The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 988-996

A MATRIX OCCUPANCY PROBLEM[1]

BY PATRICK J. EICKER, M. M. SIDDIQUI
AND PAUL W. MIELKE, JR.

Colorado State University

1. Introduction. Suppose that for $i = 1, \cdots, s$, x_i balls, $1 \leqq x_i \leqq T$, are randomly placed in the ith row of an $s \times T$ matrix. It is assumed that: (i) each cell contains at most one ball, and (ii) the balls in any row are distributed independently of the balls in any other row or set of rows. A column of the matrix will be said to have weight j, $0 \leqq j \leqq s$, if exactly j of the s cells in that column are occupied, the other s-j cells being empty. Let C_j, $j = 0, 1, \cdots, s$, denote the number of columns with weight j. The exact (univariate) probability distribution of C_s, the number of full columns, was first given by Mielke and Siddiqui (1965) after being implicitly used by Cowan *et al.* (1963) to describe a temporal attack pattern of three asthmatics. The purpose of the present paper is to study the probability distribution of the basic variables associated with the matrix from which the (multivariate) distribution of the vector $C = (C_0, \cdots, C_s)$ can be derived easily.

We construct an $s \times T$ matrix corresponding to the original matrix replacing "a ball" by a one and "empty" by a zero. Any column of the new matrix is a permutation of j zeros and s-j ones for some $j = 0, 1, \cdots, s$. The entire discussion which follows is in terms of the new matrix. We shall refer to a cell with a zero in it as a 0-cell, and a cell with a one in it as a 1-cell.

Let

$$\mathscr{P} = \{p : p = (a_1, \cdots, a_s), a_j = 0 \quad \text{or} \quad 1, j = 1, \cdots, s\}$$
$$\mathscr{P}_i = \{p \in \mathscr{P} : p = (a_1, \cdots, a_s), \textstyle\sum_j a_j = i\}, \qquad i = 0, 1, \cdots, s.$$

There are 2^s elements in $\mathscr{P}$ and $\binom{s}{k}$ elements in $\mathscr{P}_k$. Note that $\mathscr{P}_0$ and $\mathscr{P}_s$ each have only one element which we denote as p_0 and p_s respectively. For each $p \in \mathscr{P}$, let N_p be the number of columns with structure p, then

$$\eta = \{N_p : p \in \mathscr{P}\}$$

is a set of 2^s random variables. Also,

$$C_j = \textstyle\sum_{p \in \mathscr{P}_j} N_p, \qquad j = 0, 1, \cdots, s,$$

is the number of columns with weight j. For example, if $s = 2$, then

$$\eta = \{N_{00}, N_{01}, N_{10}, N_{11}\} \quad \text{and} \quad C_0 = N_{00}, \quad C_1 = N_{10} + N_{01}, \quad C_2 = N_{11}.$$

In Section 2 we obtain the exact distribution of η. In later sections we investigate the asymptotic properties of this distribution under various assumptions on the behavior of x_i's as $T \to \infty$.

Received October 17, 1969; revised November 18, 1971.

[1] This research was supported in part by National Institutes of Health Grant 5 T1 GM 725.

It will be shown that the 2^s random variables in η satisfy $s+1$ linearly independent linear constraints so that there are only 2^s-s-1 linearly independent random variables in η. It is convenient to delete $s+1$ random variables with largest (or smallest) weight. Let

$$\mathscr{P}^* = \mathscr{P} - \mathscr{P}_s - \mathscr{P}_{s-1},$$

then $\eta^* = \{N_p : p \in \mathscr{P}^*\}$ may be taken as a maximal linearly independent subset of η.

2. Exact probability distributions. One of the $s+1$ constrains on N_p's is

$$\sum_{p\in\mathscr{P}} N_p = T, \tag{2.1}$$

as the T columns of the matrix are partitioned into 2^s different types contained in $\mathscr{P}$. To obtain the other s constraints we introduce the following notation: let, for $i = 0, 1, \cdots, s$, $k = 1, \cdots, s$,

$$\mathscr{P}_{ik}^* = \{p \in \mathscr{P}_i : p = (a_1, \cdots, a_s), a_k = 0\}.$$

In words, $\mathscr{P}_{ik}^*$ is the set of those permutations which have weight i and *a* zero in the kth place. Note that, for each k, the set $\mathscr{P}_{sk}^*$ is empty, while $\mathscr{P}_{s-1,k}^*$ contains only one permutation.

Since, for $k = 1, \cdots, s$, there are $T - x_k$ 0-cells in the kth row of the matrix, we have the following s constraints

$$\sum_{i=0}^{s} \sum_{p\in\mathscr{P}_{ik}^*} N_p = T - x_k, \qquad k = 1, \cdots, s. \tag{2.2}$$

Since x_k's themselves have no mutual constraints all those linear constraints are linearly independent. Thus the random variables in η satisfy $s+1$ linearly independent linear constraints.

Note that if a permutation p has weight i then it will appear as an index of summation in s-i of (2.2). For example if $p = (1, 1, 1, 0, \cdots, 0)$ then $p \in \mathscr{P}_{3k}^*$ for $k = 4, 5, \cdots, s$. The following lemma is then obtained easily.

LEMMA 2.1. *Let* $\mathscr{P}_0 = \{p_0\}$, $\mathscr{P}_s = \{p_s\}$ *and* $\mathscr{P}_{s-1,k}^* = \{p_{s-1,k}\}$, *then*

$$\begin{aligned} N_{p_s} &= \textstyle\sum_{i=1}^{s} x_i - (s-1)T + \sum_{i=0}^{s-2} \sum_{p\in\mathscr{P}_i} (s-i-1)N_p, \\ N_{p_{s-1,k}} &= T - x_k - \textstyle\sum_{i=0}^{s-2} \sum_{p\in\mathscr{P}_{ik}^*} N_p, \qquad k = 1, \cdots, s. \end{aligned} \tag{2.3}$$

REMARK 1. For $s = 1$ (2.3) gives $N_1 = x_1$, $N_0 = T - x_1$ so that $\mathscr{P}^*$ is empty.

REMARK 2. For $s = 2$, (2.3) gives

$$\begin{aligned} N_{11} &= x_1 + x_2 - T + N_{00}, \\ N_{10} &= T - x_2 - N_{00}, \\ N_{01} &= T - x_1 - N_{00}, \end{aligned}$$

so that $\mathscr{P}^* = \{N_{00}\}$, i.e., there is only one linearly independent random variable.

We can now state the following theorem.

THEOREM 2.1 (Exact distribution). *For any set of integers* $\{j_p, p \in \mathscr{P}^*\}$

$$P(N_p = j_p, p \in \mathscr{P}^*) = [\prod_{i=1}^{s} \binom{T}{x_i}]^{-1} \frac{T!}{\prod_{p \in \mathscr{P}^*} j_p! \, j_{p_s}! \prod_{k=1}^{s} j_{p_{s-1,k}}!} .$$

where $j_{p_s} = N_{p_s}$, *and* $j_{p_{s-1,k}} = N_{p_{s-1,k}}$, $k = 1, \cdots, s$, *are determined in terms of* $\{j_p, p \in \mathscr{P}^*\}$ *by* (2.3) *when the letter* N *is replaced by* j, *and where the factorial of a negative integer is taken to be* $+\infty$.

PROOF. Obvious combinatorial considerations.

REMARK 3. For $s = 2$ we have

$$P(N_{00} = j_{00}) = [\binom{T}{x_1}\binom{T}{x_2}]^{-1} \frac{T!}{j_{00}! \, (T - x_1 - j_{00})! \, (T - x_2 - j_{00})! \, (x_1 + x_2 - T + j_{00})!} ,$$

and the positive probabilities are associated only when $\max(0, T - x_1 - x_2) \leqq j_{00} \leqq \min(T - x_1, T - x_2)$. This distribution may be recognized as a hypergeometric distribution. In the discussion of the asymptotic distributions in the following sections we will notice parallels with the asymptotic distributions of this hypergeometric distribution, namely, normal, Poisson, and binomial. For ready reference we note the following properties of this hypergeometric distribution:

$$EN_{00} = T(1 - x_1/T)(1 - x_2/T) ,$$
$$\operatorname{Var} N_{00} = (T - 1)^{-1} x_1 x_2 (1 - x_1/T)(1 - x_2/T) .$$

Asymptotic normality. Assume that x_i, $i = 1, 2$, depend on T and $x_i/T \to \alpha_i$, $0 < \alpha_i < 1$, as $T \to \infty$. Then, if $T \to \infty$, the limiting distribution of $T^{-\frac{1}{2}}(N_{00} - EN_{00})$ is $N(0, \sigma^2)$, where $\sigma^2 = \alpha_1 \alpha_2 (1 - \alpha_1)(1 - \alpha_2)$.

Limiting Poisson. Assume that as $T \to \infty$, $x_i/T \to 1$, $T - x_i \to \infty$, $i = 1, 2$, . and $EN_{00} \to \lambda < \infty$. Then, if $T \to \infty$, the limiting distribution of N_{00} is Poisson with mean parameter λ.

Limiting binomial. Assume that $T - x_1 = c$, a constant, and $x_2/T \to \alpha_2$, $0 < \alpha_2 < 1$, as $T \to \infty$. Then, if $T \to \infty$, the distribution of N_{00} converges to the binomial distribution with sample size parameter c and probability parameter $1 - \alpha_2$.

The Moments. To obtain the moments of N_p's, a useful device is to introduce the indicator functions of p's. Let, for $p \in \mathscr{P}$,

$$\delta_i = \delta_i(p) = 1 \quad \text{if column } i \text{ has structure } p,$$
$$= 0 \quad \text{otherwise,}$$

so that $N_p = \sum_{i=1}^{T} \delta_i$. If $I = \{1, 2, \cdots, s\}$ and $I_{0p} \subset I$ is the set of indices of 0-cells of p, and $I_{1p} = I - I_{0p}$ then

$$P(\delta_i = 1) = \prod_{j \in I_{1,p}} \frac{x_j}{T} \prod_{j \in I_{0p}} \left(1 - \frac{x_j}{T}\right) , \qquad i = 1, \cdots, T .$$

Note that δ_i, $i = 1, \cdots, T$ are identically distributed (although not mutually independent). The joint distribution of any set of $\delta_i = \delta_i(p)$'s can be obtained through straightforward combinatorial arguments. Means, variances, covariances, and higher moments of N_p's can then be evaluated. Thus

$$\mu_p = EN_p = TP(\delta_i = 1)\,,$$
$$\operatorname{Var} N_p = \mu_p\Big[1 + (T-1)\prod_{j\in I_{1p}}\Big(\frac{x_j - 1}{T-1}\Big)\prod_{j\in I_{0p}}\Big(1 - \frac{x_j}{T-1}\Big) - \mu_p\Big]\,,$$

and, for $p_1 \neq p_2$,

$$\operatorname{Cov}(N_{p_1}, N_{p_2}) = \mu_{p_2}\Big[(T-1)\prod_{j\in I_{1p_1}I_{1p_2}}\Big(\frac{x_j - 1}{T-1}\Big)\prod_{j\in I_{1p_1}I_{0p_2}}\Big(\frac{x_j}{T-1}\Big)$$
$$\times \prod_{j\in I_{0p_1}I_{1p_2}}\Big(1 - \frac{x_j - 1}{T-1}\Big)\prod_{j\in I_{0p_1}I_{0p_2}}\Big(1 - \frac{x_j}{T-1}\Big) - \mu_{p_1}\Big]\,.$$

3. Asymptotic normality. In this section it will be assumed that (i) $s \geqq 2$, (ii) x_i, $i = 1, \cdots, s$, are functions of T and, as $T \to \infty$, $x_i/T \to \alpha_i$, $0 < \alpha_i < 1$. Then, as $T \to \infty$,

$$T^{-1}EN_p \to \beta_p = \prod_{j\in I_{1p}}\alpha_j \prod_{j\in I_{0p}}(1 - \alpha_j)\,,$$
$$T^{-1}\operatorname{Var} N_p \to \sigma_p^2 = \beta_p + [s - 1 - \textstyle\sum_{j\in I_{1p}}\alpha_j^{-1} - \sum_{j\in I_{0p}}(1-\alpha_j)^{-1}]\beta_p^2\,,$$

and, for $p_1 \neq p_2$,

$$T^{-1}\operatorname{Cov}(N_{p_1}, N_{p_2}) \to \sigma_{p_1p_2}$$
$$= \beta_{p_1}\beta_{p_2}[s - 1 - \textstyle\sum_{j\in I_{1p_1}I_{1p_2}}\alpha_j^{-1} - \sum_{j\in I_{0p_1}I_{0p_2}}(1-\alpha_j)^{-1}]\,.$$

It can also be shown that

$$\lim_{T\to\infty} T^{-2}E(N_p - \mu_p)^4 = 3\sigma_p^4\,.$$

Introduce, for each $p \in \mathscr{P}$,

$$Z_p = T^{-\frac{1}{2}}(N_p - \mu_p)\,,$$

then $\{Z_p : p \in \mathscr{P}^*\}$ is a set of $2^s - s - 1$ linearly independent random variables.

THEOREM 3.1 (Asymptotic normality). *Under the conditions* (i) *and* (ii) *stated above the set* $\{Z_p : p \in \mathscr{P}^*\}$ *has a limiting distribution which is multivariate normal with mean vector zero and the covariance matrix* $\sigma = (\sigma_{p_1p_2}; p_1 \in \mathscr{P}^*, p_2 \in \mathscr{P}^*)$, $\sigma_{pp} = \sigma_p^2$.

PROOF. Rosén (1967b) develops conditions ((C1)—(C4) of his paper) for the asymptotic normality for sums of dependent random vectors, these conditions being an immediate generalization of the conditions given by him in an earlier paper (1967a) for dependent random variables. To verify these conditions for the matrix occupancy problem, we introduce the double sequence of random vectors

$$\begin{array}{c} X_1^{(1)}, X_2^{(1)}, \cdots, X_{T_1}^{(1)} \\ \vdots \\ X_1^{(k)}, X_2^{(k)}, \cdots, X_{T_k}^{(k)} \\ \vdots \quad \vdots \qquad \vdots \end{array}$$

where $T_k \to \infty$ as $k \to \infty$, and $X_j^{(k)}$ is a $2^s - s - 1$ dimensional vector with components

$$X_{j,\nu}^{(k)} = T_k^{-\frac{1}{2}}\left(\delta_j^{p_\nu} - \frac{\mu_{p_\nu}}{T_k}\right), \qquad \nu = 1, 2, \cdots, 2^s - s - 1,$$

$$\begin{aligned} \delta_j^{p_\nu} &= 1 \quad \text{if } j\text{th column in the } s \times T_k \text{ array has structure } p_\nu \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Here $p_1, p_2, \cdots$ is simply indexing the elements of $\mathscr{P}^*$. The condition (ii) stated in the beginning of this section now reads $\lim_{k\to\infty} x_i^{(k)}/T_k = \alpha_i$, $0 < \alpha_i < 1$, $i = 1, \cdots, s$, where x_i is designated as $x_i^{(k)}$ to indicate explicitly the dependence of x_i's on T_k. Then the vectors $S_1^{(k)} = \sum_{j=1}^{T_k} X_j^{(k)}$ have components $Z_{p_\nu}^{(k)}$, $\nu = 1, \cdots, 2^s - s - 1$, where

$$Z_{p_\nu}^{(k)} = T_k^{-\frac{1}{2}}(N_{p_\nu} - \mu_{p_\nu}).$$

Rosén states his conditions on the sums

$$S_\gamma^{(k)} = \sum_{j=1}^{[\gamma T_k]} X_j^{(k)}, \qquad 0 \leqq \gamma < 1.$$

His "smallness" conditions (C1) and (C4), stated in terms of second moments, are implied by the existence of the fourth moments $\lim_{k\to\infty} E[Z_{p_\nu}^{(k)}]^4 = 3\sigma_{p_\nu}^4$, $\nu = 1, \cdots, 2^s - s - 1$. The dependence conditions are as follows:

(C2) There is a matrix function $M(\gamma)$, defined and continuous for $0 \leqq \gamma < 1$, such that

$$\lim_{\Delta\to+0} \Delta^{-1} \limsup_{k\to\infty} E\|E\{(S_{\gamma+\Delta}^{(k)} - S_\gamma^{(k)}) \mid S_\gamma^{(k)}\} - \Delta M(\gamma) S_\gamma^{(k)}\| = 0, \qquad 0 \leqq \gamma < 1;$$

(C3) There is a matrix function $D(\gamma)$, defined and continuous for $0 \leqq \gamma < 1$, such that

$$\lim_{\Delta\to+0} \Delta^{-1} \limsup_{k\to\infty} E\|E\{(S_{\gamma+\Delta}^{(k)} - S_\gamma^{(k)})(S_{\gamma+\Delta}^{(k)} - S_\gamma^{(k)})' \mid S_\gamma^{(k)}\} - \Delta D(\gamma)\| = 0, \qquad 0 \leqq \gamma < 1.$$

Here, for a square matrix $A = (a_{ij})$, A' is its transposed matrix and $\|A\|^2 = \sum_{i,j} a_{ij}^2$. Under our assumptions, for each $i = 1, \cdots, s$, the $x_i^{(k)}$ balls are thrown in the ith row randomly. If $S_\gamma^{(k)}$ is given, the distribution of balls in the first $[\gamma T]$ columns is fixed. The distribution of balls in columns $[\gamma T] + 1$ to $[(\gamma + \Delta)T]$ given $S_\gamma^{(k)}$ is easily obtained similar to the distribution given in Theorem 2.1. Applying condition (ii) to the first two moments of this distribution, we find that (C2) and (C3) are satisfied by

$$M(\gamma) = 0 \quad \text{and} \quad D(\gamma) = \sigma, \qquad 0 \leqq \gamma < 1.$$

From Rosén's Theorem 1 (1967b) we conclude that $S_\gamma^{(k)}$, $0 \leqq \gamma \leqq 1$, is asymptotically ($k \to \infty$) normally distributed with mean vector 0 and covariance matrix $\Lambda(\gamma)$, where $\Lambda(0) = 0$, $\Lambda(1) = \lim_{\gamma \uparrow 1} \Lambda(\gamma)$, and, for $0 \leqq \gamma < 1$, $\Lambda(\gamma)$ satisfies the differential equation

$$\frac{d}{d\gamma}\Lambda(\gamma) = M(\gamma)\Lambda(\gamma) + (M(\gamma)\Lambda(\gamma))' + D(\gamma) .$$

Setting $M(\gamma) = 0$ and $D(\gamma) = \sigma$, we obtain $\Lambda(\gamma) = \gamma\sigma$, so that $\Lambda(1) = \sigma$.

Asymptotic distributions for any type or weight of column may be obtained from Theorem 3.1 by the use of the appropriate normal theory.

4. Other asymptotic distributions. In the previous section it was assumed that all the ratios x_i/T had limits which were strictly between zero and one. This implied that the means of the random variables were of the same order as T. In this section several theorems will be proved in which it is assumed that the means approach a finite limit as T becomes large. The method of proof is to show that the moments of the law of the random variable approach the moments of the limiting law. For this reason the following expression is useful:

$$\begin{aligned}(\textstyle\sum_{i=1}^T \delta_i)^n = {} & \textstyle\sum_{i_1=1}^T \delta_{i_1}^n + \sum_{1\leqq j_1 \leqq n-j_1} \dfrac{n!}{j_1!(n-j_1)!} \sum_{i_1=1}^T \sum_{i_2=1}^T {}_{i_1 \neq i_2}\, \delta_{i_1}^{j_1}\delta_{i_2}^{n-j_1} \\ & + \textstyle\sum_{1\leqq j_1\leqq j_2 \leqq n-j_1-j_2} \sum \dfrac{n!}{j_1!j_2!(n-j_1-j_2)!} \\ & \times \textstyle\sum_{i_1=1}^T \sum_{i_2=1}^T \sum_{i_3=1}^T {}_{i_1\neq i_2 \neq i_3}\, \delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\delta_{i_3}^{n-j_1-j_2} \\ & + \cdots + n! \textstyle\sum_{i_1=1}^T \sum_{i_2=1}^T \cdots \sum_{i_n=1}^T {}_{i_1\neq i_2\neq\cdots\neq i_n}\, \delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}, \quad \text{if} \quad T \geqq n. \end{aligned} \tag{4.1}$$

An explanatory word on the above sums in necessary. $\sum_{1\leqq j_1\leqq j_2\leqq n-j_1-j_2} \sum$ means that the sum is to be taken over all pairs of integers, j_1 and j_2, which satisfy $1 \leqq j_1 \leqq j_2 \leqq n - j_1 - j_2$. $\sum_{i_1=1}^T \sum_{i_2=1}^T \sum_{i_3=1}^T {}_{i_1\neq i_2\neq i_3}$ means that the sum is to be taken over all triplets of integers between one and T, such that no pair is equal. This sum is to be taken so that no term appears more than once.

THEOREM 4.1. *Let p be a permutation of weight less than $s-1$. If* $\lim_{T\to\infty} x_i/T = 1$ *and* $\liminf (T - x_i) = +\infty$ *for all $i \in I_{0p}$ and if* $\lim_{T\to\infty} \mu_p = \lambda$ *such that* $0 < \lambda < \infty$, *then the law of N_p converges to the law of a Poisson random variable with parameter λ.*

PROOF. $N_p = \sum_{i=1}^T \delta_i$ where

$$\begin{aligned}\delta_i &= 1 \qquad \text{if column } i \text{ has structure } p, \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

If $T \geqq n$, then $E(N_p{}^n)$ may be evaluated using (4.1). Let $B_k = B_k(j_1, j_2, \cdots, n - \sum_{r=1}^{k-1} j_n)$ be a function of k arguments such that B_k is the number of distinct permutations of its arguments. Then

$$E(N_p{}^n) = \sum_{k=0}^{n-1} \sum_{1 \leq j_1 \leq \cdots \leq n - \sum_{r=1}^k j_r} \sum \cdots \sum \frac{n! \binom{T}{k+1} E(\delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_{k+1}}^{n - \sum_{r=1}^k j_r})}{(\prod_{r=1}^k j_r!)(n - \sum_{r=1}^k j_r)!} .$$

Consider

$$\begin{aligned} E(\delta_{i_1}^{j_1} \delta_{i_2}^{j_2} & \cdots \delta_{i_{k+1}}^{n-\sum_{r=1}^k j_r}) \cdot \binom{T}{k+1} \\ &= \prod_{r=0}^k \prod_{j \in I_{1p}} \frac{x_j - r}{T - r} \prod_{j \in I_{0p}} \left(1 - \frac{x_j}{T - r}\right) \cdot \binom{T}{k+1} \\ &= \frac{1}{(k+1)!} \prod_{r=0}^k \left(\frac{T}{T-r}\right)^s \left(1 - \frac{r}{T}\right) \Big[T \prod_{j \in I_{1p}} \frac{x_j}{T} \prod_{j \in I_{0p}} \left(1 - \frac{x_j}{T}\right) \\ &\quad - r \prod_{j \in I_{1p}} \frac{x_j}{T} \prod_{j \in I_{0p}} \left(1 - \frac{x_j}{T}\right) \left(\sum_{j \in I_{1p}} \frac{T}{x_j} + \sum_{j \in I_{0p}} \frac{T}{T - x_j}\right) \\ &\quad + o\left(\frac{1}{T}\right) \Big] . \end{aligned}$$

If the limit is taken, it is seen that $\lim_{T\to\infty} E(N_p{}^n)$ is the nth moment of a Poisson random variable with parameter λ.

The following theorem is a slight generalization of Theorem 4.1 with essentially the same proof.

THEOREM 4.2. *Let $p \in \mathscr{P}$. If $\lim_{T\to\infty} \mu_p = \lambda$ such that $0 < \lambda < \infty$; if there exist indices j_1 and j_2 which have the property that either* (a) *j_1 and/or j_2 are in I_{1p} and $\lim_{T\to\infty} x_{j_i}/T = 0$ or* (b) *j_1 and/or j_2 are in I_{0p} and $\lim_{T\to\infty} x_{j_i}/T = 1$; and if $\liminf (T - x_i) = \liminf x_i = +\infty$, $1 \leqq i \leqq s$; then the law of N_p converges to the law of a Poisson random variable with parameter λ.*

Theorems 4.1 and 4.2 deal with the asymptotic distribution of columns of a particular structure. It seems that in practice, a theorem dealing with the asymptotic distribution of columns of a particular weight or weights would be more useful. The following theorem and corollaries deal with this situation. Their proofs are nearly the same as the proof of Theorem 4.1.

THEOREM. 4.3. *Let $r \leqq s - 2$. If $\lim_{T\to\infty} x_i/T = 1$ and $\liminf (T - x_i) = +\infty$ for all $1 \leqq i \leqq s$ and if $\lim_{T\to\infty} \mu_p = \lambda_p < \infty$ for all $p \in \mathscr{P}_r$ and there exists $p \in \mathscr{P}_r$ such that $\lambda_p > 0$, then the law of $\sum_{p \in \mathscr{P}_r} N_p$ converges to the law of a Poisson random variable with parameter $\sum_{p \in \mathscr{P}_r} \lambda_p$.*

COROLLARY 4.1. *Let $r \leqq s - 2$. If $\lim_{T\to\infty} x_i/T = 1$ and $\liminf (T - x_i) = +\infty$ for all $1 \leqq i \leqq s$ and if $\lim_{T\to\infty} \mu_p = \lambda_p < \infty$ for all $p \in \mathscr{P}_n$ and all $n \leqq r$ and if there exists $p \in \mathscr{P}_n$, $n \leqq r$ such that $\lambda_p > 0$, then the law of $\sum_{n=0}^r \sum_{p \in \mathscr{P}_n} N_p$ converges to the law of a Poisson random variable with parameter $\sum_{n=0}^r \sum_{p \in \mathscr{P}_n} \lambda_p$.*

Note that under the hypotheses of Corollary 4.1, $\lambda_p = 0$ for all $p \in \mathscr{P}_n$ if $n < r$. Hence $\sum_{n=0}^r \sum_{p \in \mathscr{P}_n} \lambda_p = \sum_{p \in \mathscr{P}_r} \lambda_p$.

The following theorem deals with another situation in which the mean approaches a finite limit. The proofs are essentially the same as the proof of Theorem 4.1.

THEOREM 4.4. *Let* $p \in \mathscr{P}$. *If there is an index* j *such that* $j \in I_{1p}$ *and* $x_j \equiv c$ *(a constant) and if* $\lim_{T\to\infty} x_i/T = \alpha_i$, $0 < \alpha_i < 1$ *for all* $i \neq j$; *then the law of* N_p *converges to the law of a binomial random variable with parameters* c *and* $\prod_{i \in I_{1p}; i \neq j} \alpha_i \prod_{i \in I_{0p}} (1 - \alpha_i)$.

5. Example. Three asthmatic patients are observed for a period of 120 days. They have 9, 10, and 22 attacks respectively during the 120 days. If the patients react independently then the probability distribution of the number of days on which at least two patients have attacks may be obtained from Theorem 2.1 with $s = 3$, $T = 120$, $x_1 = 9$, $x_2 = 10$, and $x_3 = 22$. Table 1 gives this exact probability distribution with $P_j = P(N_{110} + N_{101} + N_{011} + N_{111} \geqq j)$; that is the probability that there are at least j days on which two or more patients have attacks. For purposes of comparison, these probabilities computed from the asymptotic results are also included. Table 2 gives the probabilities computed from the normal approximation, Theorem 3.1. Table 3 shows the probabilities computed from the Poisson approximation, Corollary 4.1. Since the expected number of columns of weight two or more is small (3.96) compared to the total number of columns, it would be expected that the Poisson approximation would be closer than the normal approximation. The main differences in the approxi-

TABLE 1

j	P_j	j	P_j
0	1.00×10^{0}	10	4.83×10^{-4}
1	9.93×10^{-1}	11	5.40×10^{-5}
2	9.50×10^{-1}	12	4.35×10^{-6}
3	8.21×10^{-1}	13	2.47×10^{-7}
4	6.00×10^{-1}	14	9.53×10^{-9}
5	3.54×10^{-1}	15	2.39×10^{-10}
6	1.63×10^{-1}	16	3.61×10^{-12}
7	5.79×10^{-2}	17	2.97×10^{-14}
8	1.56×10^{-2}	18	1.09×10^{-16}
9	3.18×10^{-3}	19	1.17×10^{-19}

TABLE 2

j	P_j
0	9.9389×10^{-1}
1	9.6947×10^{-1}
2	8.9251×10^{-1}
3	7.2774×10^{-1}
4	4.8923×10^{-1}
5	2.5463×10^{-1}
6	9.7840×10^{-2}
7	2.6990×10^{-2}
8	5.2300×10^{-3}
9	7.0000×10^{-4}

TABLE 3

j	P_j	j	P_j
0	1.0000×10^{0}	10	7.5951×10^{-3}
1	1.8091×10^{-1}	11	2.6260×10^{-3}
2	9.0532×10^{-1}	12	8.3791×10^{-4}
3	7.5573×10^{-1}	13	2.4808×10^{-4}
4	5.5835×10^{-1}	14	6.8479×10^{-5}
5	3.6302×10^{-1}	15	1.7701×10^{-5}
6	2.0839×10^{-1}	16	4.3007×10^{-6}
7	1.0638×10^{-1}	17	9.8561×10^{-7}
8	4.8691×10^{-2}	18	2.1371×10^{-7}
9	2.0149×10^{-2}	19	4.3968×10^{-8}

mate distributions from the exact distribution is that the critical tail probabilities are too small and large for the normal and Poisson approximations respectively.

Acknowledgment. The authors wish to thank the referee for drawing their attention to Rosén's papers.

REFERENCES

COWAN, D. W., THOMPSON, H. J., PAULUS, H. J., and MIELKE, P. W. (1963). Bronchial asthma associated with air pollutants from the grain industry. *J. Air Pollution Control Assoc.* **13** 546–552.

MIELKE, P. W. and SIDDIQUI, M. M. (1965). A combinatorial test for independence of dichotomous responses. *J. Amer. Statist. Assoc.* **60** 437–441.

ROSÉN, B. (1967a). On the central limit theorem for sums of dependent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7** 48–82.

ROSÉN, B. (1967b). On asymptotic normality of sums of dependent random vectors. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7** 95–102.