FUNCTIONS OF ORDER STATISTICS

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Two theorems on the asymptotic normality of linear combinations of functions of order statistics are given. Theorem 1 requires a “smooth” scoring function but the underlying df need not be continuous even and can also depend on the sample size. Theorem 2 allows general scoring functions but places additional restrictions on the df. Applications included.

0. Introduction. Let \( \mathcal{G} \) denote the class of left continuous functions on \( (0, 1) \) that are of bounded variation on \( (\theta, 1 - \theta) \) for all \( \theta > 0 \). (Associated with any \( g \) in \( \mathcal{G} \) is a Lebesgue-Stieltjes signed measure \( g \) on \( (0, 1) \) and its total variation measure \( |g| \).) Let \( c_{n1}, \ldots, c_{nn} \) for \( n \geq 1 \) be known constants. Let \( d_{n1}, \ldots, d_{nn} \) for \( n \geq 1 \) be known constants of a greater order of magnitude that will be associated with the points \( 0 < p_1 < \cdots < p_n < 1 \). Let \( [\cdot] \) denote the greatest integer function. We will consider the limiting distribution of

\[
T_n = n^{-1} \sum_{i=1}^{n} c_{ni} g_n(\xi_{ni}) + \sum_{k=1}^{n} d_{nk} g_n(\xi_{n[n(k-1)+1]})
\]

where each \( g_n \) is a function in \( \mathcal{G} \) and where \( 0 \leq \xi_{n1} \leq \cdots \leq \xi_{nn} \leq 1 \) are the order statistics of \( n \) independent Uniform \((0, 1)\) rv’s.

Remark 1. If \( g_n = h(F_n^{-1}) \) for some sequence of df’s \( F_n \), in the class \( \mathcal{G} \) of all df’s, then \( T_n \) has the same distribution as does \( n^{-1} \sum_{i=1}^{n} c_{ni} h(\alpha_{ni}) + \sum_{k=1}^{n} d_{nk} h(\alpha_{n[n(k-1)+1]}) \) where \( \alpha_{n1} \leq \cdots \leq \alpha_{nn} \) are the order statistics of a random sample of size \( n \) from \( F_n \).

For the mean or trimmed mean we have \( \kappa = 0 \), but for the Winsorized mean \( \kappa = 2 \) and \( p_1 \) and \( 1 - p_n \) are the Winsorizing percentages.

Theorem 1 and 2 below are both major theorems. Theorem 1 seems to be the first good theorem for general \( g \), though it does require somewhat regular scores. Nonetheless, the rather minor Example 1 is probably sufficient for most applications. Theorem 2 employs a different representation. It allows very general scores, though \( g \) must now satisfy smoothness conditions. This second theorem is the one that relates to other theorems for this problem; in particular, it is an improvement on the theorems of [4]. See [4] for a discussion of the literature. The present Theorem 2 and the theorems of [1] and [8] are approximately equally strong; though [1] is probably slightly weaker.

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412
Theorem 1 and its Corollary 2 are much cleaner than any of these. Never use Theorem 2 unless Corollary 2 and Theorem 1 fail.

Theorem 1, Example 1 and Corollary 2 are the main results. Also the parallel paper [7] applies the present technique to give the best results yet obtained for the trimmed and Winsorized means.

We stress that the df $F$ need not be continuous.

1. **The first main theorem.** Define a function $J_n$ on $[0, 1]$ to equal $c_{ni}$ for $(i - 1)/n < t \leq i/n$ and $1 \leq i \leq n$ with $J_n(0) = c_{n1}$. Let $d_1, \ldots, d_k$ be finite nonzero constants. We will consider $n^k(T_n - \mu_n)$ where

$$
\mu_n = \int_0^1 g_n J_n dI + \sum_{k=1}^k d_k g_n(p_k);
$$

here $I$ denotes the identity function and $\int \cdot dI$ denotes integration with respect to Lebesgue measure.

For fixed $b_1, b_2$ and $M$ define a "scores bounding function" $B$ by

$$
B(t) = Mt^{-b_1}(1 - t)^{-b_2} \quad \text{for } 0 < t < 1.
$$

For fixed $\delta > 0$ define

$$
D(t) = Mt^{-i + b_1 + \delta}(1 - t)^{-i + b_2 + \delta} \quad \text{for } 0 < t < 1.
$$

Let $g$ denote a fixed function in $\mathcal{G}$ and let $J$ denote a fixed measurable function on $(0, 1)$. Define $q(t) = [t(1 - t)]^{1/2}$ on $[0, 1]$.

**Assumption 1.** (Boundedness) Let $|g| \leq D$, all $|g_n| \leq D$, $|J| \leq B$ and all $|J_n| \leq B$ on $(0, 1)$.

**Assumption 2.** (Smoothness) Except on a set of $t$'s of $|g|$-measure 0 we have both $J$ is continuous at $t$ and $J_n \rightarrow J$ uniformly in some small neighborhood of $t$ as $n \rightarrow \infty$.

**Assumption 3.** (Convergence) $\int_0^1 Bqd|g_n - g| \rightarrow 0$ as $n \rightarrow \infty$.

**Assumption 4.** ($\kappa > 0$) For $1 \leq k \leq \kappa$ we have $n^k(d_{nk} - d_k) \rightarrow 0$ as $n \rightarrow \infty$.

In some small neighborhood of each of $p_1, \ldots, p_\kappa$ the functions $g_n'$ for $n \geq 1$ form an equiuniformly continuous family and $g_n(p_k) \rightarrow g(p_k)$ and $g_n'(p_k) \rightarrow g'(p_k)$ as $n \rightarrow \infty$ for each $1 \leq k \leq \kappa$. (If $g_n = g$ for all $n$ we require at each of $p_1, \ldots, p_\kappa$ only that $g'(p_k)$ exist.)

Define

$$
\sigma^2 = \sum_{k=1}^k \sum_{j=1}^k (s \wedge t - st)J(s)J(t) dg(s) dg(t) + 2 \sum_{k=1}^k d_k g'(p_k) \int (t \wedge p_k - tp_k) J(t) d(t) + \sum_{j=1}^k \sum_{k=1}^k d_k g'(p_j) g'(p_k)(p_j \wedge p_k - p_j p_k).
$$

**Theorem 1.** If Assumptions 1, 2, 3 and 4 hold, then

$$
n^k(T_n - \mu_n) \rightarrow_d N(0, \sigma^2)
$$

with $\mu_n$ of (2) and $\sigma^2$ of (3) finite.
PROOF. We may assume that $0 < \xi_{n1} < \cdots < \xi_{nn} < 1$ are the special Uniform (0, 1) order statistics described in the Appendix. These have empirical df $\Gamma_n$, empirical process $U_n = n^t(\Gamma_n - I)$ on $[0, 1]$ and satisfy $\rho(U_n, U) \to_{a.s.} 0$ as $n \to \infty$; here $U$ is a special Brownian bridge and $\rho$ denotes the uniform metric. Other properties of these special $\xi_{ni}$’s and $U$ will be cited from the Appendix as needed.

We give the proof first for $\kappa = 0$. For $0 \leq t \leq 1$ define

$$\phi_n(t) = -\int_t^1 J_n dI$$

so that $c_{ni}/n = [\phi_n(i/n) - \phi_n((i - 1)/n)]$

for $1 \leq i \leq n$.

Summing by parts gives

$$T_n = \int_0^1 \phi_n(\Gamma_n^{-1})J_n dI = \sum_{i=0}^{n-1} g_n(\xi_{ni})[\phi_n(i/n) - \phi_n((i - 1)/n)]$$

$$= -\phi_n(0)g_n(\xi_{n1}) - \sum_{i=1}^{n-1} \phi_n(i/n)[g_n(\xi_{ni+1}) - g_n(\xi_{ni})]$$

$$= a.s. -\phi_n(0)g_n(\xi_{n1}) - \int_{\xi_{n1}}^{\xi_{nn}} \phi_n(\Gamma_n) d\mu_n$$

where the second integral representation uses the fact that it is a.s. true that no $\xi_{ni}$ takes on one of the countable number of values at which $g_n$ is discontinuous. Integration by parts gives

$$\mu_n = \int_0^1 g_n d\phi_n + \int_{\xi_{n1}}^{\xi_{nn}} g_n J_n dI$$

$$= a.s. -\int_{\xi_{n1}}^{\xi_{nn}} g_n d\phi_n + \phi_n(\xi_{n1})g_n(\xi_{n1}) - \phi_n(\xi_{nn})g_n(\xi_{nn}) + \int_{\xi_{n1}}^{\xi_{nn}} \phi_n(\Gamma_n) d\mu_n.$$ 

Thus

$$n^t(T_n - \mu_n) = -(S_n + \gamma_{n1} + \gamma_{n2} + \gamma_{n3})$$

where

$$S_n = \int_{\xi_{n1}}^{\xi_{nn}} A_n U_n d\mu_n = \int_0^1 A_n^* U_n d\mu_n$$

with $A_n = [\phi_n(\Gamma_n) - \phi_n]/(\Gamma_n - I)$, where $A_n^*$ is equal to the difference quotient $A_n$ on $[\xi_{n1}, \xi_{nn}]$ and is equal to 0 otherwise, and where

$$\gamma_{n1} = n^t g(\xi_{n1})[\phi_n(0) - \phi_n(\xi_{n1})], \quad \gamma_{n2} = n^t g(\xi_{nn})\phi_n(\xi_{nn}),$$

$$\gamma_{n3} = n^t \int_{\xi_{n1}}^{\xi_{nn}} \phi_n(\Gamma_n) d\mu_n.$$ 

(For each fixed $\omega$ in the probability space, at the at most $n$ points where $\Gamma_n = I$ we define $A_n$ to be 0.)

Define

(4) \[ S = \int_0^1 JU d\mu + \sum_{k=1}^n d_k g'(p_k)U(p_k) \]

so that $S$ is a $N(0, \sigma^2)$ rv; and $\sigma^2$ is finite by Assumption 1. (We remind the reader that we still have $\kappa = 0$ for the time being.)

Let $\chi_{ne}$ denote the indicator function of the set $S_{n,e}$ of Lemma A. 3; and
note also the definition of \( \hat{\beta}_t \) made there. Now
\[
|\chi_{sz} S_n - S| \leq \gamma_n + \gamma_n \equiv \int_0^1 |\chi_{sz} A_n \ast U_n | d|g_n - g| + \int_0^1 |\chi_{sz} A_n \ast U_n - JU| d|g|.
\]
Also, by Assumption 1, when \( b_1, b_2 > 0 \) we have
\[
|A_n| = |\int_0^1 J_x dI/(\Gamma_n - I)| \leq \int_0^1 B dI/(\Gamma_n - I) \leq B \vee B(\Gamma_n);
\]
and since \( B \) is a reproducing \( u \)-shaped function (see Definition A. 3), Lemma A. 3 gives the key equation of this proof
\[
(5) \quad \chi_{sz} |A_n \ast| \leq B_{\beta_t} \leq M \beta B
\]
for some constant \( M \) and for \( B_{\beta_t} \) as in Definition A. 3. Clearly, (5) also holds for all other values of \( b_1 \) and \( b_2 \).

For a.e. fixed \( \omega \) the functions \( U \) and \( U_n \) for \( n \geq 1 \) are uniformly bounded by \( M_{\omega \eta} \); see Remark A. 5. Thus a.e.
\[
|\gamma_n| \leq M_\omega M \int_0^1 B d|g_n - g| \to 0 \quad \text{as} \quad n \to \infty
\]
using (5) and Assumption 3. Thus \( \gamma_n \to_{a.e.} 0 \) as \( n \to \infty \).

Integrate by parts and use Assumption 1 to find \( \int_0^1 B d|g| < \infty \). For fixed \( \omega \) the integrand of \( \gamma_n \) is dominated by the \( |g| \)-integrable function \((M_\omega + 1) M_{\omega \eta} B_{\eta} \); and the integrand converges a.e. \( |g| \) to 0 since \( \rho(U_n, U) \to_{a.e.} 0 \) as \( n \to \infty \) and since \( A_n(t) \to J(t) \) a.e. \( |g| \) as \( n \to \infty \) by Assumption 2 and the representation
\[
A_n = \int_0^1 J_x(s) d|s|/(\Gamma_n - I).
\]
Thus \( \gamma_n \to_{a.e.} 0 \) as \( n \to \infty \) by using the dominated convergence theorem once for each \( \omega \).

Thus \( \chi_{sz} S_n \to_{a.e.} S \) as \( n \to \infty \). Thus \( S_n \to_{\omega} S \) as \( n \to \infty \); since \( E\chi_{sz} \to 1 \) as \( n \to \infty \) for each \( \varepsilon > 0 \) by Lemma A. 3.

In case \( \varepsilon = 0 \) it remains only to show that \( \gamma_{s1}, \gamma_{s2} \) and \( \gamma_{s3} \) are negligible. Case 1: \( b_1 < 1 \). Then
\[
|\gamma_{s1}| \leq n^4 D(\xi_{s1}) \int_0^1 B dI \quad \text{so that on the event} \quad \xi_{s1} < (\frac{1}{2} \wedge p_i) \quad \text{we have by Assumption 1 that} \quad |\gamma_{s1}| \leq M n^4 \xi_{s1}^{1+\delta} \quad \text{for some constant} \quad M \in.
\]
Since \( n^4 \xi_{s1} = O_p(1) \) and since \( P(\xi_{s1} < (\frac{1}{2} \wedge p_i)) \to 1 \) as \( n \to \infty \), we have \( \gamma_{s1} \to_{\omega} 0 \) as \( n \to \infty \). Case 2: \( b_1 \geq 1 \). Then on the event \( \xi_{s1} < (\frac{1}{2} \wedge p_i) \) we have
\[
|\gamma_{s1}| = |\gamma_{s1}|^{1-\delta}_n \int_0^1 |\gamma_{s1}|^{1+\delta}_n c_{s1} \leq M_\delta [n^{\delta - 1} + n^{-1} \sum n^{\epsilon} \xi_{s1}^{(i)(i(n)^{-\delta})}] \leq M_\delta [n^{\delta - 1} + \int_0^t t^{-\delta} \, dt \cdot \xi_{s1}^{1+\delta}] \]
(replace \( t^{-\delta} \) by \( \log t \) when \( b_1 = 1 \)) and also
\[
|\gamma_{s1}| \leq M n^4 \xi_{s1}^{1+\delta} + \xi_{s1}^{1+\delta} = M \xi_{s1}^{1+\delta} \frac{n^4 \xi_{s1}^{1+\delta}}{(n^4 \xi_{s1}^{1+\delta})^i}
\]
for constants \( M_\delta, M_\eta, M_\zeta \). So again \( \gamma_{s1} \to_{\omega} 0 \) as \( n \to \infty \). Analogously \( \gamma_{s2} \to_{\omega} 0 \) as \( n \to \infty \) by considering the cases \( b_2 < 1 \) and \( b_2 \geq 1 \). Also \( n^4 \int_0^1 |g_j J_x| d|s| \leq n^4 \int_0^1 M [t(1 - t)]^{1+\delta} dt \to_{\omega} 0 \) as \( n \to \infty \); so that \( \gamma_{s3} \to_{\omega} 0 \) as \( n \to \infty \) without resort to cases.
This completes the proof for $\kappa = 0$. Assumption 4 was not used.

Suppose now $\kappa > 0$. Now $\gamma_{n1} + \gamma_{n2} + \gamma_{n3} \to 0$ as $n \to \infty$ by the above proof. Without loss set $\kappa = 1$, $d_{n1} = d_{n2}$, $d_i = d$ and $p_i = p$. In addition to the term considered above, $n^3(T_n - \mu_n)$ contributes $R_n = n^3 [d_{nU} (\frac{\xi}{n} (1 + \frac{1}{n}) - \delta g_n(p))]$ and $S$ contributes $R = -dg'(p)U(p)$. But using only Assumption 4 we have

$$|R_n - R| \leq |n^3 (d_n - d)g_n (\Gamma_n^{-1}(p))| + |dn^3 [g_n (\Gamma_n^{-1}(p)) - g_n(p)]$$

$$+ dg'(p)U(p)| \to_{a.s.} 0$$

since $\rho(V_n, V) \to_{a.s.} 0$ as $n \to \infty$ with $V_n = n^3 (\Gamma_n^{-1} - I)$ and $V = -U$ by (A. 4). [ ]

**Remark 2.** For the special Uniform $(0, 1)$ order statistics and the special Brownian bridge $U$ of the Appendix, we have in fact shown that

$$n^3(T_n - \mu_n) \to_p - \int_0^1 J Udg - \sum_{k=1}^e d_k g'(p_k)U(p_k) \quad \text{as} \quad n \to \infty .$$

**Remark 3.** A random vector converges in probability if and only if each coordinate does. Thus the obvious vector analog of Theorem 1 follows trivially from Remark 2.

**Example 1.** Let $\alpha_1, \ldots, \alpha_n$ be a random sample from an arbitrary df $F$ for which $E|\alpha|^r < \infty$ for some $r > 0$. Let

$$T_n = n^{-1} \sum_n^1 J(t_{ni}) \alpha_{ni}$$

where $\max_{1 \leq i \leq n} |t_{ni} - i/n| \to 0$ as $n \to \infty$ and where for some $a > 0$

$$a[(i/n) \wedge (1 - i/n)] \leq t_{ni} \leq 1 - a[(i/n) \wedge (1 - i/n)] \quad \text{for} \quad 1 \leq i \leq n .$$

Suppose $J$ is continuous except at a finite number of points at which $F^{-1}$ is continuous, and suppose

$$|J(t)| \leq M[t(1 - t)]^{-\frac{1}{2} + 1/r + \delta}$$

for $0 < t < 1$ for some $\delta > 0$. Then

$$n^3(T_n - \int_0^1 J F^{-1} dI \to_d N(0, \sigma^2)$$

as $n \to \infty$

with finite variance

$$\sigma^2 = \int_0^1 \int_0^1 (s \wedge t - st) J(s)J(t) dF^{-1}(s) dF^{-1}(t) .$$

This result is deficient in that $J_n$ has not been replaced by $J$ in the centering constant; this is true likewise of Theorem 1. But this is a purely deterministic problem, and it seemed advisable to separate it off. We now provide one possible solution. Suppose $J'$ exists and is continuous on $(0, 1)$ with

$$|J'(t)| \leq M[t(1 - t)]^{-\frac{1}{2} + 1/r + \delta}$$

for $0 < t < 1$ ;
and strengthen the “max-condition” to
\[ n \max_{1 \leq i \leq n} |t_{ni} - i/n| = O(1) \quad \text{as } n \to \infty. \]

Then our conclusion may be strengthened to
\[ n^t(T_n - \frac{1}{n} JF^{-1} dI) \to_d N(0, \sigma^2) \quad \text{as } n \to \infty. \]

**Proof.** We will apply Theorem 1 with \( \kappa = 0 \), \( g_n = g = F^{-1} \) and \( b_1 = b_2 = b \equiv \frac{1}{2} - 1/r - \delta \). Since \( E|\alpha|^r < \infty \) we have

\[ t|F^{-1}(t)|^r \leq \frac{1}{b} |F^{-1}(s)|^r ds \leq \frac{1}{b} \int_{-\infty}^{\infty} |s|^r dF(s) \to 0 \quad \text{as } t \to 0 \]

using \( F^{-1} \circ F(t) \leq t \) for \( -\infty < t < \infty \). Thus \( |g| \leq D \) with the choice \( -\frac{1}{2} + b + \delta = -1/r \). By the “a-condition” on the \( t_{ni} \)’s we have \( |J_n| \leq B \equiv M_n[t(1-t)]^{-1+1/r+\delta} \) for some constant \( M_n \); see Definition A.3. Thus Assumption 1 holds. The continuity of \( J \) and the “max-condition” on the \( t_{ni} \)’s together imply Assumption 2. The first result now follows from Theorem 1. By the mean value theorem and the “strengthened max-condition” we have

\[ n^t|J_n(t) - J(t)||g(t)| \leq n^{-1} M_n |J'(t^*)||g(t)| \leq n^{-1} M_n M[t^*(1-t^*)]^{-1+1/r+\delta} M[t(1-t)]^{-1/r} \leq n^{-1} M_n M^2 M_n[t(1-t)]^{-1+\delta} \]

where the final inequality uses Definition A.3 and the “a-condition” bound on \( t^* \). The proof is easily completed using \( n^t \int_{1/n}^{1-1/n} [t(1-t)]^{-1+\delta} dt \to 0 \) as \( n \to \infty \). \( \square \)

**Example 1a.** Let \( \alpha_1, \ldots, \alpha_n \) be a random sample from a df having \( E|\alpha|^r < \infty \) for some \( r > 2 \). Then

\[ n^t(\bar{\alpha} - E(\alpha)) \to_d N(0, \text{Var}[\alpha]) \quad \text{as } n \to \infty. \]

This example shows that the ordinary central limit theorem “just fails” to be a corollary to Theorem 1.

**Example 1b.** Let \( \alpha_1, \ldots, \alpha_n \) be a random sample from the \( N(0, 1) \) df \( \Phi \). For integral \( r > 0 \) let

\[ T_n = n^{-1} \sum [\Phi^{-1}(i/n + 1)]^r \alpha_{ni}. \]

(Or use \((3i - 1)/(3n + 1)\) in place or \( i/n + 1 \) as some would advocate.) Then

\[ n^t(T_n - E(\alpha^{r+1})) \to_d N(0, E(\alpha^{2r+2}) - E(\alpha^{r+1})) \quad \text{as } n \to \infty. \]

**Example 2 (The linearly trimmed mean).** Let \( 0 < a < \frac{1}{2} \) be fixed, and let \( a_n = \lfloor na \rfloor \). Let \( \alpha_1, \ldots, \alpha_n \) be a random sample from \( F_\theta = F(\cdot - \theta) \) where \( F \) is any df symmetric about 0. For \( n \) even we follow Crow and Siddiqui (1967) and define

\[ T_n = \sum_{a_n+1}^{n/2} [2(i - a_n) - 1](\alpha_{ni} + \alpha_{n-i+1})/2(n/2 - a_n)^2. \]
(We omit \( n \) odd.) Then

\[
n^3(T_n - \theta) \to_d [4/(1 - 2a)^2] 2^{\frac{1}{2}} \int_a^1 (t - a) W(t) dF^{-1}(t) \quad \text{as} \quad n \to \infty
\]

where \( W \) is Brownian motion.

**Proof.** Let \( c_n = c_{n,n-i+1} = n[2(i - a_n) - 1]/2(n/2 - a_n)^2 \) for \( 1 \leq i \leq n/2 \). Then \( J(t) \) equals \( 0 \) or \( 4(t - a)/(1 - 2a)^2 \) according as \( 0 \leq t \leq a \) or \( a \leq t \leq \frac{1}{2} \) while \( J(t) = J(1 - t) \) for \( \frac{1}{2} \leq t \leq 1 \). Then by Theorem 1 and \( n^{\frac{1}{3}} \int_0^1 (J_n - J) F^{-1} dl \to 0 \) as \( n \to \infty \) we have

\[
n^3(T_n - \theta) \to_d - \frac{1}{2} \int_0^1 UJ dF^{-1} = 2^{\frac{1}{2}} \frac{1}{2} WJ dF^{-1}
\]

using the symmetry of \( F \), where

\[
W(t) = -[U(t) + U(1 - t)]/2^{\frac{1}{2}} \quad \text{for} \quad 0 \leq t \leq 1
\]

is Brownian motion with \( \text{Var} [W(1)] = 1 \). \( \square \)

**Example 3.** Let \( \alpha_1, \ldots, \alpha_n \) be independent Bernoulli \((\theta)\) rv's. Let \( g = F^{-1} \). Thus \( g(t) \) equals \(-\infty, 0, 1\) for \( t = 0, 0 < t \leq 1 - \theta, 1 - \theta < t \leq 1 \). Let \( J(t) \) equal \( 0, 1 \) for \( 0 \leq t < \frac{1}{2}, \frac{1}{2} \leq t \leq 1 \) and let \( c_n = J(i/n) \). Then \( T_n \) equals \( \frac{1}{2} \) if more than \( \frac{1}{2} \) of the \( \alpha_i \)'s are positive; while \( T_n \) equals the proposition of positive \( \alpha_i \)'s if less than \( \frac{1}{2} \) of the \( \alpha_i \)'s are positive.

(a) Suppose \( \theta = \frac{1}{2} \). Then \( n^3(T_n - \mu_n) = n^3(T_n - \frac{1}{2}) \) is asymptotically distributed as a rv having a \( N(0, \frac{1}{2}) \) density on \(( -\infty, 0)\) and having point mass \( \frac{1}{2} \) at \( 0 \). Note that \( J \) is not continuous a.e. \( |g| \); and hence the hypotheses of Theorem 1 fail to hold.

(b) Suppose \( \theta < \frac{1}{2} \). Then \( n^3(T_n - \mu_n) = n^3(T_n - \frac{1}{2}) \) is asymptotically \( N(0, 0) \) by Theorem 1.

(c) Suppose \( \theta < \frac{1}{2} \). Then \( n^3(T_n - \mu_n) = n^3(T_n - \theta) \) is asymptotically \( N(0, \theta(1 - \theta)) \) by Theorem 1.

**Remark 4.** The rv \( T_n \) of (1) will be asymptotically normal provided there is "not too much weight in the tails." If \( |g| \leq D \), then we only allow \( |J| \leq B \) so that the rv \( S \) of (4) will have finite variance. This tradeoff between \( g \) and \( J \) can be different in each tail (in that \( b_1 \) need not equal \( b_2 \)).

**2. Variations on the First Main Theorem.** In this section we consider (Corollary 2) the important weakening of Assumption 3 and (Corollary 1) the rather minor dropping of the existence of a limit \( g \). For these we strengthen Assumption 2.

**Assumption 2'.** (Strong smoothness) Let \( J \) be continuous on \((0, 1)\) and suppose that \( J_n \to J \) uniformly on \([\theta, 1 - \theta]\) as \( n \to \infty \) for each \( \theta > 0 \).

We say that \( g_n \) converges weakly to \( g \) if \( g_n(t) \to g(t) \) as \( n \to \infty \) at each continuity point \( t \) of \( g \).
FUNCTIONS OF ORDER STATISTICS

Assumption 3'. (Weak convergence) Let \( g_n \) converge weakly to \( g \) as \( n \to \infty \).

Corollary 1. Suppose Assumptions 1, 2', and 4 (omit the conditions on \( g \)) hold. Then
\[
n! (T_n - \mu_n)/\sigma_n \xrightarrow{d} N(0, 1)
\]
provided \( \lim \inf_{n \to \infty} \sigma_n^2 > 0 \). (Replace \( g \) by \( g_n \) in (3) to obtain \( \sigma_n^2 \).)

Proof. In the proof of Theorem 1 let \( S_{\nu_n} = \int \frac{1}{2} JU \, dg_n + \sum_i d_k g_n' (p_k) U (p_k) \). Then when \( k = 0 \) we have
\[
\chi_{n!} |S_n - S_{\nu_n}| = |\int \frac{1}{2} \chi_{n!} [A_n^* U_n - JU] \, dg_n|
\leq \rho_0 (U, U) \int \frac{1}{2} q M, B d |g_n| + \rho_0 (U, 0) \int \frac{1}{2} \chi_{n!} q |A_n^* - J| d |g_n|
= o_{P}(1) O(1) + O_{P}(1) \int \frac{1}{2} \chi_{n!} q |A_n^* - J| d |g_n| .
\]
Integration by parts and Assumption 1 shows that for any \( \eta > 0 \) there exists \( \theta > 0 \) such that \( (\int_{\theta}^{\theta} + \int_{1-\theta}^{1-\theta}) q M, B d |g_n| < \eta / 2 \) for all \( n \). Also on \( [\theta, 1 - \theta] \) the difference quotient \( A_n^* \) converges uniformly to \( J \) by Assumption 2'. Hence on subsequences \( n' \) for which \( \int_{\theta}^{\theta} d |g_n| \) remains bounded we have \( \int_{\theta}^{\theta} \chi_{n!} q |A_n^* - J| d |g_n| \to 0 \). Thus \( \int_{\theta}^{\theta} \chi_{n!} q |A_n^* - J| d |g_n| \to 0 \) for all \( \epsilon > 0 \). Hence \( \chi_{n!} (S_n - S_{\nu_n}) \to 0 \). We can divide by \( \sigma_n \) without destroying \( \to_n \) provided \( \sigma_n \) is bounded away from 0. \( \square \)

Corollary 2. Suppose Assumptions 1, 2', 3' and 4 hold. Then
\[
n! (T_n - \mu_n) \xrightarrow{d} N(0, \sigma^2)
\]
as \( n \to \infty \)
with \( \mu_n \) of (2) and \( \sigma^2 \) of (3) finite.

Proof. In addition to the proof of Corollary 1 we need only show that \( \int_{\theta}^{\theta} U J d (g_n - g) \to 0 \) as \( n \to \infty \). But we simply repeat the type of argument used in Corollary 1. That is, truncate off \( (\int_{\theta}^{\theta} + \int_{1-\theta}^{1-\theta}) U J d |g_n - g| \) since it makes a small contribution with high probability; and then use weak convergence of \( g_n \) to \( g \) on \( [\theta, 1 - \theta] \) and the continuity of \( J \) on \( [\theta, 1 - \theta] \) to show that \( \int_{\theta}^{\theta} U J d (g_n - g) \to 0 \) as \( n \to \infty \) for any \( \theta > 0 \) by the Helly-Bray theorem. \( \square \)

3. The second main theorem. We will start fresh in regard to notation. The Appendix is rather more heavily relied on.

We now fix \( g \) in the class of all left continuous functions and consider
\[
T_n = n^{-1} \sum_{i=1}^{n} c_{n!} g (\xi_{n!}) .
\]
We suppose throughout that there exist functions \( C_n \) on \( (0, 1) \) and a signed measure \( \nu \) on \( (0, 1) \) such that
\[
\int_{\{a,b\}} \nu = \int_{\{a,b\}} C_n d \nu
\]
for \( 1 \leq i \leq n \) where \( \int_{\{a,b\}} d \nu = \int_{\{a,b\}} d \nu \).

Example 4. Let \( \nu \) denote Lebesgue measure and let \( C_n(t) = c_{n,i} \) for \( (i-1)/n < t \leq i/n \) and \( 1 \leq i \leq n \). Call these “simply generated scores.”

Example 5. Let \( \nu \) put mass 1 at \( t = \frac{1}{2} \) and zero mass elsewhere. Let \( C_n(\frac{1}{2}) = 1 \). Then \( T_n \) is \( g \) evaluated at a sample median (at the sample median for \( n \) odd).

In general
\[
T_n = \int_0^t g(\Gamma_n^{-1})C_n \, d\nu.
\]

Let \( C \) denote a fixed measurable function on \((0, 1)\) and let
\[
\mu = \int_0^1 gC \, d\nu
\]
and
\[
\sigma^2 = \int_0^1 \int_0^t (s \wedge t - st)g'(s)g'(t)C(s)C(t) \, dv(s) \, dv(t)
\]
provided these exist. Let * now restrict functions on \((0, 1)\) to \([1/n, 1-1/n]\).

If \( C_n^* \) replaces \( C \) in (11) and (12) the resulting quantities will be called \( \mu_n \) and \( \sigma_n^2 \).

Now
\[
n^1(T_n - \mu) = T_n^* + \gamma_{n1} + \gamma_{n2} + \theta_n
\]
where
\[
T_n^* = \int V_n A_n \, d\nu,
\]
\[
A_n = [g(\Gamma_n^{-1}) - g]/(\Gamma_n^{-1} - I),
\]
\[
\gamma_{n1} = n^{-1} c_{n,1} g(\hat{z}_{n,1}), \quad \gamma_{n2} = n^{-1} c_{n,2} g(\hat{z}_{n,2})
\]
and
\[
\theta_n = n^1 \int (C_n^* - C)g \, dv.
\]

(Left continuity is used to define \( A_n \) at the at most finite number of points, for each fixed \( \omega \), at which it might otherwise be undefined.)

(E1) (i) For all large \( n \) we have \( |C_n| \leq \phi \) a.e. \( |\nu| \) where \( \int q |g'| |\phi d| |\nu| < \infty \) for some \( q \) in \( \mathcal{C} \).

(ii) \( \int q |A_n^* - g'| |\phi d| |\nu| \to_p 0 \) as \( n \to \infty \), for this same \( q \).

(E1a) \( g \) is absolutely continuous on \((\varepsilon, 1 - \varepsilon)\) for all \( \varepsilon > 0 \). \( g' \) exists a.e. \( |\nu| \) and \( |g'| \leq R \) a.e. Lebesgue measure where \( R \) is a reproducing u-shaped (or increasing) function. For all large \( n \) we have \( |C_n| \leq \phi \) a.e. \( |\nu| \) where \( \int q R |\phi d| |\nu| < \infty \) for some \( q \) in \( \mathcal{C} \).

(E2) \( \gamma_{n1} + \gamma_{n2} \to_p 0 \) as \( n \to \infty \).

(E2a) \( |g| \leq M[1 - I]^{-\alpha} \) for some \( M > 0 \) and \((c_{n,1} \vee c_{n,2}) = o(n^{1-\alpha}) \).

(E2b) \( |g| \leq M(1 - I)^{-\alpha} \) for some \( M > 0 \) and \( c_{n,1} = o(n^1) \) and \( c_{n,2} = o(n^{1-\alpha}) \).
(E3) \( C_n \to C \text{ a.e. } |\nu| \) as \( n \to \infty \).

(E4) \( n^k \int_0^1 (C_n^* - C) g \, d\nu \to 0 \) as \( n \to \infty \).

(Roughly speaking, use \( R \) increasing in (E1a) and use (E2b) for distributions on \([0, \infty)\).) See below for conditions (E1b) and (E1c).

**Theorem 2.** Conditions (E1a), (E1b) and (E1c) each imply (E1). Conditions (E2a) and (E2b) each imply (E2). If (E1), (E2), (E3) and (E4) hold, then

\[
n^k (T_n - \mu) \to_d N(0, \sigma^2)
\]

with \( \mu \) of (11) and \( \sigma^2 \) of (12) finite. If only (E1) and (E2) hold, then

\[
n^k (T_n - \mu_n)/\sigma_n \to_d N(0, 1)
\]

provided \( \lim \inf_{n \to \infty} \sigma_n > 0 \).

**Proof.** We first prove that \( n^k (T_n - \mu) \to_d N(0, \sigma^2) \). Let \( T = \int_0^1 V g'C \, d\nu \), which is a \( N(0, \sigma^2) \) rv with \( \sigma^2 \) finite by (E1) (i). Also

\[
|T_n - T| = \int_0^1 \left| (V_n^* - V)(A_n^* - g' + g')C_n + V(A_n^* - g')C_n \right| \, d\nu \leq \rho_q(V_n^*, V) \int_0^1 q |A_n^* - g'| \phi d|\nu| + \int_0^1 q |g'| \phi d|\nu| + \rho_q(V, 0) \int_0^1 q |A_n^* - g'| \phi d|\nu| + \rho_q(V, 0) \int_0^1 q |g'| \phi d|\nu|.
\]

Now \( \rho_q(V_n^*, V) = o_p(1), \rho_q(V, 0) = o_p(1) \) and \( \int_0^1 q |g'| C_n - C d|\nu| \to 0 \) by (E1) (i), (E3) and the dominated convergence theorem. Thus \( T_n \to_p T \) under (E1) and (E3). Referring to (13), (E2) and (E4) show that \( n^k (T_n - \mu) \to_p T \).

The proof for \( n^k (T_n - \mu_n)/\sigma_n \) is even easier. Note that we can divide by \( \sigma_n \) without destroying \( \to_p \) since \( \sigma_n \) is bounded away from 0.

We now show that (E2a) implies (E2). Using Stirling’s approximation

\[
E(t_n^2) = c_n^2 \int_0^1 \left( 1 - t \right)^{2n} n(1 - t)^{s-1} \, dt/n \\
= M_c c_n^2 \Gamma(n - 2\alpha)/\Gamma(n + 1 - 4\alpha) \leq M_c c_n^2 / n^{2\alpha} \to 0.
\]

Likewise (E2b) implies (E2).

We now show that (E1a) implies (E1). Now (E1) (i) is trivial. Since \( g' \) exists a.e. \( |\nu| \) we have for every \( \omega \) that the difference quotient \( A_n \) converges to \( g' \) a.e. \( |\nu| \) (since \( \rho(\Gamma_n^{-1}, I) \to_{\text{a.e.}} 0 \)). Also

\[
|A_n| = \left| \int_0^1 g'(s) \, ds/(\Gamma_n - I) \right| \leq \int_0^1 R(s) \, ds/(\Gamma_n - I) < R \vee R(\Gamma_n).
\]

Thus for the set \( S_{n,t} \) of Lemma A.3

\[
\chi_{|x|} A_n^* \leq R_{\beta,t} \leq M_x R.
\]

Thus we may for each fixed \( \omega \) apply the dominated convergence theorem to conclude

\[
\chi_{|x|} \int_0^1 q |A_n^* - g'| \phi d|\nu| \to_{\text{a.e.}} 0.
\]
Thus $\int_0^1 q|A_n^* - g'|d|\nu| \to_p 0$.

To show that (E1b) implies (E1), work Remark A. 4 into the first paragraph of this proof. That (E1c) implies (E1) is also easy. Conditions (E1b) and (E1c) are in the spirit of [1]. They are not stated until after Theorem 2 because they are long, and because I feel that (E1a) is a better condition. □

(E1b) (i) $g$ is absolutely continuous on $(\epsilon, 1 - \epsilon)$ for all $\epsilon > 0$. (ii) For all positive $\beta$ in some neighborhood of 0 there exists $0 < M_\beta < \infty$ such that $|g'(s)/g'(t)| \leq M_\beta$ whenever $\beta t \leq s \leq t + \beta$ and $t \leq \frac{1}{2}$ and whenever $\beta(1 - t) \leq 1 - s \leq (1 - t) + \beta$ and $t \leq \frac{1}{2}$. (iii) For all large $n$ we have $|C_n| \leq g$ a.e. $|\nu|$ where $\int_0^1 q|g'|d|\nu| < \infty$ for some $q$ in $\mathcal{D}$.

(E1c) (i) $g$ satisfies a Lipschitz condition on $(\epsilon, 1 - \epsilon)$ for all $\epsilon > 0$ and $g'$ exists a.e. $|\nu|$. (ii) There exists $\delta > 0$ such that for all positive $\beta$ in some neighborhood of 0 there exists $0 < M_\beta < \infty$ such that $|g'(s)/g'(t)| < M_\beta$ whenever $\beta t \leq s \leq t + \delta$ and $t \leq \delta$ and whenever $\beta(1 - t) \leq 1 - s \leq (1 - t) + \beta$ and $t \leq \delta$; and $|\nu|((\delta_0, 1 - \delta_0)) < \infty$ for some $0 < \delta_0 < \delta$. Or there exists $\delta_0 > 0$ such that $c_{n_i} = 0$ for $i \leq n_0$ and $i \geq n(1 - \delta_0)$; and $|\nu|((\delta_0, 1 - \delta_0)) < \infty$. (iii) For all large $n$ we have $|C_n| \leq g$ a.e. $|\nu|$ where $\int_0^1 q|g'|d|\nu| < \infty$ for some $q$ in $\mathcal{D}$.

**Remark 5.** A version of Theorem 2 in which $g$ depends on $n$ may be proved with the aid of Corollary 3 of [6].

**APPENDIX**

**A UNIFORMLY CONVERGENT EMPIRICAL PROCESS**

**A1. Definition of the basic rv's $\xi_1, \ldots, \xi_n$ and processes $U_n$, $U$.** There is a probability space $(\Omega, \mathcal{F}, P)$ with the following very special random quantities defined on it.

$\xi_1, \ldots, \xi_n$ are independent Uniform $(0, 1)$ rv's on $\Omega$ and $0 < \xi_{n_1} < \cdots < \xi_{n_n} < 1$ for all $\omega$ in $\Omega$, where the $\xi_{n_i}$'s denote the ordered $\xi_i$'s. We let $\Gamma_n$ denote the empirical df of these rv's.

$U$ denotes a Brownian bridge on $\Omega$. That is $\{U(t) : 0 \leq t \leq 1\}$ is a normal process with all sample paths continuous, $E(U(t)) = 0$ for $0 \leq t \leq 1$ and the covariance function of the $U$ process is

$$s \wedge t - st.$$ 

Note that $V \equiv -U$ is also a Brownian bridge.

For $n \geq 1$ we define the "uniform empirical process" $U_n$ by

$$(A1) \quad U_n(t) = n[\Gamma_n(t) - t] \quad \text{for} \quad 0 \leq t \leq 1.$$ 

We also define the "uniform quantile process" $V_n$ by

$$(A2) \quad V_n(t) = n[\Gamma_n^{-1}(t) - t] \quad \text{for} \quad 0 \leq t \leq 1.$$
(the inverse of a df will always be the left continuous one).

For functions \( f_1, f_2 \) on \((0, 1)\) let \( \rho(f_1, f_2) = \sup_{0 < t < 1} |f_1(t) - f_2(t)| \). The special property referred to is

\[
(A3) \quad \rho(U_n, U) \to 0 \quad \text{as } n \to \infty
\]

and

\[
(A4) \quad \rho(V_n, V) \to 0 \quad \text{as } n \to \infty.
\]

Equation (A3) says that every sample path of the \( U_n \) process converges uniformly as \( n \to \infty \) to the corresponding sample path of the \( U \) process.

That the \( U_n \)'s exist was shown in Pyke and Shorack (1968). The \( \xi_i \)'s are introduced here in the hope of making this approach more easily understood. Lemmas A1, A2 and A3 and Theorem A1 were proved or referenced in [3].

Though suppressed in the notation, the \( \xi_i \)'s are actually a triangular array.

**A2. The basic convention.** Let \( Z_1, \ldots, Z_n \) be independent rv's having df \( F \) in the class \( \mathcal{F} \) of all df's. Let \( F_n \) denote their empirical df. Let

\[
(A5) \quad n^1(F_n - F) \quad \text{on } (-\infty, \infty)
\]

be the "empirical process"; and let

\[
(A6) \quad n^1(F^{-1}_n - F^{-1}) \quad \text{on } (0, 1)
\]

be the "quantile process."

**Proposition A1 (The inverse transformation).** Let \( \xi \) be any Uniform \((0, 1)\) rv. For an arbitrary \( F \) in \( \mathcal{F} \) define \( X = F^{-1}(\xi) \). Then for all real \( x \) we have

\[
(A7) \quad [X \leq x] = [\xi \leq F(x)].
\]

Also, \( X \) has df \( F \).

For \( 1 \leq i \leq n \) define \( X_i = F^{-1}(\xi_i) \), where the \( \xi_i \)'s are our special Uniform \((0, 1)\) rv's. Then \( X_1, \ldots, X_n \) are independent rv's having df \( F \). Moreover, from (A7) we see that the \( X_i \)'s have empirical process

\[
(A8) \quad U_n(F) \quad \text{on } (-\infty, \infty)
\]

and quantile process

\[
(A9) \quad n[F^{-1}(G_n^{-1}) - F^{-1}] \quad \text{on } (0, 1).
\]

Thus the processes (A8) and (A9) have the same finite dimensional distributions as do the processes (A5) and (A6).

Also, a statistic \( T_n = T_n(Z_1, \ldots, Z_n) \) has the same distribution as does the statistic \( T_n = T_n(X_1, \ldots, X_n) \). We can use the second representation in attempts to show \( T_n \) is asymptotically normal; and if the statistics we consider
can be represented as functionals on certain stochastic processes, (A3) and (A4) will prove fundamental in our proofs.

Because of the above correspondence, we will simply identify the \( X_i \)'s with the \( Z_i \)'s. With this convention we then have

\[
\text{(A10)} \quad n^1(\xi_n - F) = U_n(F) \quad \text{and} \quad \Gamma_n^{-1} = F^{-1}(\Gamma_n^{-1}) .
\]

Also, the \( i \)th order statistic \( X_{ni} \) is equal to \( F^{-1}(\xi_{ni}) \) for \( 1 \leq i \leq n \). Note that all random quantities are now defined on \( \Omega \).

Let \( I \) denote the identity function and let \( \int_S dI \) denote an integral with respect to Lebesgue measure. We write \( \rightarrow_d \) to denote convergence in distribution. Let \( \chi(S) \) denote the indicator function of the set \( S \). For function \( f \) on \((0, 1)\) and a signed measure \( \nu \) on \((0, 1)\) with total variation measure \( |\nu| \) define

\[
\|f\|_\nu = \int_0^1 |f| d|\nu| .
\]

**A3. The \( q \) functions.**

**Definition A1.** Let \( \mathcal{C}(\mathcal{F}) \) denote the class of positive, strictly increasing continuous functions \( q \) on \([0, 1]\) for which \( \int_0^1 q^{-2} dI < \infty \). Let \( \mathcal{C} \) denote the class of all \( q \) such that \( q(t) = q(1 - t) = \bar{q}(t) \) for \( 0 \leq t \leq \frac{1}{2} \) and some \( \bar{q} \) in \( \mathcal{C}(\mathcal{F}) \). Let \( \mathcal{C}^{(\mathcal{F})} \) denote the class of all \( q \) such that \( q(1 - I) \) is in \( \mathcal{C}(\mathcal{F}) \).

For functions \( f_1, f_2 \) and \( q \) on \((0, 1)\) we let

\[
\rho_q(f_1, f_2) = \sup_{0 < t < 1} |(f_1(t) - f_2(t))/q(t)| .
\]

**Remark A1.** By far the most important members of \( \mathcal{C} \) are \([I(1 - I)]^{1-\delta} \)
for \( \delta > 0 \). A function \( q \) having \( q(t) = q(1 - t) = -t^\delta \log t \) for \( 0 \leq t \leq e^{-2} \)
is also in \( \mathcal{C} \).

**Remark A2.** For all \( q \) in \( \mathcal{C} \) we have

\[
[n^1q(1/n)]^{-1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty ,
\]
since \( q^{-2}(1/n)/n \leq \int_0^{1/n} q^{-2} dI \).

**A4. Some properties.**

**Proposition A2 (Glivenko-Cantelli Lemma).** As \( n \rightarrow \infty \) we have \( \rho(\Gamma_n, F) \rightarrow 0 \)
uniformly in all \( F \) in \( \mathcal{F} \). Also

\[
\rho(\Gamma_n^{-1}, I) = \rho(\Gamma_n, I) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty .
\]

**Lemma A1.** Let \( q \) in \( \mathcal{C}(\mathcal{F}) \). Then

\[
P(|U_n(t)| \leq q(t) \quad \text{for} \quad 0 \leq t \leq \theta) \geq 1 - \int_0^\theta q^{-2} dI
\]
for \( n \geq 0 \) where \( U_0 \equiv U \).

**Remark A3.** \( \rho_q(U, 0) = O_p(1) \) for any \( q \) in \( \mathcal{C} \).
Definition A2. We will say that "* restricts function $f$ on $[0, 1]$ (or $(0, 1)$ etc.) to the interval $[a, b]$ (or $(a, b]$ etc.)" if we mean $f^*(t)$ equals $f(t)$ for $a \leq t \leq b$ and equals 0 otherwise. We write $\int_* f^* dI$ for $\int_0^1 f^* dI$.

Let $N$ denote an integer and let $n = n(N)$ be a sequence of integers such that $n \to \infty$ as $N \to \infty$. Let $(K_n(t): 0 \leq t \leq 1)$ be a process on $\Omega$ for which

(i) For any $\varepsilon > 0$ there exists $\beta = \beta_\varepsilon > 0$ such that

$$P(K_n(t) \leq \beta t \quad \text{for all} \quad t \leq 1/N) > 1 - \varepsilon \quad \text{and}$$

(ii) $\rho(K_N, K) \to a.s. 0$ as $N \to \infty$ where $K$ is a positive, increasing continuous function on $[0, 1]$ having $K \leq MI$ for some $M > 0$.

Lemma A2. If (i) and (ii) hold, then for any $q$ in $\mathcal{C}(\mathcal{A})$ we have

$$\rho_q(U_n(K_n)^*, U(K)) \to 0 \quad \text{as} \quad N \to \infty ;$$

where * restricts functions on $[0, 1]$ to $[1/N, 1]$.

Lemma A3. Given $\varepsilon > 0$ there exists $0 < \beta = \beta_\varepsilon < 1$ and a subset $S_{n, \varepsilon}$ of $\Omega$ having $P(S_{n, \varepsilon}) > 1 - \varepsilon$ on which

$$1 - (1 - t)\beta \leq \Gamma_n(t) \leq (1 - t) \quad \text{for} \quad 0 \leq t \leq 1,$$

$$\beta t \leq \Gamma_n(t) \quad \text{for all} \quad t \leq 1,$$

$$\Gamma_n(t) \leq 1 - \beta(1 - t) \quad \text{for all} \quad t \leq 1,$$

$$\beta t \leq \Gamma_n^{-1}(t) \leq 1 - \beta(1 - t) \quad \text{for} \quad 0 \leq t \leq 1,$$

$$\Gamma_n^{-1}(t) \leq t/\beta \quad \text{for} \quad t \geq 1/n \quad \text{and}$$

$$1 - (1 - t)/\beta \leq \Gamma_n^{-1}(t) \quad \text{for} \quad t \leq 1 - 1/n .$$

We may also require that for all $n$ exceeding some $n_1$ we have

$$|\Gamma_n - I| \leq \beta \quad \text{and} \quad |\Gamma_n^{-1} - I| \leq \beta \quad \text{on} \quad [0, 1]$$

provided $\omega$ is in $S_{n, \varepsilon}$.

Remark A4. On the set $S_{n, \varepsilon}$ of the previous lemma we have for $n \geq n_1$ and for all $s$ between $t$ and $\Gamma_n^{-1}(t)$ that $\beta t \leq s \leq t + \beta$ for $t \leq 1/2$ and $\beta(1 - t) \leq 1 - s \leq (1 - t) + \beta$ for $t \leq 1/2$.

Theorem A1 ($\rho_q$-convergence of $U_n$ and $V_n$). For any $q$ in $\mathcal{C}$ we have

$$\rho_q(U_n, U) \to 0 \quad \text{and} \quad \rho_q(V_n^*, V) \to 0 \quad \text{as} \quad n \to \infty$$

where * restricts functions on $[0, 1]$ to $[1/n, 1 - 1/n]$.

Remark A5. $\rho_q(U_n, 0) = O_p(1)$ and $\rho_q(V_n^*, 0) = O_p(1)$ for any $q$ in $\mathcal{C}$.

Lemma A4. For all $q$ in $\mathcal{C}$ there exists a continuous function $\zeta$ on $[0, 1]$ such that $\zeta(0) = 0$ and $\zeta q$ is in $\mathcal{C}$.

Proof. For example, when $q = [I(1 - I)]^{1-\delta}$, let $\zeta = [I(1 - I)]^{\delta/2}$. 

It suffices to show that if \( f \) is a continuous, strictly positive, strictly decreasing function on \((0, 1)\) such that \( \int_0^1 f \, dI < \infty \), then there exists a positive, continuous decreasing function \( \phi \) on \((0, 1)\) such that \( \phi(t) \to \infty \) as \( t \to 0 \) and \( \int_0^1 \phi \, dI < \infty \).

Let \( a_n = \int_0^1 f \, dI \) and let \( \phi_n = \phi(1/n) \) for any fixed positive, continuous, decreasing function \( \phi \) on \((0, 1)\) such that \( \phi(t) \to \infty \) as \( t \to 0 \). If \( b_n = \int_0^1 f \phi \, dI \), then \( b_n \leq \phi_{n+1} a_n \) for all \( n \). It thus suffices to show that if \( \sum a_n \) is any convergent series, then there exists a sequence \( c_n \to \infty \) such that \( \sum c_n a_n < \infty \).

Let \( \varepsilon_m = a^{2m} \) with \( 0 < a < 1 \). Choose a strictly increasing sequence \( n_m \) such that \( \sum_{k=n_m}^{n_{m+1}} a_k < \varepsilon_m \) for all \( m \). Let \( c_k = \varepsilon_{m-1} \) for \( n_m \leq k < n_{m+1} \) and \( m \geq 1 \) with \( c_1 = \cdots = c_{n_1-1} = 0 \). Then

\[
\sum_{k=1}^{\infty} c_k a_k = \sum_{m=1}^{\infty} \sum_{k=n_m}^{n_{m+1}-1} c_k a_k \leq \sum_{m=1}^{\infty} \varepsilon_{m-1} \sum_{k=n_m}^{\infty} a_k \\
\leq \sum_{m=1}^{\infty} \varepsilon_{m-1} = \sum_{m=1}^{\infty} a^{2m} < \infty .
\]

This proof was suggested by Jac Fabius. 

**A5. Reproducing u-shaped function.**

**Definition A3.** A positive function \( R \) on \((0, 1)\) will be called “u-shaped” if for some \( 0 < a < 1 \) the function is decreasing on \((0, a]\) and increasing on \((a, 1)\). We introduce the notation \( R_\beta \) for

\[
R_\beta = R(\beta t) \quad \text{for} \quad 0 < t \leq \frac{1}{2} ,
\]

\[
= R(1 - \beta(1 - t)) \quad \text{for} \quad \frac{1}{2} < t < 1 .
\]

If for all \( \beta \) in a neighborhood of 0 there exists a constant \( M_\beta \) such that \( R_\beta \leq M_\beta R \) on \((0, 1)\), then \( R \) will be called a “reproducing u-shaped function.”

For any \( \theta \geq 0 \) the function \( R = [I(1 - I)]^{\theta} \) is a reproducing u-shaped function.

In certain problems there is a natural asymmetry.

**Definition A4.** For increasing (decreasing) functions \( R \) on \((0, 1)\) we define \( R_\beta = R(1 - \beta(1 - t)) \) (= \( R(\beta t) \)). If for all \( \beta \) in a neighborhood of 0 we have \( R_\beta \leq M_\beta R \) on \((0, 1)\) for some \( M_\beta \), then \( R \) will be called a “reproducing increasing (decreasing) function.”

Coupled with Lemma A3 this concept yields a powerful tool.

**REFERENCES**


