j Pf \in L_\infty(A)$ whenever $f \in L_\infty(X, \Sigma, m)$, so we may use the beginning of the proof to obtain (N, r are fixed):

$$\lim_n b_{Nnr}(f) = \textstyle\sum_{j=0}^{N} \langle \lambda, T_A(PT_{A'})^j Pf\rangle / \lambda(A) .$$

By Theorem 1.1 λT_A is invariant for P_A and $\lambda = \sum_{j=0}^{\infty} (\lambda T_A)(PT_{A'})^j$, by uniqueness of the invariant measure (P is ergodic). Hence

$$\lim_{N\to\infty} \lim_{n\to\infty} b_{Nnr}(f) = \langle \lambda, Pf\rangle / \lambda(A) = \langle \lambda, f\rangle / \lambda(A) .$$

Therefore $a_{nr}(f) \to \langle \lambda, f\rangle/\lambda(A)$ if and only if $\lim_{N\to\infty} \lim_{n\to\infty} \sup c_{Nnr}(f) = 0$ (assuming $f \geqq 0$). At the beginning of the proof it is shown that $a_{nr}(f) \to \langle \lambda, f\rangle/\lambda(A)$ for $f \in L_\infty(X)$ supported in A. Hence we have only to show the same convergence for $f \in L_\infty(X)$ supported in B_k. Thus we assume $0 \leqq f \in L_\infty(B_k)$, and it is enough to show $a_{n1}(f) \to \langle \lambda, f\rangle/\lambda(A)$. By the definition of the B_k's, there is an I and constant M such that $(f \geqq 0) f \leqq M \sum_{i=0}^{I} (PT_{A'})^i P1_A$. Thus:

$$\begin{aligned} M^{-1} c_{Nn1}(f) &= \{\langle \mu, \textstyle\sum_{j=N+1}^{n-1} P^{n-j} T_A(PT_{A'})^j Pf\rangle + \langle \mu, (PT_{A'})^n Pf\rangle\} / M\mu P^n(A) \\ &\leqq \langle \mu, \textstyle\sum_{j=N+1}^{n-1} P^{n-j} T_A(PT_{A'})^j P \sum_{i=0}^{I} (PT_{A'})^i P1_A\rangle / \mu P^n(A) \\ &\quad + \langle \mu, (PT_{A'})^n P \textstyle\sum_{i=0}^{I} (PT_{A'})^i P1_A\rangle / \mu P^n(A) . \end{aligned}$$

The P standing before summation on i will be represented as $P = PT_A + PT_{A'}$. We also use $\sum_{i=0}^{I} (PT_{A'})^i P1_A \leqq 1$ and obtain:

$$\begin{aligned} M^{-1} c_{Nn1}(f) &\leqq \textstyle\sum_{i=0}^{I} \langle \mu, \sum_{j=N+1}^{n-1} P^{n-j} T_A(PT_{A'})^{j+i+1} P1_A\rangle / \mu P^n(A) \\ &\quad + \langle \mu, \textstyle\sum_{j=N+1}^{n-1} P^{n-j} T_A(PT_{A'})^j P1_A\rangle / \mu P^n(A) \\ &\quad + \textstyle\sum_{i=0}^{I} \langle \mu, (PT_{A'})^{n+i+1} P1_A\rangle / \mu P^n(A) + \langle \mu, (PT_{A'})^n P1_A\rangle / \mu P^n(A) \\ &= \textstyle\sum_{i=0}^{I} c_{N+i+1,n,i+2}(1_A) + c_{Nn1}(1_A) . \end{aligned}$$

Since $a_{nr}(1_A) \to 1$, $\lim_N \limsup_{n\to\infty} c_{Nnr}(1_A) = 0$. Hence $\lim_{n\to\infty} \limsup_{n\to\infty} c_{Nn1}(f) = 0$. We have proved $\langle \mu P^{n+1}, f\rangle/\mu P^n(A) \to \langle \lambda, f\rangle/\lambda(A)$ for $f \in L_\infty(A_k)$, and the theorem now follows directly, since $\mu P^{n+1}(A)/\mu P^n(A) \to 1$. □

REMARK. Krengel's Example 3.1 in [10] shows that the assumptions of Theorem 3.1 *do not* imply that $\langle \mu P^n, f\rangle/\mu P^n(A) \to \langle \lambda, f\rangle/\lambda(A)$ for *every* $f \in L_\infty(X)$ supported in a set of finite λ-measure. In that example P is a Markov chain on $\{0, 1, 2, \cdots\}$ having the strong ratio limit property, μ is the Dirac measure at 0, $A = \{0\}$, f is the characteristic function of a set of finite invariant measure, and $\lim_{n\to\infty} \sup \langle \mu P^n, f\rangle / \mu P^n(A) = \infty$.

EXAMPLE 3.1. The conditions of Theorem 3.1 are satisfied for a Harris process of period 2 and some μ. (However, with initial distribution μ the chain is *aperiodic*).

Let P be the symmetric random walk on the integers with $P_{i\,i\pm1} = \frac{1}{2}$. Define $A = \{0, 1\}$ and $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ where δ_i is the Dirac measure at i. The invariant measure gives mass 1 to each i. P has period 2, but simple calculations show:

$$\mu P^{2m}(A) = \binom{2m}{m} 2^{-2m}\,; \qquad \mu P^{2m+1}(A) = \binom{2m+1}{m} 2^{-2m-1}$$

$$B = \{0\} \Rightarrow \mu P^{2m}(B) = \tfrac{1}{2}\binom{2m}{m} 2^{-2m}\,; \qquad \mu P^{2m+1}(B) = \tfrac{1}{2}\binom{2m+1}{m} 2^{-2m-1}$$

and conditions of Theorem 3.1 are easily checked.

For a Harris process the sets A_k can be identified, as done by Jain. We shall need the following lemma.

LEMMA 3.2. *If $A \in \Sigma$ then for every $j \geqq 1$ $\sum_{m=0}^{j-1} (PT_{A'})^m P1_A \geqq P^j 1_A$.*

PROOF. By induction. For $j = 1$ we have equality.

$$\begin{aligned} P^{j+1} 1_A \leqq P \textstyle\sum_{m=0}^{j-1} (PT_{A'})^m P1_A &= \textstyle\sum_{m=1}^{j} (PT_{A'})^m P1_A \\ &+ PT_A \textstyle\sum_{m=0}^{j-1} (PT_{A'})^m P1_A \leqq \sum_{m=1}^{j} (PT_{A'})^m P1_A + P1_A\,. \end{aligned}$$

The last inequality follows from $\sum_{m=0}^{\infty} (PT_{A'})^m P1_A \leqq 1$ (see [4] Lemma VI.B).

COROLLARY 3.1. *Let P be an ergodic Harris process, and let $A \in S$ satisfy the conditions of Theorem* 3.1 *with the measure μ.*

(a) *If P is aperiodic, then for G, $H \in S$ and any integer r*

$$\lim_{n\to\infty} \mu P^{n+r}(G)/\mu P^n(H) = \lambda(G)/\lambda(H)\,.$$

(b) *If P is periodic and $A \subset F \in S$ then the above convergence holds for $G, H \subset F$.*

PROOF. (b) $\Rightarrow$ (a) since for P aperiodic $A \cup G \cup H = F \in S$ by Levitan's [13], Theorem 2.2.

We prove (b). $F \in S$ so there exist an $\varepsilon > 0$ and $k > 0$ such that $q_k(x, y) \geqq \varepsilon$ for every $x, y \in F$. By Lemma 3.2

$$\sum_{m=0}^{k-1} (PT_{A'})^m P1_A(x) \geqq P^k 1_A(x) \geqq Q_k 1_k(x) = \int q_k(x, y) 1_A(y) \lambda(dy)\,.$$

If $x \in F$ the right-hand side $\geqq \varepsilon\lambda(A)$ since $A \subset F$. Thus

$$\sum_{m=0}^{k-1} (PT_{A'})^m P1_A \geqq \varepsilon\lambda(A) 1_F$$

and F has the property required from B_k in the proof of Theorem 3.1.

REMARKS. (1) If P is Harris and given by a transition possibility, (i.e. μP is defined for every μ), then if μ and A are as in Theorem 3.1 (even if $\mu \perp m$) then the corollary may be applied. This is because for Harris processes

$$\lambda(A) > 0 \Rightarrow \sum_{m=0}^{\infty} (PT_{A'})^m PT_A 1(x) = 1 \qquad \text{for } every\ x,$$

Lemmas 3.2, 3.1 hold *everywhere* and and the proof is the same.

(2) Corollary 3.1 is stronger than Levitan's result [13], Theorem 3.2, where it is proved (for P induced by a transition probability) only for μ a Dirac measure of a point $q \in F$.

(3) Corollary 3.1 does not follow from Jain's results [9], as Jain treats only the aperiodic case and Example 3.1 shows that the periodic case also happens.

EXAMPLE 3.2. The conditions of Theorem 3.1 are satisfied for a point transformation preserving an infinite measure (so it is not a Harris process).

Let $X = \{0, 1, \cdots\}$ and let P be a conservative and ergodic Markov chain on X preserving an infinite measure λ and having the strong ratio limit property. Let Ω be the infinite product space $\prod_{n=0}^{\infty} X$, Σ the product σ-algebra and T the shift transformation on Ω. T is conservative and ergodic and preserves an infinite σ-finite measure [14] Section 4. Take $A = \{0\} \times X \times X, \cdots$, and μ the probability measure on Ω corresponding to the initial probability concentrated at 0. Using (4.3) and (4.1) of [14] together with the strong ratio limit property of P it is not difficult to show that μ, T and A satisfy the conditions of Theorem 3.1.

If P is ergodic and conservative with invariant measure λ, then the adjoint process P^* is defined, is conservative and ergodic and has the same invariant measure [4], Chapter VII. If $A \in \Sigma$ satisfies $0 < \lambda(A) < \infty$, we wish to apply Theorem 3.1 to P and P^* simultaneously. In order to do so, we assume that the conditions of Theorem 3.1 hold for P and P^*. Since going from P to P^* interchanges the roles of functions and measures, the conditions for P^* can be stated in terms of the action of P on measures supported in A (with bounded Radon-Nikodym derivatives with respect to λ).

Combining all the conditions together in terms of P we obtain the conditions of the following theorem, from which we derive Orey's theorem. In order to simplify the statement of the theorem we drop the fact that the conditions need hold only for measures η supported in A with $d\eta/d\mu$ bounded.

THEOREM 3.2. *Let P be a conservative and ergodic Markov process with invariant measure λ, and let $A \in \Sigma$ with $0 < \lambda(A) < \infty$. Let $\mu \ll \lambda$ be a probability measure supported in A such that for every probability measure η supported in A*

$$\lim_{n\to\infty} \eta P^{n+1}(B)/\mu P^n(A) = \lambda(B)/\lambda(A) \qquad \text{for} \quad B \subset A\,.$$

Then for every finite measure $\nu \ll \lambda$ there is a sequence of measures $\nu_j \uparrow \nu$, satisfying

$$\lim_{n\to\infty} \langle \nu_j P^{n+1}, f\rangle/\mu P^n(A) = \nu_j(X) \langle \lambda, f\rangle/\lambda(A)$$

for every $f \in L_\infty(A_k)$. ($\{A_k\}$ is defined in Theorem 3.1.)

SKETCH OF PROOF. The conditions of the theorem imply that μ can be replaced by any probability measure supported in A, so we may assume $d\mu/d\lambda = \lambda(A)^{-1}1_A$. Also, every $\eta \ll m$ supported in A clearly satisfies the conditions of Theorem 3.1.

For $f \in L_\infty(A_k)$. Let $\tilde{\mu}$ be defined by $d\tilde{\mu}/d\lambda = f$. Put $d\eta_B/d\lambda = \lambda(B)^{-1}1_B$.

$$\frac{\tilde{\mu}P^{*n+1}(B)}{\tilde{\mu}P^{*n}(A)} = \frac{\int P^{*n+1}1_B \cdot f d\lambda}{\int P^{*n}1_A \cdot f d\lambda} = \frac{\lambda(B)\langle \eta_B P^{n+1}, f\rangle}{\lambda(A)\langle \mu P^n, f\rangle} \to_{n\to\infty} \frac{\lambda(B)}{\lambda(A)}$$

for $B \subset A$, so by Theorem 3.1 applied to P^* we have $B_k \uparrow X$ with $\lambda(B_k) < \infty$ and

$$\langle \tilde{\mu}P^{*n+1}, u\rangle / \langle \tilde{\mu}P^{*n}, v\rangle \to_n \langle \lambda, u\rangle/\langle \lambda, v\rangle \qquad \text{for} \quad 0 \leqq u, v \in L_\infty(B_k)\,.$$

($\{B_k\}$ depends only on A.) To conclude the proof take $v = d\mu/d\lambda$ and

$$u_j = \frac{d\nu_j}{d\lambda} \in L_\infty(B_k) \quad \text{with } u_j \uparrow \frac{d\nu}{d\lambda} \qquad \text{and use}$$

$$\langle \mu P^n, f\rangle/\mu P^n(A) \to \langle \lambda, f\rangle/\lambda(A)\,.$$

COROLLARY 3.2. *Under the assumptions of the previous theorem, there exists a sequence* $D_k \uparrow X$, $D_0 = A$, $\lambda(D_k) < \infty$ *such that*

$$\lim_{n\to\infty} \frac{\int u \cdot P^{n+r} f d\lambda}{\int v \cdot P^n g d\lambda} = \frac{\langle \lambda, u\rangle\langle \lambda, f\rangle}{\langle \lambda, v\rangle\langle \lambda, g\rangle}$$

for every $u, v, f, g \in L_\infty(D_k)$ *and integer* r.

PROOF. We define $D_k = A_k \cap B_k$. All the sets depend only on A. If $d\nu = u d\lambda$ and $d\tau = v d\lambda$, then from the previous proof, we have

$$\frac{\int uP^{n+r} f d\lambda}{\int vP^n g d\lambda} = \frac{\nu\langle P^{n+r}, f\rangle}{\langle \mu P^{n+r-1}, f\rangle} \prod_{j=n}^{n+r-2} \frac{\langle \mu P^{j+1}, f\rangle}{\langle \mu P^j, f\rangle} \cdot \frac{\langle \mu P^n, f\rangle}{\langle \mu P^{n-1}, g\rangle} \cdot \frac{\langle \mu P^{n-1}, g\rangle}{\langle \tau P^n, g\rangle}$$

which tend to

$$\frac{\nu(X)}{\tau(X)} \frac{\langle \lambda, f\rangle}{\langle \lambda, g\rangle} = \frac{\langle \lambda, u\rangle\langle \lambda, f\rangle}{\langle \lambda, v\rangle\langle \lambda, g\rangle}\,.$$

COROLLARY 3.3. *Let P be an aperiodic Harris process given by a transition probability, and let $F \in S$ satisfy Jain's condition* (F) [9]: *There exists a point $\theta \in F$ such that for $x \in F$ and $E \subset F$*

$$\lim_{n\to\infty} P^{n+1}(x, E)/P^n(\theta, F) = \lambda(E)/\lambda(F)\,.$$

Then for any two probability measures η, $\nu \ll \lambda$ with $d\eta/d\lambda$ and $d\nu/d\lambda$ bounded with supports in S and any $G, H \in S$

$$\lim_{n\to\infty} \nu P^{n+r}(G)/\eta P^n(H) = \lambda(G)/\lambda(H)\,.$$

PROOF. We first show the existence of a set $A \subset F$ satisfying the conditions of Theorem 3.2.

Condition (F) implies $P^{n+1}(s, F)/P^n(\theta, F) \to 1$ on F so by Egoroff's theorem there is a set $A \subset F$ on which the convergence is uniform, and especially

$$\sup_{n \geqq n_0} \sup_{x \in A} P^{n+1}(x, F)/P^n(\theta, F) < \infty .$$

Hence for μ a probability measure supported in A and $B \subset A$ we obtain (by Lebesgue's theorem)

$$\begin{aligned}\lim_n \mu P^{n+1}(B)/P^n(\theta, F) &= \lim_n \int_A P^{n+1}(x, B)/P^n(\theta, F)\mu(dx) \\ &= \int_A \{\lim_n P^{n+1}(x, B)/P^n(\theta, F)\}\mu(dx) = \lambda(B)/\lambda(F) .\end{aligned}$$

Hence A satisfies the hypothesis of Theorem 3.2.

Since P^* is also an aperiodic process and the kernels are given by $q_k^*(x, y) = q_k(y, x)$ [4], Chapter VII, we have $S(P) = S(P^*)$. For given $G, H \in S$ we can choose A_k to include $A \cup G \cup H \in S$ (see Corollary 3.1), hence the required result now follows Corollary 3.2.

COROLLARY 3.4. (Orey's Theorem [12].) *Let X be the set of nonnegative integers and $P_{ij}^{(n)}$ the n-step transition probabilities of an ergodic conservative Markov chain.*
If $\lim_{n\to\infty} P_{00}^{(n+1)}/P_{00}^{(n)} = 1$, *then for every* $i, j, m, h \geqq 0$ *and integer* r $P_{ij}^{(n+r)}/P_{mh}^{(n)} \to \lambda_j/\lambda_h$ *where* $\{\lambda_i\}$ *is the invariant measure of the chain.*

REMARKS. (1) Corollary 3.3 does not follow from [9] as it follows only from condition (F). Orey's theorem follows directly.

(2) If P is an aperiodic Harris process without a transition probability condition (F) is modified as follows (and Corollary 3.3 applies): There exists a probability measure $\mu_0 \ll \lambda$ supported in F such that for $E \subset F$

$$\lim_{n\to\infty} P^{n+1} 1_E(x)/\mu_0 P^n(F) = \lambda(E)/\lambda(F) \quad \text{a.e. on} \quad F .$$

(3) Krengel's Example 3.1 of [10] cited above shows that there are probability measures ν for which $\limsup_{n\to\infty} \nu P^n(A)/\mu P^n(A) = \infty$, even when the conditions of Theorem 3.2 are satisfied: Take for P the adjoint of that example, $d\nu/d\lambda = f/\langle\lambda, f\rangle$, $A = \{0\}$ and $d\mu/d\lambda = 1_A/\lambda(A)$. Thus

$$\frac{\nu P^n(A)}{\mu P^n(A)} = \frac{\lambda(A)}{\langle\lambda, f\rangle} \cdot \frac{\langle f P^n, 1_A\rangle}{\langle 1_A P^n, 1_A\rangle} = \frac{\lambda(A)\langle 1_A P^{*n}, f\rangle}{\langle\lambda, f\rangle\langle 1_A P^{*n}, 1_A\rangle} .$$

(Here $\langle u, f\rangle = \int u f d\lambda$). The lim sup of the right-hand side is ∞, as the adjoint P^* is the process of Krengel's example.

THEOREM 3.3. *Let P, λ, A and μ satisfy the hypothesis of Theorem 3.2, and $\{A_k\}$ as in Theorem 3.1. If $\nu \ll \lambda$ is a probability measure satisfying* $\limsup_{n\to\infty} \nu P^{n+1}(A)/\mu P^n(A) \leqq 1$, *then for every* $f, g \in L_\infty(A_k)$ *and integer* r

$$\lim_{n\to\infty} \langle \nu P^{n+r}, f\rangle / \langle \mu P^n, g\rangle = \langle \lambda, f\rangle / \langle \lambda, g\rangle \, .$$

PROOF. It is enough to assume $g = 1_A$ and $r = 1$ by virtue of Theorem 3.1. Let $\nu_j \uparrow \nu$ be the sequence of measures defined by Theorem 3.2.

$$\liminf_{n\to\infty} \frac{\langle \nu P^{n+1}, f\rangle}{\mu P^n(A)} \geqq \lim_{n\to\infty} \frac{\langle \nu_j P^{n+1}, f\rangle}{\mu P^n(A)} = \frac{\nu_j(X)\langle \lambda, f\rangle}{\lambda(A)} \, .$$

Letting $j \to \infty$ $\liminf \langle \nu P^{n+1}, f\rangle / \mu P^n(A) \geqq \langle \lambda, f\rangle / \lambda(A)$. If $0 \leqq f \leqq 1$ is in $L_\infty(A)$, then this last result is applied to $1_A - f$, and using the hypothesis

$$\limsup_{n\to\infty} \frac{\langle \nu P^{n+1}, f\rangle}{\mu P^n(A)} = \limsup_{n\to\infty} \left\{ \frac{\nu P^{n+1}(A)}{\mu P^n(A)} - \frac{\langle \nu P^{n+1}, 1_A - f\rangle}{\mu P^n(A)} \right\}$$

$$\leqq 1 - \liminf \frac{\langle \nu P^{n+1}, 1_A - f\rangle}{\mu P^n(A)} \leqq 1 - \frac{\langle \lambda, 1_A - f\rangle}{\lambda(A)} = \frac{\langle \lambda, f\rangle}{\lambda(A)} \, .$$

Hence $\langle \mu P^{n+1}, f\rangle / \mu P^n(A) \to \langle \lambda, f\rangle / \lambda(A)$ for $f \in L_\infty(A)$. Since $\langle \mu P^{n+1}, f\rangle / \mu P^n(A) \to \langle \lambda, f\rangle / \lambda(A)$, we may repeat the proof of Theorem 3.1, with ν in the numerator instead of μ, and the result is thus extended to each A_k.

REMARK. Our Theorem 3.3 is similar to Jain's result [9], Theorem 1', but we are not concerned with Dirac measures, for which P is not necessarily defined, nor do we assume condition (C). Our proof is entirely analytic, although Lemma 3.1 has a probabilistic meaning. The next theorem is similar to [9], Theorem 3, with application of the above remark.

THEOREM 3.4. *Let P conservative and ergodic Markov process with invariant (σ-finite) measure λ, and let $A \in \Sigma$ satisfy $0 < \lambda(A) < \infty$. The convergence*

$$\lim_{n\to\infty} \langle \nu P^{n+r}, f\rangle / \langle \eta P^n, g\rangle = \langle \lambda, f\rangle / \langle \lambda, g\rangle$$

holds for every integer r, $0 \leqq f, g \in L_\infty(A_k)$ and any two probability measures ν and η weaker than λ if and only if there exists a probability measure μ supported in A such that:

(I) *μ satisfies the hypothesis of Theorem* 3.2.

(II) *For every probability measure $\nu \ll \lambda$*

$$\limsup_{n\to\infty} \nu P^n(A) / \mu P^n(A) < \infty \, .$$

PROOF. Take any probability measure μ supported on A, then (I), (II) are clearly necessary ($A_0 = A$).

Sufficiency: Let μ satisfy (I) and (II), then $\mu P^n(A) > 0$ for n greater than some fixed N (since $\mu P^{n+1}(A) / \mu P^n(A) \to 1$). For $n \geqq N$ we define $\phi_n(x) = P^n 1_A(x) / \mu P^n(A)$. We identify $L_1(\lambda)$ with the space of measures $\ll \lambda$ and consider ϕ_n as a functional on $L_1(\lambda)$. By (II) $\{\langle \nu, \phi_n\rangle : n \geqq N\}$ in bounded for every $\nu \ll \lambda$, and thus the functionals on $L_1\{\phi_n \colon n \geqq N\}$ are bounded at every element in $L_1(\lambda)$. By Theorem 3.2 $\lim_{n\to\infty} \langle \nu, \phi_n\rangle = \nu(X)$ for ν in a dense subspace of $L_1(\lambda)$. Hence the conditions of Theorem II. 3.6 of [1] are satisfied, and $\lim \langle \nu, \phi_n \rangle$ exists for every measure ν, and by continuity of the limit is $\nu(X)$. Hence if ν is a probability measure, it satisfies the hypothesis of Theorem 3.3 (remembering $\mu P^{n+1}(A)/\mu P^n(A) \to 1$). This clearly implies the needed convergence. □

REMARK. Condition (II) can be replaced by:
(II)′ There exist integers N and M such that

$$P^n 1_A(x)/\mu P^n(A) \leqq M \text{ a.e.} - \lambda \qquad \text{for} \quad n \geqq N .$$

(II)′ $\Rightarrow$ (II) by integrating with respect to ν, and (II) $\Rightarrow$ (II)′ by the principle of uniform boundedness (the $\{\phi_n\}$ functionals are bounded at any point, and thus $\{||\phi_n||: n \geqq N\}$ is bounded).
It is condition (II)′ which appears in [9], Theorem 3. Jain's theorem can be proved as a corollary. We omit the rather lengthy proof.

We now show an application of Theorem 3.4 to non-Harris processes.

DEFINITION 3.1. An ergodic Markov process P with finite invariant measure $\lambda \sim m$ (and we assume $\lambda(X) = 1$) *is mixing* if $\langle P^n 1_B, 1_A \rangle \to \lambda(B)\lambda(A)$ for every $A, B \in \Sigma$. (P is conservative by [3] page 38.)

THEOREM 3.5. *Let P be an ergodic Markov process with λ an invariant Probability measure m. If there exists $A \in \Sigma$ with $\lambda(A) > 0$ such that $\langle P^n 1_B, 1_B \rangle \to \lambda(B)^2$ for every $B \subset A$, then P is mixing.*

PROOF. If $B \subset A$ then the condition implies $\langle P^n(1_B - \lambda(B)), 1_B - \lambda(B) \rangle \to 0$ so by a theorem of Foguel [2] $P^n 1_B \to \lambda(B)$ weakly in $L_2(\lambda)$ (P is a contraction of $L_2(\lambda)$ by [4] Chapter VII). Hence for any $\mu \ll \lambda$ a probability measure if $d\mu/d\lambda \in L_2$ then $\mu P^n(B) \to \lambda(B)$. By standard approximations this is true for any μ. Fix μ supported in A, and the conditions of Theorem 3.4 are clearly satisfied. Hence we have a sequence $A_k \uparrow X$ such that for any $\nu \ll \lambda$ and $E \subset A_k$ we have $(\nu(X) = 1) \nu P^n(E)/\mu P^n(A) \to \lambda(E)/\lambda(A)$. $\mu P^n(A) \to \lambda(A)$ so $\nu P^n(E) \to \lambda(E)$. If $E \in \Sigma$, put $E_k = E \bigcap A_k$ so $E_k \uparrow E$. Suppose $d\nu/d\lambda$ is bounded. Standard approximations in L_2 yield $\nu P^n(E) \to \lambda(E)$. Take for $F \in \Sigma$ $d\nu/d\lambda = \lambda(F)^{-1} 1_F$ so

$$\langle P^n 1_E, 1_F \rangle = \lambda(F) \nu P^n(E) \to \lambda(E)\lambda(F) .$$

Thus P is mixing.

REMARK. The last theorem applies also to point transformations. If $\lambda(X) = \infty$ then no point tranformation satisfies the conditions of Theorem 3.4, as can be shown using Corollary 1 of [7].

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