A NOTE ON THE ZERO-ONE LAW

BY JULIUS R. BLUM AND PRAMOD K. PATHAK

University of New Mexico

Let \( \mathcal{M} = \{\mu_n : n \geq 1\} \) be a sequence of probability measures defined on
the measurable space \((\mathcal{R}_n, \mathcal{B}_n)\) and suppose that the measures \(\{\mu_n : n \geq 1\}\)
satisfy the following condition (B): \(\forall \varepsilon > 0, k \geq 1, m \geq 1\), there exists
an \(n \geq m\) such that \(\|\mu_k - \mu_n\| < \varepsilon\). We show that if \(A \in \bigotimes_n^n \mathcal{B}_n\)
and if \(A\) is permutation invariant then \(\mu(A) = 0\) or 1. The zero-one laws of Hewitt
and Savage [Trans. Amer. Math. Soc. 80 (1955) 470-501] and Horn and
our result.

1. Let \( \mathcal{M} = \{\mu_n : n \geq 1\} \) be a sequence of probability measures defined on
the measurable space \((\mathcal{R}_n, \mathcal{B}_n)\). Consider the product space \(\bigotimes_n^n \mathcal{R}_n, \bigotimes_n^n \mathcal{B}_n, \mu = \bigotimes_n^n \mu_n\)
and suppose that the measures \(\{\mu_n : n \geq 1\}\) satisfy the following
condition.

(B) For each \(\varepsilon > 0, k \geq 1, m \geq 1\), there exists an \(n \geq m\) such that
\(\|\mu_k - \mu_n\| < \varepsilon\).

The main object of this note is to establish the following zero-one law.

(1.1) Theorem. Consider the probability space \(\bigotimes_n^n \mathcal{R}_n, \bigotimes_n^n \mathcal{B}_n, \mu = \bigotimes_n^n \mu_n\)
and suppose that the probability measures \(\{\mu_n : n \geq 1\}\) satisfy condition (B). Let
\(A \in \bigotimes_n^n \mathcal{B}_n\) and suppose that \(A\) is invariant under all permutations of finitely many
coordinates. Then \(\mu(A) = 0\) or 1.

This theorem is an extension of the zero-one laws in Hewitt-Savage [2] and
Horn and Schach [3]. The substitution \(\mu_1 = \mu_2 = \cdots\) in the above theorem
yields the Hewitt-Savage zero-one law while the assumption that \(\forall k \geq 1\) and
\(m \geq 1\) there is an \(n \geq m\) such that \(\mu_k = \mu_n\) yields the zero-one law due to
Horn-Schach [3].

To prove the theorem we need the following preliminary results.

(1.2) Lemma. Let \((\Omega, \mathcal{A}, \mu)\) be a probability space and let \(\{\mathcal{A}_n : n \geq 1\}\) be a
decreasing sequence of sub-\(\sigma\)-algebras of \(\mathcal{A}\). Let \(A \in \mathcal{A}\). Suppose that \(\forall \varepsilon > 0\)
and \(n \geq 1\) there exists a \(B_n \in \mathcal{A}_n\) such that \(\mu(A \triangle B_n) < \varepsilon\). Then there is a set
\(B \in \mathcal{A}_\infty = \bigcap_\infty \mathcal{A}_n\) such that \(\mu(A \triangle B) = 0\).

Proof. For each \(n \geq 1\) choose a \(B_n \in \mathcal{A}_n\) such that \(\mu(A \triangle B_n) < 1/2^n\). Now
let \(B = \lim sup B_n\).

We say that a set \(A \in \mathcal{A}_\infty\) is "tail-approximable" if \(\forall \varepsilon > 0, \forall n \geq 1\) there
exists a \(B_n\) such that \(\mu(A \triangle B_n) < \varepsilon\). The preceding lemma shows that if \(\mathcal{A}_\infty\)
is trivial (i.e. \(\mu(B) = 0\) or \(1 \forall B \in \mathcal{A}_\infty\)) then every "tail-approximable" set is also

\(^1\) Research supported by the NSF Grant GP-25736.
trivial. This result is stronger than some of the well-known zero-one laws and enables us to prove a somewhat stronger version of the zero-one law given above in Theorem 1.1.

(1.3) Lemma. Let \( \{ \mu_k : k \geq 1 \} \) and \( \{ \nu_k : k \geq 1 \} \) be probability measures and suppose for some \( \varepsilon > 0 \) that \( ||\mu_k - \nu_k|| < \varepsilon \). Then \( ||X^n_k \mu_k - X^n_k \nu_k|| < n \varepsilon \).

Proof. This is based on induction and the observation that if \( \mu \) is a probability measure and \( \nu \) a signed measure with \( ||\nu|| < \varepsilon \) then \( ||\mu \times \nu|| < \varepsilon \).

(1.4) Proof of Theorem 1.1. Let \( B^n = X^n_{k+1} B_k \). Then \( B^1 \supset B^2 \supset \ldots \) is a decreasing sequence of \( \sigma \)-algebras. From the classical zero-one law it follows that \( B^n = \lim \ B^\infty \) is a trivial \( \sigma \)-algebra. Consequently it suffices to show that every permutation invariant set (i.e. invariant under all permutations of finitely many coordinates) is tail-approximable. Now let \( A \) be permutation invariant. Let \( \varepsilon > 0 \) and \( n \geq 1 \). Let \( A_m \) be a cylinder set based on \( B_1 \times B_2 \times \ldots \times B_m \) such that \( \mu(A \triangle A_m) < \varepsilon/2m \). For each \( k, 1 \leq k \leq m \), choose \( k(m) \geq \max \{ m + 1, n \} \) such that \( ||\mu_k - \mu_{k(m)}|| < \varepsilon/2m \) and such that \( 1(m), 2(m), \ldots, m(m) \) are all different. For each \( (x_1, x_2, \ldots) \in X^n_r B_k \) define the permutation, \( T \), which interchanges every \( x_k, 1 \leq k \leq m \), with \( x_{k(m)} \). It is easily seen, as a consequence of Lemma 1.3, that \( ||\mu - \mu T^{-1}|| < \varepsilon/2 \). So

\[
\mu(A \triangle T(A_m)) \leq \mu T^{-1}(A \triangle T(A_m)) + ||\mu - \mu T^{-1}|| < \varepsilon/2 + \mu T^{-1}(A \triangle T(A_m)).
\]

Since \( A \) is permutation invariant,

\[
\mu(A \triangle T(A_m)) < (\varepsilon/2) + \mu(A \triangle A_m) < \varepsilon.
\]

Since \( T(A_m) \) is a \( B^n \)-measurable cylinder set, it follows that \( A \) is tail-approximable. Therefore \( \mu(A) = 0 \) or \( 1 \).

REFERENCES

