The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 968-976

CENTRAL LIMIT THEOREMS FOR SUMS OF DEPENDENT VECTOR VARIABLES[1]

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We prove the following central limit theorems for sums of mutually dependent random vector variables: Given that a sequence of random vector variables satisfies a certain type of decoupling condition (and two milder restrictions), we present a Lindeberg-Feller condition which we show to be both necessary and sufficient for central limit behavior. The decoupling condition and one of the two milder conditions is then applied to a Markov process with stationary transition mechanism.

1. Introduction. The central limit theorem (clt) for sums of independent random variables (rv's) has had a long and varied history in pure and applied probability theory, and extensions of it to the dependent rv case are of great interest. Indeed, many of the physical applications of the clt, e.g., to statistical mechanics and turbulence theory, have assumed that the clt is valid for such dependent rv sums, apparently without prior justification (Khinchin [5], Batchelor [1]).

It is the purpose of the present paper to prove, for mutually dependent real vector rv's, a clt which is mathematically distinct from other recent versions (Rosén [7], [8], Serfling [10], Philipp [6]), and which may perhaps be more useful for certain physical applications (Cocke [2]).

There are two essential ingredients to our version of the clt: (I) A Lindeberg-Feller condition, which guarantees that each of the groups of rv's used in the proof makes a uniformly small contribution to the total sum in the limit, and (II) a decoupling condition, which delineates the manner in which the sequences of groups of rv's become mutually independent from each other in the limit. A third condition (III) is also introduced, but we show that it is only a mild restriction on the random process itself.

We also prove a second theorem, which states that the Lindeberg-Feller condition (I) is also a necessary condition for convergence to the normal distribution, in much the same sense as in the independent variable case.

In Section 2 we set up the notation used and discuss further the conditions (I), (II), and (III). Section 3 deals with the clt itself, and the necessity of condition (I) is demonstrated. We also state a lemma in connection with the decoupling condition (II), as well as a generalization of our clt.

In Section 4, the clt is applied to a Markov process with stationary transition mechanism. It is shown in this example that the clt is easy to interpret, and

Received August 2, 1971; revised September 23, 1971.

[1] This research was partially supported by a National Science Foundation Science Development Grant to the University of Arizona.

the results intuitively appealing. In Section 5 we discuss some implications and limitations of our results and compare them with other recent work.

2. Notation and assumptions. We consider the semi-infinite class of mutually dependent rv's $\{X_i\}_1^\infty$, each of which is an N-dimensional vector, $X_i = (X_{i(1)}, X_{i(2)}, \cdots, X_{i(N)})$. The probability function of the first n of these variables is denoted

$$P\{X_1 \in A_1, \cdots, X_n \in A_n\} = F^{(n)}(A_1, \cdots, A_n) .$$

The averages $a_i = E(X_i)$, and $E(X_i X_j)$ are assumed finite, but the existence of higher-order moments is not assumed.

It is shown in Section 3 that the conditions (I), (II), and (III), discussed below, are sufficient to insure that the vector sum $S_n = \sum_{i=1}^n (X_i - a_i)$, suitably normalized, becomes distributed according to the multivariate normal distribution in the limit $n \to \infty$. The general method of proof of this version of the clt involves showing that one can leave out some of the variables in such a way that the entire sum S_n becomes more and more closely approximated by the sum of "almost all" of them.

In addition, this must occur in such a way that "almost all" the X_i break up into sequences of groups which become group-wise independent from each other as $n \to \infty$. It is then a simple matter to revise one of the standard methods of proof of the classical clt to prove the theorem in this case. This process is similar to a method used by Rosenblatt [9] in proving a clt for certain special types of mixing processes, and is generally rather well known in probability theory.

Let ω_n be the set of all positive integers $\leqq n$. Then for each n, we denote the omitted set of variables by $Q_n \subset \omega_n$; i.e., X_i is omitted from the sum in the proof of the theorem if $i \in Q_n$. The included set is then $P_n = \bar{Q}_n \cap \omega_n$. Further, we consider that P_n is composed of $r(n)$ subsets P_{nk}, where $P_n = \bigcup_{k=1}^{r(n)} P_{nk}$. In most applications (see Section 4) it will happen that each P_{nk} becomes infinite, and also that Q_n becomes infinite, but that the ratio of the number of omitted variables to the number of included ones goes to zero.

The set of included variables is denoted symbolically by $\mathbf{X}_p^{(n)} = \{X_i : i \in P_n\}$, with corresponding subsets $\mathbf{X}_p^{(nk)}$. The set of all variables up to n is written symbolically as $\mathbf{X}^{(n)}$. We denote $a_i = E(X_i)$, and

$$\begin{aligned} X_{nk} &= \textstyle\sum_{i \in P_{nk}} (X_i - a_i) , & B_{nk}^2 &= E(X_{nk}^2) , \\ B_n^2 &= \textstyle\sum_{k=1}^{r(n)} B_{nk}^2 , & \xi_{nk} &= X_{nk}/B_n^2 , \\ \xi_n &= \textstyle\sum_{k=1}^{r(n)} \xi_{nk} , & \bar{\xi}_n &= \textstyle\sum_{i \in Q_n} (X_i - a_i)/B_n^2 . \end{aligned}$$

Since $X_{nk(a)}$ is an N-dimensional vector, X_{nk}^2 denotes the scalar product. Note that B_{nk}^2 is the trace of the $N \times N$ covariance matrix $E(X_{nk(a)} X_{nk(b)})$, and therefore the vector variable $\xi_{n(a)}$, considered formally as a sum of independent vector rv's $\xi_{nk(a)}$, is normalized such that its covariance matrix $C^{(n)}$ has unit trace.

We denote the marginal probability function of the set of vector variables $\mathbf{X}_p^{(n)}$ by

$$G^{(n)}(A) = P\{\mathbf{X}_p^{(n)} \in A\} ,$$

and correspondingly for the subsets

$$G^{(nk)}(A) = P\{\mathbf{X}_p^{(nk)} \in A\} .$$

The probability functions for the vector variables ξ_{nk} are written

$$F_{nk}(A) = P\{\xi_{nk} \in A\} .$$

We abbreviate the difference

$$\delta G^{(n)} = G^{(n)} - \prod_{k=1}^{r(n)} G^{(nk)} .$$

Using the above notation, we now present the conditions used in Section 3.

The first condition is a Lindeberg-Feller condition, such as used in proofs of the classical clt ([4] 227, [3] 491): For any $\tau > 0$, we must have

$$\text{(I)} \qquad \sum_{k=1}^{r(n)} \int_{|x|>\tau} x^2 F_{nk}(dx) \to 0 \qquad (n \to \infty) .$$

As in the independent variable case, this implies that each of the vector variables X_{nk} contributes a vanishingly small amount to the sum S_n/B_n in the limit $n \to \infty$.

The second condition is a decoupling condition which entails the asymptotic independence of the X_{nk} as $n \to \infty$. We require

$$\text{(II)} \qquad \int |\delta G^{(n)}(d\mathbf{X}_p^{(n)})| \to 0 \qquad (n \to \infty) .$$

The separating set Q_n must thus decouple the variables X_{nk} in the limit. In Section 3 we show that this condition is implied by a regularity condition resembling ones used by Serfling [10] and Philipp [6], but is in many ways stronger than theirs. Our condition (II) itself also seems rather strong, but it is easily satisfied for the Markov process discussed in Section 4. The integral in condition (II) is simply the "total variation" of $\delta G^{(n)}$.

The final condition is much more of a restriction on the sets Q_n, and is a rather weak restriction on the random process itself:

$$\text{(III)} \qquad E\,|\bar{\xi}_n| \to 0 \qquad (n \to \infty) .$$

Generally, this is true if, as mentioned before, the membership of P_n becomes infinite sufficiently faster than Q_n. Again, our Markov process provides a natural illustration of this occurrence.

3. The theorems and a lemma. We now prove the clt for the entire sum $\xi_n + \bar{\xi}_n$, assuming conditions (I), (II), and (III). Again, no moments of order higher than second are assumed to exist.

We use the method of characteristic functions, and introduce quadratic forms $\tilde{t}C^{(n)}t$ and $\tilde{z}C^{-1}z$, where

$$C^{(n)}_{(a)(b)} = \sum_{k=1}^{r(n)} \int x_{(a)} x_{(b)} F_{nk}(dx) ,$$

and C^{-1} is the matrix inverse of the matrix $C = \lim_n C^{(n)}$, which is assumed to exist. (Remember that tr $C^{(n)} = 1$, identically.)

We denote the multivariate normal distribution with covariance matrix C as

$$N(u, C) = (2\pi)^{-N/2} |C|^{-1} \int_{-\infty}^{u_1} \cdots \int_{-\infty}^{u_N} dz_1 \cdots dz_N \exp(-\tilde{z}C^{-1}z/2)$$

THEOREM 3.1. *If there exists a sequence P_{nk} such that conditions* (I), (II), *and* (III) *hold and $C = \lim_n C^{(n)}$ exists, then the distribution function for the normed vector sum $S_n/B_n = \xi_n + \bar{\xi}_n$ satisfies*

$$P\{\xi_n + \bar{\xi}_n < u\} \to N(u, C) \qquad (n \to \infty).$$

PROOF. We define the characteristic functions with vector argument t

$$f_{nk}(t) = E[\exp(i\xi_{nk} \cdot t)] = \int \exp(ix \cdot t) F_{nk}(dx)$$
$$\phi_{on}(t) = \prod_{k=1}^{r(n)} f_{nk}(t)$$
$$\phi_n(t) = E\{\exp[i(\xi_n + \bar{\xi}_n) \cdot t]\}$$

Note that $E(\xi_{nk}) = 0$, $E[(\xi_n \cdot t)^2] = \tilde{t}C^{(n)}t$, and therefore the proof of the classical central limit theorem as in Gnedenko ([4] 228) or Feller ([3] 491) goes through for the characteristic function $\phi_{on}(t)$ because of condition (I), even though the distributions F_{nk}, for different n, comprise different groups of variables. The only changes needed in Gnedenko's proof, for example, are a result of the multidimensionality of our rv's ξ_{nk}, and simply involve substitutions of the form $xt \to x \cdot t$, and the change $t^2 \to \tilde{t}C^{(n)}t$ in his expression for the quantity ρ_n([4] 230).

Therefore, condition (I) implies that

$$\phi_{on}(t) \to \exp(-\tilde{t}Ct/2) \qquad (n \to \infty).$$

We now show that (II) and (III) imply that $\phi_n(t) - \phi_{on}(t) = \Delta\phi_n(t) \to 0$ $(n \to \infty)$. Consider that

$$\phi_{on}(t) = \int \exp(i\xi_n \cdot t) \prod_{k=1}^{r(n)} G^{(nk)}(d\mathbf{X}_p^{(nk)})$$

and therefore

$$\Delta\phi_n(t) = \int \exp(i\xi_n \cdot t)[\exp(i\bar{\xi}_n \cdot t) - 1]F^{(n)}(d\mathbf{X}^{(n)}) + \int \exp(i\xi_n \cdot t)\delta G^{(n)}(d\mathbf{X}_p^{(n)}).$$

Thus

$$|\Delta\phi_n(t)| \leqq \int |\exp(i\bar{\xi}_n \cdot t) - 1| F^{(n)}(d\mathbf{X}^{(n)}) + \int |\delta G^{(n)}(d\mathbf{X})|.$$

But by Schwarz' inequality and the relation $|e^{ia} - 1| \leqq |a|$ one may show that

$$|\Delta\phi_n(t)| \leqq |t| E|\bar{\xi}_n| + \int |\delta G^{(n)}(d\mathbf{X})|$$

which goes to zero by (II) and (III) on any finite domain of t. Thus

$$\phi_n(t) \to \exp(-\tilde{t}Ct/2) \qquad (n \to \infty),$$

and the theorem is proved.

We now prove a converse theorem showing the necessity of condition (I) for convergence to the multivariate normal distribution. We note that (I) is stated in terms of the sums X_{nk} instead of the original rv's X_i, and so the analogy to the independent variable case is not complete. However, the following theorem shows that condition (I) is necessarily associated with central-limit convergence within the scheme we have set up.

Further, the additional assumption needed to prove the converse theorem is almost the same as in the independent rv case ([3] 492). It is required that

$$\max_k B_{nk}/B_n \to 0 \qquad (n \to \infty)\,, \tag{3.1}$$

a condition which is perfectly natural for central limit behavior.

THEOREM 3.2. *If* P_{nk} *exists such that* (3.1), (II), *and* (III) *are true and if* $P\{\xi_n + \bar{\xi}_n < u\} \sim N(u, C^{(n)})$ $(n \to \infty)$, *then it follows that* (I) *is true also.*

PROOF. Note first that (II) and (III) imply that $\Delta\phi_n(t) \to 0 (n \to \infty)$ as in the proof of Theorem 3.1. Thus the hypotheses of this theorem show that

$$\phi_{on}(t) \sim \exp(-\bar{t}C^{(n)}t/2) \qquad (n \to \infty)\,. \tag{3.2}$$

Now as remarked by Feller ([3] 492), (3.2) implies that

$$\log \phi_{on}(t) \sim \sum_{k=1}^{r(n)} [f_{nk}(t) - 1]\,,$$

and we may continue with his version of the proof for the independent variable case, provided we substitute $t = (0, \cdots, t_{(a)}, 0, \cdots, 0)$. His (6.12) then becomes, in our notation,

$$\begin{aligned} C^{(n)}_{(a)(a)} - \sum_{k=1}^{r(n)} \int_{|x| \leqq \tau} x^2_{(a)} F_{nk}(dx) &= \sum_{k=1}^{r(n)} \int_{|x| > \tau} x^2_{(a)} F_{nk}(dx) \\ &\leqq \frac{4}{t^2 \tau^2} + o(1)\,, \end{aligned}$$

which latter quantity can be made arbitrarily small for large $t_{(a)}$. Since x is of finite dimension, the proof is complete.

It is worthwhile noting that condition (II) is implied by a regularity condition which resembles ones discussed by Serfling [10] and Philipp [6]. We have the following lemma, the proof of which is obvious.

LEMMA. *Let* A_{nk} *be any set in the Euclidean space of dimension* $N \times$ *the number of points in* P_{nk}. *Then if for all sequences of such sets* $A_{n1}, A_{n2}, \cdots, A_{nr}$, *there exists a number* $\psi(n)$ *such that* $\psi(n) \to 0$ $(n \to \infty)$ *and*

$$|G^{(n)}(A_{n1}, \cdots, A_{nr}) - \prod_{k=1}^{r(n)} G^{(nk)}(A_{nk})| \leqq \psi(n) \prod_{k=1}^{r(n)} G^{(nk)}(A_{nk})\,,$$

then (2.2) *holds.*

We may note an interesting generalization of Theorems 3.1 and 3.2. For the purposes of constructing the probability functions F_{nk}, it may be convenient not to use the real marginal distributions of ξ_{nk} derived from $F^{(n)}$, but some other distributions which approach the "real" ones as $n \to \infty$. Denoting expectation values defined by them by $E'(\ )$, we may redefine ξ_{nk} so that $E'(\xi_{nk}) = 0$, but this may force $E(\xi_{nk}) \neq 0$. In this case however, Theorems 3.1 and 3.2 still go through provided that

$$E(\xi_n) \to 0 \qquad (n \to \infty) \tag{3.3}$$

is introduced as an additional assumption in both cases. It is then no longer that necessary that $E(\xi_{nk}^2)$ exist, but only $E'(\xi_{nk}^2)$.

4. Application to a Markov process with stationary transition mechanism. We investigate under what conditions a Markov process with stationary transition mechanism satisfies conditions (II) and (III). We do not discuss condition (I), since it seems to depend on detailed analytic properties of the process. In any case, (I) has been shown to be a necessary condition as well as a part of the scheme of sufficient conditions. Also, the condition that $\lim_n C^{(n)}$ exist is not discussed here. It is a rather trivial condition and only arises in the multidimensional case.

We write the probability that $X_1 \in A_1, \cdots, X_n \in A_n$, as

$$P\{\textstyle\bigcap_{k=1}^n A_k\} = \int_{A_1} \cdots \int_{A_n} P_1(dX_1)P(X_1, dX_2)P(X_2, dX_3) \cdots P(X_{n-1}, dX_n) ,$$

where $P_1(A)$ is a given initial probability that $X_1 \in A$, and $P(x, A) = P(1, x, A)$ is the conditional one-step transition probability that $X_k \in A$, given that $X_{k-1} = x$. Since we are interested only in stationary transition mechanisms, $P(x, A)$ does not depend on k.

The two-step transition probability is thus

$$P(2, x, A) = \int P(x, dz)P(z, A) ,$$

and so on iteratively.

We now discuss the choice of the sets P_{nk} which we use here. In a manner similar to Rosenblatt [9], we break up the set ω_n into alternating blocks of large and small subsets, where each large subset contains $p(n)$ integers, and each small subset, $q(n)$. Formally, we write

$$\begin{aligned} I(n, k) &= k(p + q) - p + 1 , \qquad J(n, k) = k(p + q) , && k = 1, \cdots, r(n) \\ P_{nk} &= \{m : I(n, k) \leqq m \leqq J(n, k)\} , && k = 1, \cdots, r(n) \\ Q_n &= \textstyle\bigcup_{k=1}^{r+1} \{m : J(n, k) < m < I(n, k)\} . \end{aligned}$$

We use the convenient notation for the differential probability function

$$dH_{nk} = P(X_{I(n,k)}, dX_{I(n,k)+1})P(X_{I(n,k)+1}, dX_{I(n,k)+2}) \cdots P(X_{J(n,k)-1}, dX_{J(n,k)}) . \tag{4.1}$$

Let us now describe the conditions which we impose on the Markov process to insure satisfaction of conditions (II) and (III). It is sensible to consider only processes such that the effect of the distant past on the future becomes small, and we require that there exist a probability function $H(A)$ such that $\int x^2 H(dx)$

exists, and

$$P(m, x, A) = H(A) + \delta P(m, x, A) ,$$

where $\delta P \to 0 (m \to \infty)$ in such a way that, if we define $M_m = \max_x \int |\delta P(m, x, dz)|$,

$$M_m \to 0 \qquad (m \to \infty) . \tag{4.2}$$

We also ask that the absolute first moments $E|X_i|$ be bounded:

$$K = \max_i [\int |x|\, H(dx) + \int P_1(dz) \int \delta P(i-1, z, dx)\, |x|] < \infty , \tag{4.3}$$

and that we be able to choose $p(n)$, $q(n)$, $r(n)$, $\to \infty$ $(n \to \infty)$ in such a way that

$$rq/B_n \to 0 \qquad (n \to \infty) . \tag{4.4}$$

This last condition (4.4) is generally very easy to satisfy, since we might expect $B_n \sim An^{\frac{1}{2}}$, where A is a constant. Since $n \sim r(p+q)$, (4.4) would then be equivalent to $q(r/p)^{\frac{1}{2}} \to 0$.

In what follows, the notation delineated below is useful.

$$dH_k^n = H(dX_{I(n,k)}) , \tag{4.5}$$

$$dR_k^q = P(q+1, X_{J(n,k-1)}, dX_{I(n,k)}) = dH_k^n + d\delta R_k^q \tag{4.6}$$

$$dS_k^n = \int_1 P(dX_1) P(I(n, k) - 1, X_1, dX_{I(n,k)} = dH_k^n + d\delta R_k^n . \tag{4.7}$$

It is now a simple matter to prove the following theorem:

THEOREM 4.1. *A Markov process with stationary transition mechanism and initial probability function* $P_1(A)$, *such that* (4.2), (4.3), *and* (4.4) *hold, satisfies conditions* (II) *and* (III).

PROOF. We first note that (4.3) and (4.4) imply (III):

$$E|\bar{\xi}_n| \leqq 2 \sum_{i \in Q_n} E|X_i|/B_n \leqq 2Krq/B_n ,$$

which vanishes as $n \to \infty$ by (4.4).

It is straightforward, but somewhat tedious, to show that (4.2) implies (II). One may write, using (4.1) and (4.5)–(4.7),

$$\begin{aligned}\delta G^{(n)}(d\mathbf{X}_p^{(n)}) &= (\textstyle\prod_{k=1}^r dH_{nk}) dS_1^n \{\prod_{k=2}^r dR_k^q - \prod_{k=2}^r dS_k^n\} \\ &= (\textstyle\prod_{k=1}^r dH_{nk})(dH_1^n + d\delta S_1^k) \\ &\quad \times \textstyle\sum_{k=2}^r (\prod_{i=2}^{k-1} dH_i^n)(d\delta R_k^q \prod_{j=k+1}^r dR_k^q - d\delta S_k^n \prod_{j=k+1}^r dS_k^n) .\end{aligned}$$

But (4.2) then implies that

$$\int |\delta G^{(n)}(d\mathbf{X})| \leqq rM_q + \sum_{k=2}^r M_{I(n,k)} \lesssim 2rM_q .$$

Now, since $M_q \to 0 (q(n) \to \infty)$, we can always define $r(n) \to \infty$, such that $rM \to 0$. Thus (II) is fulfilled, with r defined as a function of q. Note that this will very likely be consistent with (4.4), since $n \sim rp$ and since (4.4) may be expressed as $r(q)q/B_{r(q)p} \to 0$, which then allows us to define $p(q)$. Clearly, $p(q)$

must be a rapidly increasing function, which is reasonable, since one would expect $p/q \to \infty (n \to \infty)$ if this method of partitioning the variables into P_n and Q_n is to be a sensible one.

Thus, both r, p and $n \geqq r(p+q)$ are defined as functions of q, but with some leeway. We note that for any n, one may write p, q, r integers such that $r(p+q) \leqq n < (p+q)+q$, where p, r are both monotone increasing functions of q. Therefore we are not restricted to n of the form $r(p+q)$. The uneven "tail" of length $< q$ enters only into the discussion of (III), where it may be neglected.

Thus the proof of the theorem is complete.

5. Discussion. We have seen that our method of proof has resulted in two theorems, Theorems 3.1 and 3.2, which are in many ways analogous to the theorems for sums of independent rv's. It is evident that one may also prove a Lyapunov theorem which is analogous to the independent variable one ([4] 232).

However, the analogy between dependent and independent rv's breaks down when one considers a strictly stationary process. For independent variables, the Lindeberg condition follows immediately, but for dependent variables strict stationarity is not sufficient to prove condition (I). It would be interesting to investigate more thoroughly under what circumstances condition (I) holds.

Comparison of our results with other recent work must be made. Rosén [7], [8] has proved clt's for dependent vector variables using interesting "structure-of-dependence" and uniform smallness conditions. The structure-of-dependence conditions relate to asymptotic properties of certain first and second conditional moments and state, among other things, that future first moments of the rv's are linearly related to the sum of past values of the rv's. The smallness condition may be described as a "local" form of the Lindeberg-Feller condition. These hypotheses seem to be quite different from the ones we have used, particularly with respect to our decoupling condition (II), which involves only zeroth moments of the rv's.

Serfling [10] has derived a variety of clt's for one-dimensional rv's involving moment-decoupling and regularity conditions, assuming the existence of moments of order greater than second, and assuming certain rates of divergence of quantities like $E(|S_n|/|B_n|^{2+\delta})$ for various $\delta > 0$. Also introduced are conditions on rates of decoupling. Since we have emphasized the Lindeberg condition, it is difficult to see where our assumptions might overlap.

Philipp [6] also derives a number of clt's for one-dimensional random processes, using restrictions like $\sum_n \psi(n)^{\frac{1}{2}} < \infty$ (see our Lemma in Section 3). A Lindeberg condition is also discussed and is likewise shown to be a necessary one, but in general the assumptions used are different from ours.

Acknowledgments. The author would like to thank Professors J.A. Bather, J.L. Denny, and W. Philipp for their helpful comments. It is particularly a pleasure to thank Professor M. Rosenblatt for illuminating discussions about many aspects of central limit theory.

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